On drawings and decompositions of 1-planar graphs

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Abstract

A graph is called 1-planar if it can be drawn in the plane so that each of its edges is crossed by at most one other edge. We show that every 1-planar drawing of any 1-planar graph on n vertices has at most $n - 2$ crossings; moreover, this bound is tight. By this novel necessary condition for 1-planarity, we characterize the 1-planarity of Cartesian product $K_m \times P_n$. Based on this condition, we also derive an upper bound on the number of edges of bipartite 1-planar graphs, and we show that each subgraph of an optimal 1-planar graph (i.e., a 1-planar graph with n vertices and $4n - 8$ edges) can be decomposed into a planar graph and a forest.

Keywords: 1-planar graph; planar graph; forest

1 Introduction

Graph drawings in the plane represent a natural way of modeling and investigating of graphs. For many applications (like visualization of UML diagrams, electronic design automation or displaying protein structure in bioinformatics), it is often important to find a drawing which is from a geometric point of view as simple as possible. There are various criteria of complexity of drawings; the simplest cases concern graphs with acyclic drawings (that is, forests) followed by graphs which can be drawn in the plane without crossing edges (planar graphs). If a graph is not planar, then each its drawing contains

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some crossings on edges. In this case, the first idea how to optimize the drawing of a nonplanar graph G is to find a drawing with a minimum possible number of crossings of edges (the *crossing number* $c(G)$). However, it is known that the problem to determine $c(G)$ is NP-complete (see [6]) and to compute the exact value of $c(G)$ is difficult even for graphs with relatively small number of vertices. Other measures of nonplanarity of graph drawings in the plane are based on the minimum number of planar graphs into which the given graph decomposes (the graph *thickness*), the maximum number of edge-disjoint nonplanar subgraphs contained in given graph (the graph *coarseness*, see [1]) or the minimum number of edges whose removal results in planar graph (the graph *skewness*). Yet another measure of nonplanarity (the *k-planarity*) takes into account the possibility to draw a graph in the plane in such a way that each edge is crossed at most k times. The case $k = 1$ was studied by Ringel [12] already in 1965 (in the connection with vertex-face colorings of plane graphs), but serious progress in this area has been achieved only in the last ten years.

Although the family of 1-planar graphs looks similar to the family of planar graphs, there are several fundamental differences. Planarity can be characterized by Wagner's theorem by forbidding the minors K_5 and $K_{3,3}$ or by Kuratowski's theorem by forbidding the subdivisions of K_5 and $K_{3,3}$. Note that, for any graph, there always exists its 1-planar subdivision; hence, it is impossible to characterize 1-planar graphs by a Kuratowski-type theorem. In [3] it was observed that the class of 1-planar graphs is not closed under the operation of edge contraction; hence, the family of 1-planar graphs is not minor closed. Korzhik and Mohar in [9] studied minimal non-1-planar graphs (that is, graphs which are not 1-planar, but become 1-planar after deletion of any edge). They proved that there are infinitely many minimal non-1-planar graphs and they showed that for every integer $n \geqslant 63$, there are at least $2^{(n-54)/4}$ non-isomorphic minimal non-1-planar graphs of order n (compared to planar graphs, the number of n-vertex minimal nonplanar graphs is polynomial in n). In contrast to planarity testing (which can be performed in linear time in terms of number of vertices), the problem of testing 1-planarity of graphs is NP-complete, see [9].

To prove that a certain graph is not 1-planar, one may try to show that it has a large number of edges (since any 1-planar graph on n vertices has at most $4n - 8$ edges, see e.g. [5, 11, 13]), or it has high chromatic number (note that, by [2], 1-planar graphs are 6-colorable), or it contains a small non-1-planar subgraph (cf. [4] with the characterization of complete multipartite 1-planar graphs). Another way is to show that any drawing of a graph has too many crossings. Following this approach we prove in the first part of this paper a new necessary condition for 1-planarity. We show that every 1-planar drawing of any 1-planar graph on n vertices has at most $n - 2$ crossings. From this result it follows that, if the crossing number of the graph on n vertices is at least $n - 1$, then it cannot be 1-planar. With essential help of this result, we characterize the 1-planarity of the class of Cartesian product $K_m \times P_n$ for $m \geqslant 1$ and $n \geqslant 5$. The second corollary of our necessary condition is that every 1-planar graph on n vertices is obtained by adding at most $n - 2$ edges to a planar graph. Finally, we prove that every optimal 1-planar graph (with the highest possible number of edges) can be decomposed into a planar graph and a forest.

