On double coverings of a pointed non-singular curve with any Weierstrass semigroup

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Abstract. Let H be a Weierstrass semigroup, i.e., the set $H(P)$ of integers which are pole orders at P of regular functions on $C\backslash\{P\}$ for some pointed non-singular curve (C, P). In this paper for any Weierstrass semigroup H we construct a double covering $\pi : \tilde{C} \longrightarrow C$ with a ramification point $\tilde{P}$ such that $H(\pi(\tilde{P})) = H$. We also determine the semigroup $H(\tilde{P})$. Moreover, in the case where H starts with 3 we investigate the relation between the semigroup $H(\tilde{P})$ and the Weierstrass semigroup of a total ramification point on a cyclic covering of the projective line with degree 6.

1 Introduction.

Let C be a complete nonsingular irreducible curve of genus $g \geq 2$ over an algebraically closed field k of characteristic 0, which is called a *curve* in this paper. Let $\mathbb{K}(C)$ be the field of rational functions on C. For a point P of C, we set

$$H(P) := \{\alpha \in \mathbb{N}_0 | \text{ there exists } f \in \mathbb{K}(C) \text{ with } (f)_\infty = \alpha P\},$$

which is called the *Weierstrass semigroup of the point* P where $\mathbb{N}_0$ denotes the additive semigroup of non-negative integers. A *numerical semigroup* means a subsemigroup of $\mathbb{N}_0$ whose complement in $\mathbb{N}_0$ is a finite set. For a numerical semigroup H the cardinality of $\mathbb{N}_0\backslash H$ is called the *genus* of H, which is denoted by $g(H)$. We note that $H(P)$ is a numerical semigroup of genus g. A numerical semigroup H is said to be *Weierstrass* if there exists a pointed curve (C, P) such that $H = H(P)$.

Let $(\tilde{C}, \tilde{P})$ be a pointed curve of genus $\tilde{g}$. Let us take a positive integer g with $\tilde{g} \geqq 6g + 4$. Using the property of the semigroup $H(\tilde{P})$ Torres [7] characterized the condition under which $\tilde{C}$ is a double covering of some curve C of genus g with ramification point $\tilde{P}$. In this paper when a pointed curve (C, P) of genus g is given we construct many examples of $\tilde{H}$ which is the semigroup of a ramification point of a double covering of C over the point P even if $g(\tilde{H}) < 6g + 4$. In fact,

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in Section 2 when H is any Weierstrass semigroup, i.e., there exists a pointed curve (C, P) with $H(P) = H$ we construct a double covering of a curve C with ramification point $\tilde{P}$ over P such that $g(H(\tilde{P})) \geqq 2g(H) + c(H) - 1$ where we denote by $c(H)$ the minimum of non-negative integers c satisfying $c + \mathbb{N}_0 \subseteqq H$. We note that $c(H) \leqq 2g(H)$. We can also describe the semigroup $\tilde{H} = H(\tilde{P})$. For any positive integer m a numerical semigroup H is called an *m-semigroup* if the least positive integer in H is m. An m-semigroup is said to be *cyclic* if it is the Weierstrass semigroup of a total ramification point on a cyclic covering of the projective line with degree m. If p is prime, Kim-Komeda [1] gives a computable necessary and sufficient condition for a p-semigroup to be cyclic. In Section 3 we describe a necessary and sufficient condition for a 6-semigroup to be cyclic. Moreover, for a 3-semigroup H we find the condition for the semigroup $\tilde{H} = H_n$ in Theorem 2.2 to be cyclic.

2 Weierstrass points on a double covering of a curve.

In this section when a Weierstrass semigroup H is given we construct a double covering $\pi : \tilde{C} \longrightarrow C$ with a ramification point $\tilde{P}$ such that $H(\pi(\tilde{P})) = H$. Moreover, we determine the Weierstrass semigroup of the ramification point $\tilde{P}$. For a numerical semigroup H we use the following notation. For an m-semigroup H we set

$$S(H) = \{s_0 = m, s_1, s_2, \ldots, s_{m-1}\}$$

where s_i is the minimum element h in H such that $h \equiv i \bmod m$. The set $S(H)$ is called the *standard basis* for H.

