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On Dropout Modelling for Stability Analysis of Networked Control Systems

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***Abstract*— This paper presents three discrete-time modelling approaches for networked control systems (NCSs) that incorporate time-varying sampling intervals, time-varying delays and dropouts. The focus of this work is on the extension of two existing techniques to describe dropouts, namely (i) dropouts modelled as prolongation of the delay and (ii) dropouts modelled as prolongation of the sampling interval, and the presentation of a new approach (iii) based on explicit dropout modelling using automata. Based on polytopic overapproximations of the resulting discrete-time NCS models, we provide LMI-based stability conditions for all three approaches. Herewith, we compare the extensions of the existing approaches and the newly proposed method in terms of modelling accuracy, conservatism and complexity of the stability analysis. Using an illustrative example, we provide a thorough numerical comparison of the alternative modelling approaches.**

I. Introduction

The literature on modelling, analysis and controller design of networked control systems (NCSs) expanded rapidly over the last decade. The reason for the interest in the research area is that the use of networks offers many advantages, such as low installation and maintenance costs, reduced system wiring (in the case of wireless networks) and increased flexibility of the system. However, from a control theory point of view, the presence of the network also introduces several disadvantages such as time-varying networked-induced delays, aperiodic sampling and/or packet dropouts.

Before deploying NCS in industrial environments, a deep understanding of the effects of packet dropouts, time-varying sampling intervals and time-varying delays on the stability and performance of the NCS is needed. Most of the literature that deals with stability and stabilisation of NCSs focuses only on one of these phenomena, while ignoring the others. Clearly, it is important to consider the combined presence of dropouts, time-varying sampling intervals and time-varying delays, as in any practical NCS these communication imperfections typically occur simultaneously.

Some results are available that consider at least two of these network phenomena. In [1]–[3] stability and stabilisation of NCSs with packet dropouts and delays are investigated, based on discrete-time NCS representations. However, they assume that the delay is constant, which is often not realistic. In [4], the stability and disturbance attenuation of a NCS with time-varying delays and packet dropouts are investigated, based on a switched system approach. Therein, it is assumed that both the controller and sensor act in a time-driven periodic manner resulting in the fact that the delays take values in a finite set only, which is upper-bounded by the sampling interval.

Continuous-time modelling approaches, based on (impulsive) delay-differential equations, including packet dropouts and time-varying delays, are described in [5]–[7]. For these models, both stability analysis and controller synthesis methods are proposed and time-varying delays both smaller and larger than the sampling interval are included. Recently, in [8] a discrete-time approach was proposed for stability analysis and state feedback synthesis for NCS with both delays and dropouts. Typically, in the latter approaches based on either discrete-time or continuous-time models the inclusion of packet dropouts is modelled as prolongation of the maximum sampling interval of the maximum delay. As this does not truly model the effect of dropouts and introduces spurious solutions, these approaches have some degree of conservatism in the sense that the models allow for sequences of control updates, which cannot occur in real NCSs. In [9], also the problem of stability analysis of NCS with dropouts and variable delays is considered based on a discrete-time modelling approach. The occurrence of dropouts is modelled as prolongation of the sampling interval, which also leads to spurious solutions, and in the stability analysis a common quadratic Lyapunov function (LF) approach is adopted, which is conservative when compared to a parameter-dependent LF. In addition, the effect of varying sampling intervals and a systematic method for transforming the infinite set of linear matrix inequalities (LMIs) (due to the infinite number of possibilities for the delays) into a finite set are missing in [9].

This paper will provide a general stability analysis method based on a discrete-time modelling approach for NCS including the three mentioned network phenomena: packet dropouts, varying sampling intervals and varying delays. In particular, we provide three different models that each accommodates for packet dropouts in a different way. In doing so, the above mentioned drawbacks of existing approaches will be studied in detail. We assume that delays are time-varying and take values in a bounded set, containing an infinite number of values, i.e., $\tau_k \in [\tau_{\min}, \tau_{\max}]$, time-varying sampling intervals h_k taking values in $[h_{\min}, h_{\max}]$ and dropouts for which we only bound the maximal number of successive dropouts by $\bar{\delta}$. By adopting a discrete-time hybrid automaton modelling approach (see, e.g., [10], [11]), we truly describe the effect of packet dropouts instead of incorporating it (artificially) using prolongations of the

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maximal delay or maximal sampling interval as done in the mentioned references. We compare this newly proposed explicit approach with two (adapted) methods from [8], [9]. These adaptations consider small delays, time-varying sampling intervals and mode-dependent (i.e. dropout-dependent) Lyapunov functions (LFs), which lead to LMI-based conditions for stability of the NCS that alleviate the conservatism in the existing approaches.