2 Notation

In this paper we consider connected simple graphs, unless otherwise stated. We use the standard graph theory terminology by [15]. The *degree* of a vertex v (the *size* of a face f) in a (plane) graph G is denoted by $d_G(v)$ ($d_G(f)$, respectively). A vertex of degree k is called a *k-vertex*; similarly, a face of size k is called a *k-face*. A plane *multi-triangulation* is a plane graph whose all faces have size 3 and it can contain multiple edges.

By K_n and P_n we denote the complete graph and the path on n vertices, respectively. The Cartesian product $K_m \times P_n$ is a graph (V_m^n, E_m^n) with vertex set $V_m^n = \{v_i^j \,|\, i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}\}$ and edge set $E_m^n = \{v_a^j v_b^j \,|\, j \in \{1, \ldots, n\}, a, b \in \{1, \ldots, m\}, a \neq b\} \cup \{v_i^j v_i^{j+1} \,|\, i \in \{1, \ldots, m\}, j \in \{1, \ldots, n-1\}\}$. Note that $|V_m^n| = mn$ and $|E_m^n| = \binom{m}{2}n + m(n-1)$.

For terminology related to 1-planar graphs, we will use the following notation taken from [5]. Let G be a 1-planar graph and let $D = D(G)$ be a 1-planar drawing of G (that is, a drawing of G in the plane in which every edge is crossed at most once; we will also assume that no edge is self-crossing, adjacent edges do not cross and there is no crossing that belongs to three edges). Given two non-adjacent edges pq, $rs \in E(G)$, the *crossing* of pq, rs is the common point of two arcs $\widehat{pq}$, $\widehat{rs} \in D$ (corresponding to edges pq, rs). Denote by $C = C(D)$ the set of all crossings in D and by E_0 the set of all non-crossed edges in D. The *associated plane graph* $D^\times = D^\times(G)$ of D is the plane graph such that $V(D^\times) = V(D) \cup C$ and $E(D^\times) = E_0 \cup \{xz, yz \,|\, xy \in E(D) - E_0, z \in C, z \in xy\}$. Thus, in $D^\times$, the crossings of D become new vertices of degree 4; we call these vertices *false*. Vertices of $D^\times$ which are also vertices of D are called *true*. Similarly, the edges and faces of $D^\times$ are called *false*, if they are incident with a false vertex, and *true* otherwise.

Note that a 1-planar graph may have different 1-planar drawings, which lead to non-isomorphic associated plane graphs.

It is known that every maximal planar graph on n vertices has the same number of edges. This is not true for maximal 1-planar graphs (that is, 1-planar graphs for which the addition of any edge joining two non-adjacent vertices results in a graph which is not 1-planar). Many authors independently proved that any maximal 1-planar graph on n vertices has at most $4n - 8$ edges, see e.g. [5, 11, 13]; moreover, there are maximal 1-planar graphs on n vertices with less than $4n - 8$ edges (cf. [7]). If a 1-planar graph on n vertices has $4n - 8$ edges, then it is called *optimal*.

3 Crossings in 1-planar graphs

A well-known consequence of Euler's formula is that every simple planar graph with n vertices has at most $3n - 6$ edges. Therefore:

Lemma 1. *Let G be a 1-planar graph on n vertices and m edges. Then there are at least $m - 3n + 6$ crossings in any 1-planar drawing of G.*

Corollary 2. *Let G be an optimal 1-planar graph on n vertices. Then there are at least $n - 2$ crossings in any 1-planar drawing of G.*

Lemma 3. *Any plane multi-triangulation $G = (V, E, F)$ on k vertices has $2k - 4$ faces.*

Proof. Using the handshaking lemma for the dual of G, we obtain $\sum_{f \in F(G)} d_G(f) = 2|E(G)|$. Since G is a multi-triangulation, $3|F(G)| = 2|E(G)|$. Combining this equality with Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$, we obtain $|F(G)| = 2|V(G)| - 4$. □

In the following we present an upper bound on the number of crossings in any 1-planar drawing of a 1-planar graph.

