ON DISTRIBUTIVELY GENERATED NEAR-RINGS[1]

by STEVE LIGH

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The following theorems in ring theory are well-known:

1. Let R be a ring. If e is a unique left identity, then e is also a right identity.

2. If R is a ring with more than one element such that $aR = R$ for every nonzero element $a \in R$, then R is a division ring.

3. A ring R with identity $e \neq 0$ is a division ring if and only if it has no proper right ideals.

In this note we shall show that the above theorems can be generalized to distributively generated near-rings. Examples will be given to show that the theorems do not hold for arbitrary near-rings.

1. Definitions

A *near-ring* R is a system with two binary operations, addition and multiplication, such that:

(i) The elements of R form a group R^+ under addition.

(ii) The elements of R form a multiplicative semi-group.

(iii) $x(y+z) = xy+xz$, for all $x, y, z \in R$.

In particular, if R contains a multiplicative semigroup S whose elements generate R^+ and satisfy

(iv) $(x+y)s = xs+ys$, for all $x, y \in R$ and $s \in S$,

we say that R is a *distributively generated* (d.g.) near-ring.

The most natural example of a near-ring is given by the set R of all mappings of an additive group (not necessarily abelian) into itself. If the mappings are added by adding images and multiplication is iteration, then the system $(R, +, .)$ is a near-ring. If S is a multiplicative semigroup of endomorphisms of G and R' is the sub-near-ring generated by S, then R' is a d.g. near-ring. Other examples of d.g. near-rings may be found in (**1**).

A near-ring R that contains more than one element is said to be a *division* near-ring if and only if the set R' of nonzero elements is a multiplicative group. Every division ring is an example of a division near-ring. For examples of division near-rings which are not division rings, see (**4**).

[1] Portions of this paper appear in the author's Ph.D. dissertation written under the direction of Professor J. J. Malone, Jr., at Texas A&M University.

An element a of R is *right distributive* if $(b+c)a = ba+ca$ for all $b, c \in R$. An element $x \in R$ is *anti-right distributive* if $(y+z)x = zx+yx$ for all $y, z \in R$. It follows at once that an element a is right distributive if and only if $(-a)$ is anti-right distributive. In particular, any element of a d.g. near-ring is a finite sum of right and anti-right distributive elements.

A subset B of a near-ring R is called a *right ideal* if $(B, +)$ is a subgroup of $(R, +)$ and $B \,.\, R = \{b \,.\, r\colon\ b \in B, r \in R\} \subseteq B$.

(1.1) **Lemma.** *Let R be a near-ring, then*

(i) $x \,.\, 0 = 0, x \in R$,

(ii) $x(-y) = -(xy), x, y \in R$.

In particular, if 1 *is the identity of R, then*

(iii) $x(-1) = -x, x \in R$.

These results are easy consequences of the definitions.

2. Division near-rings

In general if a near-ring has an identity 1, (-1) need not commute with all the elements. The following lemma is easy to verify:

(2.1) **Lemma.** *If R is a near-ring with identity* 1, *then* $(-1)(-1) = 1$. *Furthermore if* $(-1)r = r(-1)$ *for all* $r \in R$, *then* R^+ *is commutative.*

(2.2) **Theorem.** *The additive group* R^+ *of a division near-ring R is abelian.*

Proof. Observe that if $1+1 = 0$, then $x+x = x(1+1) = x \,.\, 0 = 0$ for each non-zero element $x \in R$ and hence R^+ is clearly abelian. If $(-1) \neq 1$, let F be the mapping of R into R given by $rF = r(-1)+(-1)r$. F is a one-to-one map. Suppose $r(-1)+(-1)r = s(-1)+(-1)s$. Then

$$s+r(-1)+(-1)r+(-1)s(-1) = 0.$$

It follows that $(-1)(r+s(-1)) = r+s(-1)$. If $r+s(-1) \neq 0$, then $(-1) = 1$, contrary to assumption. Thus $r+s(-1) = 0$ and this implies $r = s$. Now if R is finite, then F is also an onto mapping which means that for $r \in R$, there is an element $s \in R$ such that $s(-1)+(-1)s = r$ or $r(-1) = (-1)s(-1)+s$. Hence $(-1)[s(-1)+(-1)s] = (-1)r$ implies $(-1)s(-1)+s = (-1)r$ and for all $r \in R$ we have $(-1)r = r(-1)$. From (2.1), R^+ is abelian. This result was first proved by Zassenhaus (4). A proof for the infinite case can be found in (3).

Even if the additive group of a near-ring with identity 1 is commutative, (-1) need not commute multiplicatively with all elements. For example, if G is the additive abelian group of order three then the set of mappings defined on G is a near-ring whose additive group is abelian. But $(-1)f \neq f(-1)$ where f is a non-zero constant mapping. However this is true for "most" division near-rings as the following corollary shows:

(2.3) **Corollary.** *Let R be a division near-ring with identity* 1 *such that* $1 \,.\, r = r \,.\, 1$ *for all* $r \in R$, *then* $(-1)r = r(-1)$.