The remainder of this paper is organised as follows: In Section II, we introduce two discrete-time NCS models from the literature, and extend them, and propose a novel way to model packet dropouts. In Section III, we introduce our polytopic overapproximation method. Based on an overapproximated model, we provide conditions for stability in terms of LMIs in Section IV. The drawbacks of the existing approaches and the motivations for the extensions are discussed in Section V. Finally, we illustrate the results using a numerical example in Section VI and present concluding remarks in Section VII.

II. THE NCS MODEL

In this section, the discrete-time description of a NCS including unknown and time-varying delays, unknown and time-varying sampling intervals and packet dropouts is presented. The NCS is depicted schematically in Fig. 1. It consists of a linear continuous-time plant

$$\dot{x}(t) = Ax(t) + Bu^*(t), \tag{1}$$

with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, and a discrete-time static time-invariant controller, which are connected over a communication network that induces network delays (namely the sensor-to-controller delay τ^{sc} and the controller-to-actuator delay τ^{ca}). The state measurements are sampled at the sampling times s_k:

$$s_k = \textstyle\sum_{i=0}^{k-1} h_i \quad \forall k \geq 1, \qquad s_0 = 0, \tag{2}$$

which are non-equidistantly spaced in time due to the time-varying sampling intervals $h_k > 0$. The sequence of sampling instants $s_0, s_1, s_2, \ldots$ is strictly increasing in the sense that $s_{k+1} > s_k$ for all $k \in \mathbb{N}$. We denote by $x_k := x(s_k)$ the kth sampled value of x and by u_k the corresponding control value. Packet dropouts may occur (see Fig. 1) and are modelled by the parameter m_k. This parameter denotes whether or not a packet is dropped:

$$m_k = \begin{cases} 0, & \text{if } x_k \text{ and } u_k \text{ are received,} \\ 1, & \text{if } x_k \text{ and/or } u_k \text{ is dropped.} \end{cases} \tag{3}$$

In (3), we make no distinction between packet dropouts that occur in the sensor-to-controller connection and the controller-to-actuator connection in the network. This can be justified by realising that, for static state-feedback controllers, the effect of packet dropouts on the control updates implemented on the plant is the same in both cases. Indeed, for packet dropouts between the sensor and the controller no new control update is computed and thus no new control input is sent to the actuator. In the case of packet dropouts between the controller and the actuator no new control update is received by the actuator either. Finally, the zero-order-hold (ZOH) function (in Fig. 1) is applied to transform the discrete-time control input u_k to a continuous-time control input $u^*(t)$ being the actual actuation signal of the plant.

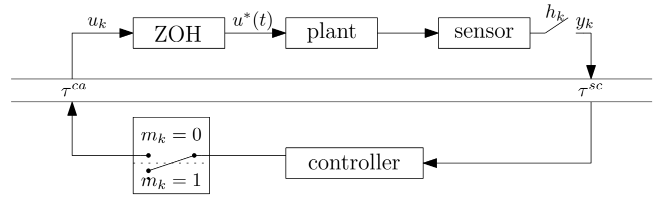

Fig. 1: Schematic overview of the NCS with variable sampling intervals, network delays and packet dropouts.