Theorem 4. *Any 1-planar drawing D of a 1-planar graph G on n vertices has at most $n - 2$ crossings. Moreover, this bound is tight.*

Proof. Let c denote the number of crossings in D. The associated plane graph $D^\times$ has $n + c$ vertices. Note that no two false vertices are adjacent in $D^\times$. Hence, we can extend $D^\times$ to a plane multi-triangulation by adding some edges into non-triangular faces of $D^\times$ in such a way that they join only true vertices.

By Lemma 3, the obtained multi-triangulation has $2n + 2c - 4$ faces. On the other hand, each false vertex is incident with four 3-faces and no two false vertices are incident with the same 3-face. Therefore, $2n + 2c - 4 \geqslant 4c$ and, consequently, $c \leqslant n - 2$. □

Corollary 5. *Let G be a graph on n vertices. If $c(G) \geqslant n - 1$, then G is not 1-planar.*

Now we demonstrate the usefulness of Corollary 5. In [8] it is proved that $c(K_5 \times P_n) = 6(n-1)$ for $n \geqslant 2$. Therefore Corollary 5 implies that $K_5 \times P_n$ is not 1-planar for $n \geqslant 5$. Consequently, $K_m \times P_n$ is not 1-planar for $m, n \geqslant 5$. Note that $K_m \times P_n$ is 1-planar for $m \leqslant 4$ and $n \geqslant 1$.

Observe that the 1-planarity of $K_5 \times P_5$ cannot be decided by any approach mentioned in the introduction.

The facts that every planar graph on n vertices has at most $3n - 6$ edges, and every 1-planar graph on n vertices has at most $4n - 8$ edges, suggest that 1-planar graphs are specific extensions of planar graphs.

Theorem 6. *Every 1-planar graph on n vertices is obtained from a planar graph by adding at most $n - 2$ edges.*

Proof. By Theorem 4, any 1-planar drawing of a 1-planar graph on n vertices contains at most $n - 2$ crossings. Every crossing arises from two different edges. If we remove one crossed edge for each crossing, then we obtain a drawing without crossings, that is, a plane graph. □

Theorem 6 implies that skewness of any 1-planar graph on n vertices is at most $n-2$. This bound is tight for optimal 1-planar graphs.

From Theorem 6, we have an upper bound for the number of edges of bipartite 1-planar graphs (cf. [10]).

Corollary 7. *Let G be a bipartite 1-planar graph on n vertices and m edges. Then $m \leqslant 3n - 6$.*

Proof. Theorem 6 implies that by removing at most $n-2$ edges from G we can get a planar graph. This planar graph is also bipartite and simple. Thus, it has at most $2n-4$ edges. Consequently, the number of edges of G is at most $3n-6$. $\square$

As another corollary of Theorem 6, we obtain the following result (proved earlier by Suzuki [14] using different methods).

Corollary 8. *Every optimal 1-planar graph G is obtained from a plane quadrangulation H by adding two (crossed) edges to each face of H.*

Proof. Let G be an optimal 1-planar graph on n vertices. From Corollary 2 and Theorem 4, it follows that any 1-planar drawing D of G contains exactly $n-2$ crossings.

Now, color the crossed edges of D with red and color the other edges with black. If we remove one (red) edge from each crossing, we obtain a plane triangulation (since this graph is simple, it has n vertices and $3n-6$ edges). This triangulation has $2n-4$ faces and $n-2$ red edges. On the other hand, no 3-face is incident with at least two red edges, hence every face is incident with exactly one red edge. Therefore, by removing these red edges, we obtain a plane quadrangulation H. $\square$

4 Edge decomposition of 1-planar graphs

We know from the previous section that the edge set of every 1-planar graph on n vertices can be decomposed into two sets A, B such that A induces a planar graph and B has at most $n-2$ edges. Since every forest on n vertices has at most $n-1$ edges, we will investigate the following question.

Question 9. Can each 1-planar graph be decomposed into a planar graph and a forest?

