On Double and Multiple Interval Graphs

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ABSTRACT

In this paper we discuss a generalization of the familiar concept of an interval graph that arises naturally in scheduling and allocation problems. We define the interval number of a graph G to be the smallest positive integer t for which there exists a function f which assigns to each vertex u of G a subset $f(u)$ of the real line so that $f(u)$ is the union of t closed intervals of the real line, and distinct vertices u and v in G are adjacent if and only if $f(u)$ and $f(v)$ meet. We show that (1) the interval number of a tree is at most two, and (2) the complete bipartite graph $K_{m,n}$ has interval number $\lceil (mn+1)/(m+n) \rceil$.

1. INTRODUCTION

A graph G is called an *interval graph* if there is a function f that assigns to each vertex u of G a closed interval of the real line R so that distinct vertices u, v of G are adjacent if and only if $f(u) \cap f(v) \neq \emptyset$. Structural characterizations of interval graphs have been provided by Lekkerkerker and Boland [7] who specified the forbidden subgraphs, Gilmore and Hoffman [2] in terms of cycles, and Fulkerson and Gross [1] in terms of matrices. Definitions not given here can be found in Ref. 5.

In this paper, we consider a generalization of the concept of an interval graph; we are motivated by scheduling and allocation problems that arise when a graph is used to model constraints on interactions between

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components of a large scale system. For a graph G, we define* the *interval number* of G, denoted $i(G)$, as the smallest positive interger t for which there exists a function f which assigns to each vertex u of G a subset $f(u)$ of R which is the union of t (not necessarily disjoint) closed intervals of R and distinct vertices u, v of G are adjacent if and only if $f(u) \cap f(v) \neq \varnothing$. The function f is called a *t-representation* of G. Thus G is an interval graph if and only if its interval number is one. Obviously every graph G with p vertices has an interval number $i(G) \leq p-1$, and thus $i(G)$ is well defined.

A number m is called an *upper bound* for a representation f of a graph G when $m > r$ for every number r in $f(u)$ and every vertex u of G.

We will frequently find it convenient to impose an additional restriction on a representation of a graph. A t-representation f of a graph G is said to be *displayed* if for every vertex u of G, there exists an open interval I_u contained in $f(u)$ so that $I_u \cap f(v) = \varnothing$ for every vertex v in G with $u \neq v$.

Recall that for any tree T, the tree T' is obtained by removing all the endvertices of T. A *caterpillar* is a tree T for which T' is a path. It was noted in Harary and Schwenk [6] that T is a caterpillar if and only if T does not contain the subdivision graph of $K_{1,3}$ as a subtree.

Theorem 1. If T is a tree, then $i(T) = 1$ if T is a caterpillar and $i(T) = 2$ if it is not.

Proof. If T is a tree and does not contain the subdivision graph of $K_{1,3}$ as a subtree, then it follows from the forbidden subgraph characterization of Ref. 7 that T is an interval graph. On the other hand, if T contains this subdivision graph, then T is not an interval graph and $i(T) \geq 2$.

Now we proceed by induction on the number of vertices to show that every tree has a displayed 2-representation. If T is the one point tree, the result is trivial. Next assume that for some $k \geq 1$, every tree on k vertices has a displayed 2-representation and let T be a tree with $k+1$ vertices.

Choose an endvertex u of T and let f be a displayed 2-representation of the tree $T-u$. Let v be the unique vertex adjacent to u in T and let I_v be an open interval contained in $f(v)$ so that $I_v \cap f(w) = \varnothing$ for every vertex w in $T-u$ with $w \neq v$. Choose a closed interval A contained in I_v.

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* Roberts [8] has studied another generalization of interval graphs. He defines the *boxicity* of a graph G as the smallest positive integer t for which there exists a function f which assigns to each vertex u of G a sequence $f(u)(1)$, $f(u)$ $2), \ldots, f(u)(t)$ of closed intervals of R so that distinct vertices u, v of G are adjacent if and only if $f(u)(i) \cap f(v)(i) \neq \varnothing$ for $i = 1, 2, 3, \ldots, t$.

Now choose an upper bound m for f and define $g(w)=f(w)$ for every vertex w in $T-u$ and $g(u)=A\cup[m, m+1]$. It is clear that g is a displayed 2-representation of T and our proof is complete. ∎

2. COMPLETE BIPARTITE GRAPHS

We now derive our main result. We use the notation $\lceil x\rceil$ to represent the smallest integer among those which are at least as large as x.

Theorem 2. The interval number of the complete bipartite graph $K_{m,n}$ is given by

$$i(K_{m,n}) = \lceil (mn+1)/(m+n)\rceil .$$

Proof. We first show that $i(K_{m,n})\geq \lceil (mn+1)/(m+n)\rceil$. Suppose that f is a t-representation of $K_{m,n}$. Without loss of generality, we may assume that for each vertex u in $K_{m,n}$, $f(u)$ is the union $A_1(u)\cup A_2(u)\cup\cdots\cup A_t(u)$ of t pairwise disjoint closed intervals.