In the model, both the varying computation time (τ_k^c), needed to evaluate the controller, and the varying network-induced delays, i.e. the sensor-to-controller delay (τ_k^{sc}) and the controller-to-actuator delay (τ_k^{ca}), are taken into account. We assume that the sensor acts in a time-driven fashion (i.e. sampling occurs at the times s_k defined in (2)) and that both the controller and the actuator act in an event-driven fashion (i.e. responding instantaneously to newly arrived data). Under these assumptions, all three delays can be captured by a single delay $\tau_k := \tau_k^{sc} + \tau_k^c + \tau_k^{ca}$, see also [12] and [13]. To include these effects in the continuous-time model, let us define the parameter $k^*(t)$ that denotes the index of the most recent control input that is available at time t as $k^*(t) := \max\{k \in \mathbb{N} | s_k + \tau_k \leq t \wedge m_k = 0\}$. The continuous-time model of the plant of the NCS is then given by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu^*(t), \\ u^*(t) &= u_{k^*(t)}. \end{aligned} \tag{4}$$

Here, we assume that the most recent control input remains active in the plant if a packet is dropped.

We assume that the delays are bounded and contained in the set $[\tau_{\min},\, \tau_{\max}]$, with $\tau_{\max} \geq \tau_{\min} \geq 0$, the sampling intervals are also bounded and lie in the set $[h_{\min},\, h_{\max}]$, with $h_{\max} \geq h_{\min}$, and that delays are equal or smaller than the sampling intervals $\tau_k \leq h_k$, for all $k \in \mathbb{N}$. Therefore, we have that $(h_k, \tau_k) \in \Theta$, for all $k \in \mathbb{N}$, where

$$\Theta = \Big\{(h, \tau) \in \mathbb{R}^2 \mid h \in [h_{\min},\, h_{\max}], \\ \tau \in [\tau_{\min},\, \min\{h, \tau_{\max}\})\Big\}. \tag{5}$$

Furthermore, the number of subsequent packet dropouts is upper bounded by $\overline{\delta}$. This means that

$$\textstyle\sum_{v=k-\overline{\delta}}^{k} m_v \leq \overline{\delta}, \tag{6}$$

and guarantees that from the sequence of control inputs $u_{k-\overline{\delta}}, u_{k-\overline{\delta}+1}, \ldots, u_k$ at least one is implemented.

In this work, we consider three different approaches for modelling dropouts in which we use an exactly discretised version of the model in (4) that includes varying delays, varying sampling intervals and packet dropouts: (i) dropouts modelled as prolongation of the delay (see, e.g., [8], [14]), (ii) dropouts modelled as prolongation of the sampling interval (see, e.g., [9]) and (iii) a novel explicit dropout modelling approach. The essence of the three methods is presented schematically in Fig. 2. In this figure, a circle

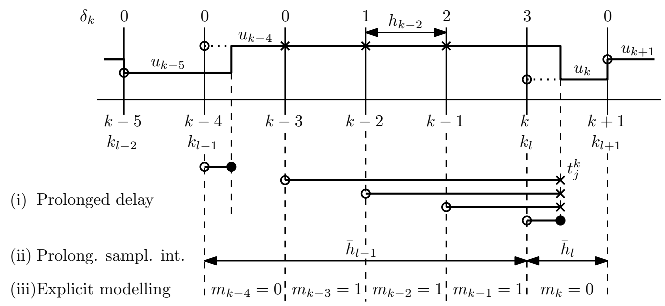

Fig. 2: Different dropout modelling approaches.

represents a sample that is not dropped and a cross a sample that is dropped. As shown, the discrete-time NCS model in methods (i) and (iii) describe the state evolution from s_k to s_{k+1}, while method (ii) discards the dropped samples and does not include the corresponding states $x(s_k)$ in the discrete-time model. This results in prolonged sampling intervals in case of method (ii). Method (i) includes the drops through prolongation of the delay using the parameters t_j^k, $j = 0, 1, \ldots, \bar{\delta} + 2$, which indicate the control update times, see [14]. Method (iii) keeps track of dropouts using an explicit automaton model. We describe these three methods in more detail in this section. In addition, we present an extension to the existing approach of [9] in section II-B.