Lemma 2.1. *Let H be an m-semigroup and n an odd integer larger than $2c(H) - 2$. We set $H_n = 2H + n\mathbb{N}_0$. Assume that $n \neq 2m - 1$.*
i) *H_n is a $2m$-semigroup with the standard basis*

$$S(H_n) = \{2m, 2s_1, \ldots, 2s_{m-1}, n, n + 2s_1, \ldots, n + 2s_{m-1}\}.$$

ii) *The genus of H_n is $2g(H) + (n - 1)/2$.*
Proof. i) Since

$$\text{Max}\{s_i - m | i = 1, \ldots, m - 1\} = c(H) - 1,$$

we get $s_i - m \leqq c(H) - 1$ for all i. Hence, we have

$$2s_i \leqq 2(c(H) - 1 + m) \leqq 4c(H) - 2 \leqq 2n$$

because of $m \leqq c(H)$ and the assumption $n \geqq 2c(H) - 1$. Therefore, we obtain the standard basis

$$S(H_n) = \{2m, 2s_1, \ldots, 2s_{m-1}, n, n + 2s_1, \ldots, n + 2s_{m-1}\}$$

for H_n, because

$$\{s \in S(H_n) | s \text{ is even } \} = \{2m, 2s_1, \ldots, 2s_{m-1}\}$$

and

$$\{s \in S(H_n) | s \text{ is odd } \} = \{n, n+2s_1, \ldots, n+2s_{m-1}\}.$$

ii) If we set

$$n \equiv r \bmod 2m \text{ with } 1 \leqq r \leqq 2m-1,$$

then we get

$$g(H_n) = \sum_{i=1}^{m-1}[(2s_i)/(2m)] + [n/(2m)] + \sum_{i=1}^{m-1}[(n+2s_i)/(2m)]$$

$$= g(H) + (n-r)/(2m) + (m-1)\cdot(n-r)/(2m) + \sum_{i=1}^{m-1}[(r+2s_i)/(2m)]$$

$$= g(H) + (n-r)/2 + \sum_{i=1}^{m-1}(s_i - i)/m + \sum_{i=1}^{m-1}[(r+2i)/(2m)]$$

$$= 2g(H) + (n-r)/2 + \sum_{i=1}^{m-1}[(r+2i)/(2m)].$$

By the way we have $r + 2i \leqq 4m-3$, and $r+2i \geqq 2m$ if and only if $i \geqq m-(r-1)/2$. Hence, we obtain

$$g(H_n) = 2g(H) + (n-r)/2 + (r-1)/2 = 2g(H) + (n-1)/2.$$

□

We construct a desired double covering $\pi : \tilde{C} \longrightarrow C$ as follows:

Theorem 2.2. *Let H be a Weierstrass m-semigroup of genus $r \geqq 0$, i.e., there exists a pointed curve (C, P) such that $H(P) = H$. For any odd $n \geqq 2c(H)-1$ we set $H_n = 2H + n\mathbb{N}_0$. Assume that $n \neq 2m-1$. Then there exists a double covering $\pi : \tilde{C} \longrightarrow C$ with a ramification point $\tilde{P}$ over P such that $H(\tilde{P}) = H_n$.*

Proof. We consider the divisor $D = ((n+1)/2)P$. Let $\mathcal{L}$ be an invertible sheaf on C such that $\mathcal{L} \simeq \mathcal{O}_C(-D)$. Then we have

$$2D \sim P + (\text{some effective divisor}) = R$$

where R is a reduced divisor. Here for any two divisors D_1 and D_2 on C $D_1 \sim D_2$ means that D_1 and D_2 are linearly equivalent. In fact, we have

$$\deg(2D-P) = 2\cdot(n+1)/2 - 1 = n \geqq 2c(H) - 1 \geqq 2r+1$$

because of $c(H) \geqq r+1$. Hence, the divisor $2D-P$ is very ample. We set $\Delta = |2D-P|$ where for a divisor E on C we denote by $|E|$ the set of effective divisors on C which are linearly equivalent to E. By Bertini's Theorem there exists a non-empty open subset U in Δ which is contained in the set