Lemma 10. *Let G be a plane quadrangulation. Then we can extend G to a plane triangulation by adding some edges to the faces of G such that the added edges induce a forest.*

Proof. By induction on the number of faces of G.

If G has two faces, then it is a cycle on four vertices. Figure 1 shows a possible extension of C_4.

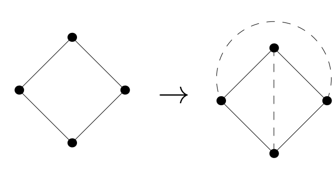

Figure 1: An extension of the smallest plane quadrangulation.

Now assume that G contains at least three 4-faces. By Euler's formula, it holds $\sum_{v\in V(G)}(d_G(v)-4)+\sum_{f\in F(G)}(d_G(f)-4)=4(|E(G)|-|V(G)|-|F(G)|)=-8$. Since every face of G has size 4, G must contain a vertex of degree 2 or 3.

First assume that G contains a 2-vertex v. Let v_1, v_2 be the vertices and f_1, f_2 be the faces incident with v. Let v_3 and v_4 be the fourth vertex incident with f_1 and f_2, respectively (see Figure 2 for illustration). Let H be a graph obtained from G by removing the vertex v. Then H is a quadrangulation with fewer faces than G, therefore we can add some edges to H which induce a forest F such that the obtained graph is a triangulation. We can modify F to a required forest in the following way.

If F contains the edge v_3v_4, then we remove this edge from F and we add two new edges v_3v and v_4v to F (see Figure 2).

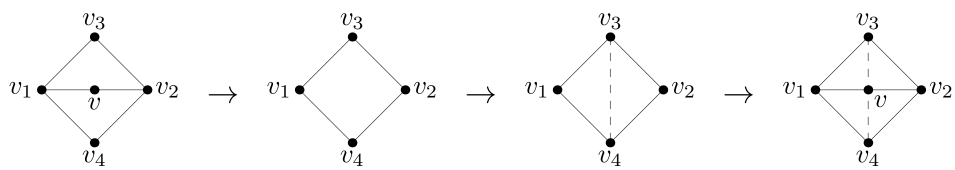

Figure 2: A modification of G and an extension of F when G has a 2-vertex.

If F contains the edge v_1v_2, then we add one more edge v_4v to F (see Figure 3).

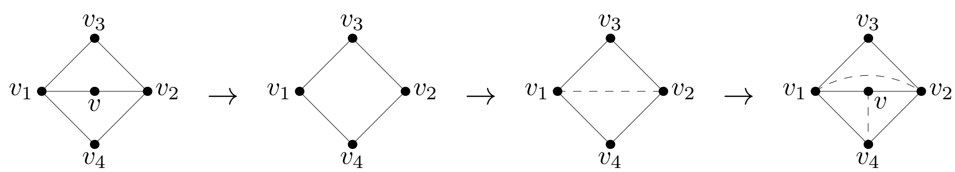

Figure 3: A modification of G and an extension of F when G has a 2-vertex.

Now assume that the minimum degree of G is 3. Let v_1, v_2, v_3 be the vertices and f_1, f_2, f_3 be the faces incident with a 3-vertex v. Let v_4, v_5 and v_6 be the fourth vertex incident with f_1, f_2 and f_3, respectively (see Figure 4 for illustration). Since G is planar, at least two of the edges v_1v_5, v_2v_6, v_3v_4 are missing in G. Without loss of generality, we assume that v_1v_5 is missing in G. Let H be a graph obtained from G by removing the vertex v and adding the edge v_1v_5. By induction hypothesis, we can extend H to a triangulation by adding some edges which induce a forest F. There are four possibilities: $v_1v_2, v_1v_3 \in F$, $v_4v_5, v_5v_6 \in F$, $v_1v_2, v_5v_6 \in F$ and $v_4v_5, v_1v_3 \in F$.

First assume that $v_1v_2, v_1v_3 \in F$ (resp. $v_4v_5, v_5v_6 \in F$). In this case we add a new edge vv_5 (resp. vv_1) to F in order to obtain a required forest (see Figure 4).