Before presenting the three techniques, let us first introduce the variable δ_k, which denotes the number of subsequent dropouts at sample instant s_k, i.e. $\delta_k = \delta$ means that $u_{k-1}, u_{k-2}, \ldots, u_{k-\delta}$ are dropped and $u_{k-\delta-1}$ is received at the plant. Note that $\delta_k \in \{0, 1, \ldots, \bar{\delta}\}$. Given the fact that $m_k \in \{0, 1\}$ (see (3)), we can write the evolution of δ_k as

$$\delta_{k+1} = m_k(\delta_k + 1). \tag{7}$$

Equation (7) expresses that δ_{k+1} is increased by one if at the kth sampling instant the packet dropped and is reset to zero if the packet arrived. The variable δ_k will be used in the second and third modelling method that are described next.

A. Dropouts as Prolongation of Delays

In [14], NCSs are studied including delays that are larger than the sampling intervals ($\tau_k > h_k$) and dropouts, which are modelled as prolongations of the transmission delays. For the sake of simplicity, this approach is adapted to the case where delays are smaller than the sampling intervals ($\tau_k \leq h_k$) here. The NCS model (4) including the assumptions on the sampling intervals, delays and dropouts in (5) and (6) can be captured in a discrete-time representation, based on an exact discretisation of (4) at the sampling instants $\{s_k\}_{k\in\mathbb{N}}$. From s_k to s_{k+1} this yields

$$x_{k+1} = e^{Ah_k}x_k + \sum_{j=0}^{\bar{\delta}+1}\int_{h_k-t_{j+1}^k}^{h_k-t_j^k} e^{As}dsBu_{k+j-\bar{\delta}-1}, \tag{8}$$

where $x_k := x(s_k)$ is the discrete-time state at the kth sampling instant s_k and $h_k \geq t_j^k \geq 0$, for all $j \in \{0, 1, \ldots, \bar{\delta}+1\}$, represent the control update instants in the sampling interval $[s_k, s_{k+1}]$, in the sense that $u_{k+j-\bar{\delta}-1}$ is active in $[s_k + t_j^k,\ s_k + t_{j+1}^k)$. See [14] for more details.

Applying a static state feedback controller $u_k = -\bar{K}x_k$ to system (8), defining the augmented state vector by $\xi_k := \begin{bmatrix} x_k^\top & u_{k-1}^\top & \ldots & u_{k-\bar{\delta}-1}^\top \end{bmatrix}^\top$ and introducing the vector ρ_k of uncertain parameters consisting of the sampling and the actuation update instants $\rho_k := (h_k, t_1^k, \ldots, t_{\bar{\delta}+1}^k)$, results in the following discrete-time NCS model

$$\xi_{k+1} = \tilde{A}_1(\rho_k)\xi_k, \tag{9}$$

where

$$\tilde{A}_1(\rho_k) = \begin{bmatrix} \Lambda(\rho_k) & M_{\bar{\delta}}(\rho_k) & M_{\bar{\delta}-1}(\rho_k) & \ldots & M_0(\rho_k) \\ -\bar{K} & 0 & 0 & \ldots & 0 \\ 0 & I & 0 & \ldots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \ldots & 0 & I & 0 \end{bmatrix}. \tag{10}$$

In (10), $\Lambda(\rho_k) = e^{Ah_k} - M_{\bar{\delta}+1}(\rho_k)\bar{K}$ and

$$M_j(\rho_k) = \int_{h_k-t_{j+1}^k}^{h_k-t_j^k} e^{As}dsB, \tag{11}$$

for all $j \in \{0, 1, \ldots, \bar{\delta}+1\}$. The actuation instants t_j^k lie in the set $t_j^k \in [t_{j,\min}, t_{j,\max}]$, where $t_{j,\min}$ and $t_{j,\max}$ are defined as

$$t_{j,\min} = \begin{cases} \tau_{\min} & \text{if } j = \bar{\delta}+1, \\ 0 & \text{if } 1 \leq j < \bar{\delta}+1, \end{cases} \tag{12}$$

and

$$t_{j,\max} = \begin{cases} h_{\max} & \text{if } 1 < j \leq \bar{\delta}+1, \\ \tau_{\max} & \text{if } j = 1. \end{cases} \tag{13}$$

Additionally, $t_0^k := 0$ and $t_{\bar{\delta}+2}^k := h_k$. See [8], [14] for the proof of the validity of the bounds on t_j^k.