$$\Delta_0 = \{E \in \Delta | E \text{ is reduced } \}.$$

We consider the non-empty open subset

$$U_1 = \{E \in \Delta | P \notin E\}.$$

Then $U \cap U_1$ is non-empty open. Take a divisor R' in $U \cap U_1$. We may set $R = P + R'$. Now we have isomorphisms

$$\mathcal{L}^{\otimes 2} \simeq \mathcal{O}_C(-2D) \simeq \mathcal{O}_C(-R) \subset \mathcal{O}_C.$$

Using the composition of the above two isomorphisms we can construct a double covering

$$\pi : \tilde{C} = \mathrm{Spec}(\mathcal{O}_C \oplus \mathcal{L}) \longrightarrow C$$

whose branch locus is R (See Mumford [6]). By Riemann-Hurwitz formula the genus of $\tilde{C}$ is

$$2r + (n-1)/2 = 2g(H) + (n-1)/2.$$

Let $\tilde{P} \in \tilde{C}$ be the ramification point of π over P. By Proposition 2.1 in Komeda-Ohbuchi [4] we obtain

$$h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}((n-1)\tilde{P})) = h^0(C, \mathcal{O}_C(((n-1)/2)P)) + h^0(C, \mathcal{L} \otimes \mathcal{O}_C(((n-1)/2)P))$$

and

$$h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}((n+1)\tilde{P})) = h^0(C, \mathcal{O}_C(((n+1)/2)P)) + h^0(C, \mathcal{L} \otimes \mathcal{O}_C(((n+1)/2)P)).$$

Since $\mathcal{L} \simeq \mathcal{O}_C(-((n+1)/2)P)$, we get

$$h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}((n-1)\tilde{P})) = h^0(C, \mathcal{O}_C(((n-1)/2)P))$$

and

$$h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}((n+1)\tilde{P})) = h^0(C, \mathcal{O}_C(((n+1)/2)P)) + 1.$$

The assumption $n \geqq 2c(H) - 1$ implies that

$$h^0(C, \mathcal{O}_C(((n+1)/2)P)) = h^0(C, \mathcal{O}_C(((n-1)/2)P)) + 1.$$

Thus, we get

$$h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(n\tilde{P})) = h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}((n-1)\tilde{P})) + 1,$$

which implies that $n \in H(\tilde{P})$. Moreover, we have $H(\tilde{P}) \supset 2H$. Thus, we get $H(\tilde{P}) \supseteqq 2H + n\mathbb{N}_0 = H_n$. By Lemma 2.1 ii) we have $g(H_n) = g(H(\tilde{P}))$, which implies that $H(\tilde{P}) = H_n$. $\square$.

Since for any $m \leqq 5$ every m-semigroup is Weierstrass (Maclachlan [5], Komeda [2], [3]), we get the following:

Corollary 2.3. *Let H be an m-semigroup for some $2 \leqq m \leqq 5$. For any odd $n \geqq 2c(H) - 1$ with $n \neq 2m - 1$ there exists a double covering with a ramification point whose Weierstrass semigroup is $2H + n\mathbb{N}_0$.*

If we take H as the semigroup generated by $3, 4$ and 5, we get the following examples:

Example 2.4. For any $g \geqq 7$ there exists a double covering with a ramification point whose Weierstrass semigroup is generated by $6, 8, 10$ and $2g - 7$.

3 Cyclic 6-semigroups.

First, we describe the condition for a 6-semigroup to be cyclic in tems of the standard basis. Using the description we determine the condition on n under which the semigroup H_n in Theorem 2.2 is cyclic when H is a 3-semigroup.