We now use f to determine a graph G. The vertices of G are the ordered pairs of the form (u, i) where u is a vertex in $K_{m,n}$ and $1\leq i\leq t$ with distinct vertices (u, i) and (v, j) adjacent in G when $A_i(u)\cap A_j(v)\neq\emptyset$. The function g defined by $g(u, i)=A_i(u)$ is a 1-representation of G so G is an interval graph. Since G is bipartite, it is triangle-free. Since G is an interval graph, it does not contain a cycle of four or more vertices as an induced subgraph. Therefore, G is a forest. Note that G has $(m+n)t$ vertices and at most $(m+n)t-1$ edges.

Now suppose that $e=\{u, v,\}$ is an edge of $K_{m,n}$. Then there exist integers i, j with $A_i(u)\cap A_j(v)\neq\emptyset$, and we may therefore define a function h from the edge set of $K_{m,n}$ to the edge set of G by setting $h(e)=h(\{u, v\})=\{(u, i), (v, j)\}$. Clearly, h is a one-to-one function and since $K_{m,n}$ has mn edges, we see that $mn\leq(m+n)t-1$, i.e., $t\geq \lceil (mn+1)/(m+n)\rceil$.

We will now show that $i(K_{m,n})\leq\lceil (mn+1)(m+n)\rceil$. Let $t=\lceil (mn+1)/(m+n)\rceil$. We will construct an interval graph G with a 1-representation g. We will then construct a t-representation f of $K_{m,n}$ by appropriately choosing, for each vertex u of $K_{m,n}$, t intervals from the range of g as the intervals whose union is $f(u)$.

We begin by labeling the vertices of $K_{m,n}$ with the symbols $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n$ so that a_i is adjacent to b_j for all i and j. Without loss of generality, we may assume $m\geq n$. Let $A=\{1, 2, 3, \ldots, m\}$ and $B=\{1, 2, 3, \ldots, n\}$.

We next construct a graph T whose vertex set is

$$\{u_k: 1 \le k \le nt\} \cup \{v_k: 1 \le k \le nt-1\} \cup \{w_{ij}: 1 \le i \le m, 1 \le j \le n\},$$

where T has the following adjacencies: v_k is adjacent to u_k and u_{k+1} for $k = 1, 2, \ldots, nt-1$ and w_{ij} is adjacent to u_j for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. The graph T is a caterpillar and, by Theorem 1, is also an interval graph. Consequently any induced subgraph of T is also an interval graph.

The next step in the construction is to color some, but not all, of the vertices of T using the elements of A as colors. We begin by assigning to $u_1, u_2, \ldots, u_{nt}$ the colors

$$1, 2, 3, \ldots, n, 1, 2, 3, \ldots, n, \ldots, 1, 2, 3, \ldots, n$$

in order. Note that each color from B is used exactly t times.

Now let $s = n - t$; then $2s \le n-1$. Suppose that S is a set of either $2s$ or $2s-1$ consecutive vertices from the sequence $v_1, v_2, \ldots, v_{nt-1}$. Consider a subset S' of S that contains s vertices, no two of which are consecutive. Then let B' be the subset of B consisting of those integers j for which there is a vertex v from S' and a vertex u adjacent to v with u having color j. It is easy to verify that B' must contain $2s$ elements, i.e., the s vertices of S' are adjacent to $2s$ distinctly colored vertices.

The next step is to assign colors to the first ms vertices in the sequence $v_1, v_2, \ldots, v_{nt-1}$. Note that $t = \lceil (mn+1)/(m+n) \rceil$ and $s = n - t$ imply that $ms \le nt - 1$. At this point, we must consider two cases depending on the parity of m. If m is even, then assign the vertices $v_1, v_2, \ldots, v_{ms}$ the colors

$$1, 2, 1, 2, \ldots, 1, 2, 3, 4, 3, 4, \ldots, 3, 4, \ldots, m-1, \\ m, m-1, m, \ldots, m-1, m$$

in order. Note that each color in A is to be used exactly s times. If m is odd, we modify this scheme as follows. We first assign color m to $v_1, v_{n+3}, v_{2n+5}, \ldots, v_{(s-1)(n+2)+1}$. Note that for each $j = 1, 2, 3, \ldots, 2s$, there are integers k, l for which u_k is adjacent to v_l, where v_l has color m and u_k has color j. Next assign to the $(m-1)s$ vertices in the sequence $v_1, v_2, \ldots, v_{ms}$, which were not assigned color m, the colors

$$1, 2, 1, 2, \ldots, 1, 2, 3, 4, 3, 4, \ldots, 3, 4, \ldots, m-2, m-1, \ldots, m-2, m-1$$

in order. Again we note that each color in A is to be used exactly s times.