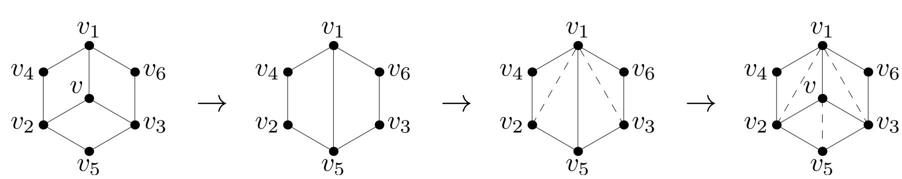

Figure 4: A modification of G and an extension of F when G has no 2-vertex.

Now assume that $v_1v_2, v_5v_6 \in F$ (if $v_1v_3, v_4v_5 \in F$, we can proceed in a similar way).

In this case, we remove the edge v_5v_6 and add two new edges vv_5 and vv_6 to F (see Figure 5).

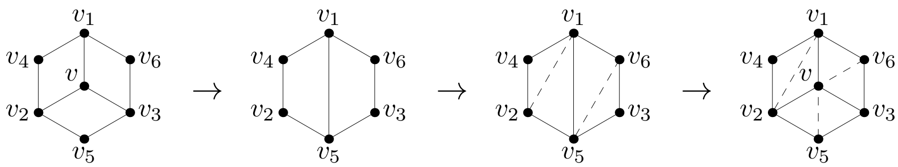

Figure 5: A modification of G and an extension of F when G has no 2-vertex.

□

Theorem 11. *The edge set of every optimal 1-planar graph can be decomposed into two sets A and B such that A induces a planar graph and B induces a forest.*

Proof. From Corollary 8, it follows that every optimal 1-planar graph G is obtained from a plane quadrangulation H by adding two edges to each face of H. Lemma 10 implies that, from each pair of crossed edges, we can choose one such that these edges induce a forest. The other edges induce a triangulation, that is, a planar graph. □

Conjecture 12. The edge set of every 1-planar graph can be decomposed into two sets A and B such that A induces a planar graph and B induces a forest.

References

[1] L.W. Beineke, G. Chartrand. The coarseness of a graph. *Compos. Math.*, 19:290–298, 1968.

[2] O.V. Borodin. A new proof of the 6-color theorem. *J. Graph Theory*, 19:507–521, 1995.

[3] Z.-Z. Chen, M. Kouno. A linear-time algorithm for 7-coloring 1-plane graphs. *Algorithmica*, 43:147–177, 2005.

[4] J. Czap, D. Hudák. 1-planarity of complete multipartite graphs. *Discrete Appl. Math.*, 160:505–512, 2012.

[5] I. Fabrici, M. Madaras. The structure of 1-planar graphs. *Discrete Math.*, 307:854–865, 2007.

[6] M.R. Garey, D.S. Johnson. Crossing number is NP-complete. *SIAM J. Algebraic Discrete Methods*, 4:312–316, 1983.

[7] D. Hudák, T. Madaras, Y. Suzuki. On properties of maximal 1-planar graphs. *Discuss. Math., Graph Theory*, 32:737–747, 2012.

[8] M. Klešč. The crossing number of $K_5 \times P_n$. *Tatra Mt. Math. Publ.*, 18:63–68, 1999.

[9] V.P. Korzhik, B. Mohar. Minimal obstructions for 1-immersions and hardness of 1-planarity testing. *J. Graph Theory*, 72:30–71, 2013.

[10] Q. Ma, S. Zhang, J. Wang. Bondage number of 1-planar graph. *Appl. Math., Irvine*, 1:101–103, 2010.

[11] J. Pach, G. Tóth. Graphs drawn with few crossings per edge. *Combinatorica*, 17:427–439, 1997.

[12] G. Ringel. Ein Sechsfarbenproblem auf der Kugel. *Abh. Math. Semin. Univ. Hamb.*, 29:107–117, 1965.

[13] H. Schumacher. Ein 7-Farbensatz 1-einbettbarer Graphen auf der projektiven Ebene. *Abh. Math. Semin. Univ. Hamb.*, 54:5–14, 1984.

[14] Y. Suzuki. Optimal 1-planar graphs which triangulate other surfaces. *Discrete Math.*, 310:6–11, 2010.

[15] D.B. West. Introduction to Graph Theory. Prentice Hall, 2001.