Lemma 3.1. *Let H be a cyclic 6-semigroup. Then there exists a pointed curve (C, P) satisfying $H(P) = H$ such that the curve C is defined by an equation of the form*

$$z^6 = \prod_{q=1}^{5} \prod_{j=1}^{i_q} (x - c_{qj})^q$$

with $\sum_{q=1}^{5} q i_q \equiv 1$ or $5 \bmod 6$ and that $f(P) = (0:1)$ where $f : C \longrightarrow \mathbb{P}^1$ is the surjective morphism defined by $f(Q) = (1 : x(Q))$. Here c_{qj}'s are distinct elements of k.

Proof. Since H is a cyclic 6-semigroup, there is a pointed curve (C, P) such that C is a cyclic covering of $\mathbb{P}^1$ of degree 6 with its total ramification point P satisfying $H(P) = H$. Hence, C is defined by an equation of the form

$$z^6 = \prod_{q=1}^{5} \prod_{j=1}^{i_q} (x - c_{qj})^q$$

where $i_1, \ldots, i_5$ are non-negative integers. If $f : C \longrightarrow \mathbb{P}^1$ is the morphism sending Q to $(1 : x(Q))$, then $f(P) = (0 : 1)$ or $(1 : c_{qj})$ for $q = 1$ or 5 and some j. Even if $f(P) = (1 : c_{qj})$, we may assume that $f(P) = (0 : 1)$ by transforming the variable x into $X = 1/(x - c_{qj})$. In this case, we get $\sum_{q=1}^{5} q i_q \equiv 1$ or $5 \bmod 6$. □

Proposition 3.2. *Let (C, P) be a pointed curve as in Lemma 3.1. Then we have*

$$S(H(P)) = \{6, \textstyle\sum_{i=1}^{5} q i_q, 2(i_1 + 2i_2 + i_4 + 2i_5), 3(i_1 + i_3 + i_5),$$
$$2(2i_1 + i_2 + 2i_4 + i_5), \textstyle\sum_{i=1}^{5} (6 - q) i_q\}.$$

Proof. We set

$$f^{-1}((1 : c_{qj})) = \{P_{qj}\} \text{ for } q = 1, 5,$$
$$f^{-1}((1 : c_{qj})) = \{P_{qj}, P'_{qj}\} \text{ for } q = 2, 4,$$
$$f^{-1}((1 : c_{qj})) = \{P_{qj}, P'_{qj}, P''_{qj}\} \text{ for } q = 3.$$

Let H be the semigroup generated by 6, $b_1 = \sum_{i=1}^{5} q i_q$, $b_2 = 2(i_1+2i_2+i_4+2i_5)$, $b_3 = 3(i_1 + i_3 + i_5)$, $b_4 = 2(2i_1 + i_2 + 2i_4 + i_5)$ and $b_5 = \sum_{i=1}^{5} (6 - q) i_q$. Since $\sum_{q=1}^{5} q i_q \equiv 1$ or $5 \bmod 6$, H is a numerical semigroup. First, we show that $H \subseteqq H(P)$. We have

$$\operatorname{div} z = -b_1 P + \textstyle\sum_{j=1}^{i_1} P_{1j} + 5 \sum_{j=1}^{i_5} P_{5j} + \sum_{j=1}^{i_2} (P_{2j} + P'_{2j})$$
$$+2 \textstyle\sum_{j=1}^{i_4} (P_{4j} + P'_{4j}) + \sum_{j=1}^{i_3} (P_{3j} + P'_{3j} + P''_{3j}),$$

$$\text{div } (x - c_{qj}) = -6P + 6P_{qj} \text{ for } q = 1, 5,$$

$$\text{div } (x - c_{qj}) = -6P + 3P_{qj} + 3P'_{qj} \text{ for } q = 2, 4,$$

$$\text{div } (x - c_{qj}) = -6P + 2P_{qj} + 2P'_{qj} + 2P''_{qj} \text{ for } q = 3.$$