When m is even, observe that each color i from A is assigned to s nonconsecutive vertices in a block of $2s-1$ consecutive vertices from the sequence $v_1, v_2, \ldots, v_{nt-1}$. When m is odd, we observe that distinct vertices that have been assigned color m are at least $n+2$ apart in the sequence $v_1, v_2, \ldots, v_{nt-1}$. Therefore, we observe that each color i from A with $i \neq m$ is assigned to s nonconsecutive vertices in a block of $2s$ or $2s-1$ consecutive vertices in the sequence $v_1, v_2, \ldots, v_{nt-1}$. For each color $i \in A$, define the set

$B(i) = \{j \in B$: There exist integers k, l with u_k adjacent to v_l for which u_k has been assigned color j and v_l has been assigned color $i\}$.

We conclude that for all values of m and for every color i from A, the set $B(i)$ contains exactly $2s$ elements.

The next step in the construction is to assign colors to some, but not all, of the vertices in $\{w_{ij}\colon 1 \le i \le m, 1 \le j \le n\}$. The construction is the same for all values of m. Let i be an element of A; assign color i to vertex w_{ij} if and only if j is an element of $B - B(i)$. Now let

$$U_1 = \{v_k\colon 1 \le k \le nt-1\} \cup \{w_{ij}\colon 1 \le i \le m, 1 \le j \le n\}$$

and let

$$U_2 = \{u_k\colon 1 \le k \le nt\}.$$

Observe that for each color i from A, exactly t vertices of U_1 have been assigned color i, and for each color j from B, exactly t vertices from U_2 have been assigned color j; furthermore, there exist adjacent vertices u', u'' with u' from U_1, u'' from U_2, u' having color i, and u'' having color j.

Now let G be the subgraph of T generated by the colored vertices and let g be a 1-representation of G. The final step in the construction is to use g to define a t-representation f of $K_{m,n}$. But this is accomplished simply by defining

$$f(a_i) = \cup\{g(u')\colon u' \text{ is a vertex from } U_1 \text{ and } u' \text{ has color } i\} \quad \text{for } i = 1, 2, \ldots, m$$

and

$$f(b_j) = \cup\{g(u'')\colon u'' \text{ is a vertex from } U_2 \text{ and } u'' \text{ has color } j\} \quad \text{for } j = 1, 2, \ldots, n.$$

It is trivial to verify that f is a t-representation of $K_{m,n}$. ∎

3. OTHER RESULTS

A preliminary version of this paper included a proof of the following result.

Theorem 3. If G has p vertices, then $i(G) \leq \lceil p/3 \rceil$.

This theorem may be established using a two-part argument in which it is proved inductively that a graph on $3n$ vertices has an n-representation and a triangle-free graph on $3n$ vertices has a displayed n-representation. The proof of the second part makes use of Turán's theorem for the maximum number of edges in a triangle-free graph.

However, the authors did not believe that the upper bound on the interval number of a graph provided by Theorem 3 was best possible. Motivated by the observation that the complete bipartite graph $K_{2n,2n}$ has $4n$ vertices and interval number $n+1$, the authors conjectured that if G is a graph with p vertices, then $i(G) \leq \lceil (p+1)/4 \rceil$.

The concept of interval number has been independently investigated by Griggs and West [4]. They obtained the formula given in Theorem 1 for the interval number of a tree as well as the upper bound given in Theorem 3. They also made the same conjecture concerning the maximum interval number of a graph with p vertices. And they also provided an upper bound on the interval number of a graph in terms of the maximum degree of a vertex in the graph. Specifically, they showed that if the maximum degree of a vertex in a graph G is d, then $i(G) \leq \lceil (d+1)/2 \rceil$. This last result allowed them to determine that the interval number of the n-cube Q_n is $\lceil (n+1)/2 \rceil$, which answered a problem posed in the preliminary version of this paper.

The authors have recently learned that Griggs [3] has established the conjecture by proving that if G has $4n-1$ vertices, then $i(G) \leq n$.

4. AN OPEN PROBLEM

Lekkerkerker and Boland [7] gave a forbidden subgraph characterization of interval graphs by listing the collection $\mathcal{I}_2$ of graphs defined by

$$\mathcal{I}_2 = \{G\colon\ i(G) = 2 \text{ but } i(H) = 1 \text{ for every proper induced subgraph } H \text{ of } G\}.$$

We propose the general problem of finding for $t \geq 3$, the collection

$$\mathcal{I}_t = \{G\colon\ i(G) = t \text{ but } i(H) \leq t-1 \text{ for every proper subgraph } H \text{ of } G\}.$$

The problem for $t = 3$ seems to both manageable and interesting since from applied viewpoint, graphs that are the intersection graphs of a family of sets each of which is the union of two intervals of the real line have practical significance, e.g., two work periods separated by a lunch break. By *double interval* graphs, we mean graphs with interval number two. Theorem 2 shows that $K_{2n,2n}$ is in $\mathscr{I}_{n+1}$ for every $n \geq 1$ and that $K_{2n-1,2n+2}$ is in $\mathscr{I}_{n+1}$ for every $n \geq 2$. In particular, we note then that a forbidden subgraph characterization of double interval graphs will include $K_{4,4}$ and $K_{3,6}$.

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