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Proof. Suppose there exists $w \in R$ such that $(-1)w = w(-1)+x$, $x \neq 0$. Then $x = w+(-1)w = (-1)((-1)w+w) = (-1)(w+(-1)w) = (-1)x$ and hence $(-1) = 1$. Thus $w = w+x$ and this implies $x = 0$, which is a contradiction.

Remark. It can be shown that if a division near-ring R has three or more elements, then the identity on the multiplicative group is the identity on R.

3. Distributively generated near-rings

(3.1) **Lemma.** *Let R be a near-ring. If $ux = x$ for all $x \in R$, and if a is anti-right distributive, then*

(i) $(x+y+z)a = za+ya+xa$,

(ii) $(xu+y+u)a = a$ *where* $x+y = y+x = 0$.

Proof. Obvious.

(3.2) **Theorem.** *If R is a d.g. near-ring and if u is a unique left identity, then u is also a right identity.*

Proof. Suppose $ux = x$ for all $x \in R$. Since R is a d.g. near-ring, we have for any $w \in R$, $w = w_1+w_2+\ldots+w_n$ where w_i is either a right or anti-right distributive element of R. Now consider $(xu+y+u)w$ where $x+y = y+x = 0$ and w is any element of R. Now applying (3.1) we have

$$
\begin{aligned}
(xu+y+u)w &= (xu+y+u)(w_1+w_2+\ldots+w_n) \\
&= (xu+y+u)w_1+(xu+y+u)w_2+\ldots+(xu+y+u)w_n \\
&= w_1+w_2+\ldots+w_n \\
&= w.
\end{aligned}
$$

The uniqueness of u implies $xu = x$ for all $x \in R$. This completes the proof.

Remark. It can be shown easily that if a near-ring has a unique right identity, then it is also a left identity. Theorem (3.2) is not true in general for arbitrary near-rings. Consider the following example: Let G be an additive group with at least three elements. Suppose $e \in G$ such that $e \neq 0$. Define $ex = x$ for all $x \in G$ and $gx = 0$ for all $g \neq e$ of G. Then $(G, +, .)$ is a near-ring (2). It is clear that e is the unique left identity but not a right identity.

The following lemma is easy:

(3.3) **Lemma.** *If D is a d.g. near-ring, then $0 \,.\, d = 0$ for all $d \in D$.*

(3.4) **Theorem.** *A necessary and sufficient condition for a d.g. near-ring D with more than one element to be a division ring is that, for all nonzero $a \in D$, $aD = D$.*

Proof. Necessity. There is an element $e \in D$ such that $ae = ea = a$ for $a \neq 0$ in D. Clearly $aD \subseteq D$. Suppose $a \neq 0$ is in D. Then there exists an element $b \in D$ such that $ab = e \in aD$. Thus $x = a(bx)$, for all $x \in D$, and so $x \in aD$. Hence $aD = D$.

Sufficiency. If a and b are nonzero elements of D, then $ab \neq 0$. For if not, there exist a_e and b_e such that $aa_e = a$ and $bb_e = a_e$. Thus

$$0 = abb_e = aa_e = a,$$

which is a contradiction. Now let r be a nonzero right distributive element of D. Then there is an element $e \in D$ such that $re = r$. But

$$r(er-r) = rer-rr = 0.$$

From the above we have $er = r$. This means that e is a two-sided identity for r. Since we know from the first part of the proof that the set of non-zero elements is closed under multiplication and multiplication is associative it only remains to prove that e is a right identity for the non-zero elements of D and every non-zero element of D has a right inverse. Let $d \neq 0$ be an element in D. Then $(de-d)r = der-dr = dr-dr = 0$. Since $r \neq 0$, we have that $de = d$. Also $dD = D$ implies there is a $d' \in D$ such that $dd' = e$. Thus we have shown that the d.g. near-ring D is a division near-ring. From (2.2) the additive group D^+ of D is abelian. It now follows (**1**, p. 93) that every element of D is right distributive and hence D is a division ring.

(3.5) **Corollary.** *A d.g. near-ring D with identity $e \neq 0$ is a division ring if and only if it has no proper right ideals.*

Proof. Necessity is quite clear. Suppose D has no proper right ideals. For each $a \neq 0$ in D, aD is a right ideal of D. Thus $aD = D$ and by (3.4) D is a division ring.

The following example shows that (3.4) can not be extended to arbitrary near-rings: Let $D = \{0, 1\}$ with addition and multiplication as defined below. Then it can be verified easily that D is a near-ring which is not a division ring.

+	0	1
0	0	1
1	1	0

.	0	1
0	0	1
1	0	1

In fact, D is the only (up to isomorphism) division near-ring for which 1 is not the identity of D.

Finally it can be shown easily that a near-ring D with identity $e \neq 0$ and $0 \,.\, x = 0$ for all $x \in D$ is a division near-ring if and only if it has no proper right ideals. Since there exist division near-rings which are not division rings (**4**), we conclude that (3.5) can not be extended to arbitrary near-rings.

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DEPARTMENT OF MATHEMATICS
TEXAS A & M UNIVERSITY
COLLEGE STATION, TEXAS 77843
U.S.A.

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