For any $m \in \{1, 2, 3, 4, 5\}$ we set

$$y_m = \prod_{q=1}^{5} \prod_{j=1}^{i_q} (x - c_{qj})^{-[-mq/6]}$$

where $[r]$ denotes the largest integer less than or equal to r for any real number r. Then we get

$$\text{div } (y_m/z^m) = -\sum_{q=1}^{5}(-mq - 6[-mq/6])i_q P + (6 - m)\sum_{j=1}^{i_1} P_{1j}$$

$$+(-6[-5m/6] - 5m)\sum_{j=1}^{i_5} P_{5j} + (-3[-2m/6] - m)\sum_{j=1}^{i_2}(P_{2j} + P'_{2j})$$

$$+(-3[-4m/6]-2m)\sum_{j=1}^{i_4}(P_{4j}+P'_{4j})+(-2[-3m/6]-m)\sum_{j=1}^{i_3}(P_{3j}+P'_{3j}+P''_{3j}).$$

Hence we obtain

$$\text{div } (y_m/z^m)_\infty = b_{6-m}P$$

for any $m \in \{1, 2, 3, 4, 5\}$. Thus, we have $H \subseteqq H(P)$, which implies that $g(H) \geqq g(H(P))$. By Hurwitz's theorem we get

$$g(H(P)) = (5i_1 + 4i_2 + 3i_3 + 4i_4 + 5i_5 - 5)/2.$$

But we have

$$g(H) \leqq \sum_{q=1}^{5}[b_q/6] = [(\sum_{i=1}^{5} qi_q)/6] + [(2(i_1 + 2i_2 + i_4 + 2i_5))/6]$$

$$+[(3(i_1 + i_3 + i_5))/6] + i_1 + i_2 + i_4 + i_5$$

$$+[(-2(i_1 + 2i_2 + i_4 + 2i_5))/6] + \sum_{q=1}^{5} i_q + [(-\sum_{i=1}^{5} qi_q)/6]$$

$$= (5i_1 + 4i_2 + 3i_3 + 4i_4 + 5i_5 - 5)/2 = g(H(P)),$$

because $\sum_{q=1}^{5} qi_q \equiv 1$ or $5 \bmod 6$. Therefore, we get the equality $g(H) = g(H(P))$, which implies that $H(P) = H$. Moreover, by the above equality the standard basis for $H(P)$ must be the desired one. □

Using the above description of a cyclic 6-semigroup in terms of the standard basis we get a computable necessary and sufficient condition for a 6-semigroup to be cyclic.

Theorem 3.3. *Let H be a 6-semigroup with*

$$S(H) = \{6, 6m_1 + 1, 6m_2 + 2, 6m_3 + 3, 6m_4 + 4, 6m_5 + 5\}.$$

Then it is cyclic if and only if we have

$$m_2 + m_5 \geqq m_3 + m_4,\ m_1 + m_5 \geqq m_2 + m_4 \text{ and } m_1 + m_4 \geqq m_2 + m_3.$$

Proof. First, assume that H is cyclic. By Lemma 3.1 and Proposition 3.2 there are non-negative integers i_1, i_2, i_3, i_4 and i_5 such that

$$\begin{cases} i_1 + 2i_2 + 3i_3 + 4i_4 + 5i_5 = 6m_1 + 1 \text{ (resp. } 6m_5 + 5) \\ 2i_1 + 4i_2 + 2i_4 + 4i_5 = 6m_2 + 2 \text{ (resp. } 6m_4 + 4) \\ 3i_1 + 3i_3 + 3i_5 = 6m_3 + 3 \\ 4i_1 + 2i_2 + 4i_4 + 2i_5 = 6m_4 + 4 \text{ (resp. } 6m_2 + 2) \\ 5i_1 + 4i_2 + 3i_3 + 2i_4 + i_5 = 6m_5 + 5 \text{ (resp. } 6m_1 + 1). \end{cases}$$

Considering i_1, i_2, i_3, i_4, i_5 to be variables the determinant of the coefficient matrix is 1296. By calculation the above system of linear equations has a unique solution

$$i_1 = m_3 + m_4 + 1 - m_1 \text{ (resp. } m_2 + m_3 - m_5),$$

$$i_2 = m_2 + m_5 - m_3 - m_4 \text{ (resp. } m_1 + m_4 - m_2 - m_3),$$

$$i_3 = m_1 + m_5 - m_2 - m_4,$$

$$i_4 = m_1 + m_4 - m_2 - m_3 \text{ (resp. } m_2 + m_5 - m_3 - m_4),$$

$$i_5 = m_2 + m_3 - m_5 \text{ (resp. } m_3 + m_4 + 1 - m_1).$$

Since all i_q's must be non-negative, we get the desired result.

We shall show the "only if"-part. Let i_q's be as in the above, which are non-negative by the assumption. Then we get the pointed curve (C, P) as in Lemma 3.1. Using Proposition 3.2 we get $H = H(P)$, which implies that H is cyclic. □

When H is a 3-semigroup, we give a criterion for the 6-semigroup H_n as in Lemma 2.1 to be non-cyclic.

Proposition 3.4. *Let H be a 3-semigroup with $S(H) = \{3, 3l_1+1, 3l_2+2\}$ and n an odd integer larger than $2c(H)-2$ and distinct from 5. We set $H_n = 2H + n\mathbb{N}_0$.*
i) *If $n \equiv 3$ mod 6, then the 6-semigroup H_n is cyclic.*
ii) *Let $n \equiv 1$ mod 6. If $2l_1 = l_2$, then the 6-semigroup H_n is cyclic. Otherwise, H_n is not cyclic.*
iii) *Let $n \equiv 5$ mod 6. If $l_1 = 2l_2 + 1$, then the 6-semigroup H_n is cyclic. Otherwise, H_n is not cyclic.*

Proof. By Lemma 2.1 i) we have

$$S(H_n) = \{6, 6l_1 + 2, 6l_2 + 4, n, n + 6l_1 + 2, n + 6l_2 + 4\}.$$

For any $i = 1, \ldots, 5$, let $s_i \in S(H_n)$ such that $s_i \equiv i$ mod 6. We set $m_i = [s_i/6]$.

First, we consider the case where $n \equiv 3$ mod 6. Then we have

$$m_1 = l_2 + [n/6] + 1, \ m_3 = [n/6] \text{ and } m_5 = l_1 + [n/6].$$

Thus, we get $m_1 + m_5 > m_2 + m_4$. Since $2l_1 \geqq l_2$ and $2l_2 + 1 \geqq l_1$, we have

$$m_2 + m_5 \geqq m_3 + m_4 \text{ and } m_1 + m_4 \geqq m_2 + m_3.$$

By Theorem 3.3 the 6-semigroup H_n is cyclic.

Second, we consider the case where $n \equiv 1 \mod 6$. Then we have

$$m_1 = [n/6],\ m_3 = l_1 + [n/6] \text{ and } m_5 = l_2 + [n/6].$$

Thus, we get $m_2 + m_5 = m_3 + m_4$. If $2l_1 > l_2$, then we have

$$m_1 + m_4 = [n/6] + l_2 < 2l_1 + [n/6] = m_2 + m_3,$$

which implies that H_n is not cyclic. Let $2l_1 = l_2$. Then we have

$$m_1 + m_4 = m_2 + m_3.$$

Moreover, we see that $c(H) = 6l_1 + 2 - 3 + 1 = 6l_1$. By the assumption we have $n \geqq 12l_1 - 1$, which implies that $[n/6] \geqq 2l_1 - 1$. Hence we obtain

$$m_1 + m_5 = l_2 + 2[n/6] \geqq l_2 + 4l_1 - 2 > l_1 + l_2 = m_2 + m_4.$$

Thus, if $2l_1 = l_2$, then H_n is cyclic.

Last, let $n \equiv 5 \mod 6$. The method similar to the case $n \equiv 1 \mod 6$ works well. □

Using the above result we get a criterion for the 6-semigroup in Example 2.4 to be cyclic.

Example 3.5. For any $g \geqq 7$ let $H(g)$ be the semigroup generated by $6, 8, 10$ and $2g - 7$. The 6-semigroup $H(g)$ is cyclic if and only if $g \equiv 2 \mod 3$.

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