B. Dropouts as Prolongation of Sampling Intervals

In [9], García-Rivera and Barreiro model dropouts as prolongations of the constant sampling intervals h, by using the dropout counter δ_k from (7). Basically, the idea is that a sampling instant s_k corresponding to a sample that is dropped ($m_k = 1$), is not used as a sampling instant in the discrete-time model of [9]. In this perspective, a new (longer) sampling interval is considered to be the difference between the sampling instants of two successive samples that are not dropped. Since the dropped samples are ignored, we introduce a new counter l and a new sampling interval $\bar{h}_l$. For each l there exists a corresponding sample k_l given by

$$k_{l+1} = \min\{k > k_l \mid m_k = 0\}, \tag{14}$$

for all $l \in \mathbb{N}$. Defining now states and inputs on these new sample instants s_{k_l} as $\bar{x}_l := x_{k_l}$ and $\bar{u}_l := u_{k_l}$, results in the following discrete-time NCS model

$$\bar{x}_{l+1} = e^{A\bar{h}_l}\bar{x}_l + \int_0^{\bar{h}_l-\bar{\tau}_l} e^{As}dsB\bar{u}_l + \int_{\bar{h}_l-\bar{\tau}_l}^{\bar{h}_l} e^{As}dsB\bar{u}_{l-1}. \tag{15}$$

In (15), $\bar{h}_l \in \bigcup_{\delta=0}^{\bar{\delta}} (\delta+1)[h_{\min},\ h_{\max}]$ is the prolongated sampling interval. By applying a static state feedback controller $\bar{u}_l = -\bar{K}\bar{x}_l$ to (15) and introducing the augmented state vector $\xi_l := \begin{bmatrix} \bar{x}_l^\top & \bar{u}_{l-1}^\top \end{bmatrix}^\top$, we obtain

$$\xi_{l+1} = \tilde{A}_2(\bar{h}_l, \bar{\tau}_l)\xi_l, \tag{16}$$

where

$$\tilde{A}_2(\bar{h}_l, \bar{\tau}_l) = \begin{bmatrix} e^{A\bar{h}_l} - \int_0^{\bar{h}_l - \bar{\tau}_l} e^{As} ds B \bar{K} & \int_{\bar{h}_l - \bar{\tau}_l}^{\bar{h}_l} e^{As} ds B \\ -\bar{K} & 0 \end{bmatrix}. \quad (17)$$

C. Explicit Dropout Modelling

Let us now present a third method using a hybrid discrete-time NCS model that explicitly describes the occurrence of packet drops. As shown in (3), at each sampling instant k a packet may drop or not. Exact discretisation of (4) at sampling instants s_k for both cases results in the following discrete-time uncertain NCS model

$$x_{k+1} = \begin{cases} e^{Ah_k} x_k + \int_0^{h_k - \tau_k} e^{As} ds B u_k \\ \qquad + \int_{h_k - \tau_k}^{h_k} e^{As} ds B u_k^{\mathrm{old}} & \text{if } m_k = 0, \\ e^{Ah_k} x_k + \int_0^{h_k} e^{As} ds B u_k^{\mathrm{old}} & \text{if } m_k = 1, \end{cases} \quad (18)$$

with $u_k := u(s_k)$ the controller output at the kth sampling instant and $u_k^{\mathrm{old}} = u_l$, where $l = \max\{l \in \mathbb{N} | s_l < s_k \wedge m_l = 0\}$, the previously computed and successfully implemented control value.

The model in (18) will form the basis for a new model. We apply a static state feedback control law of the form $u_k = -\bar{K} x_k$ to the model and define the state of the closed-loop NCS model as $\xi_k := \begin{bmatrix} x_k^\top & (u_k^{\mathrm{old}})^\top \end{bmatrix}^\top$, which results in the following discrete-time hybrid system

$$\xi_{k+1} = \tilde{A}_3(h_k, \tau_k, m_k) \xi_k, \quad (19)$$

which explicitly depends on m_k, with

$$\tilde{A}_3(h_k, \tau_k, 0) = \begin{bmatrix} e^{Ah_k} - \int_0^{h_k - \tau_k} e^{As} ds B \bar{K} & \int_{h_k - \tau_k}^{h_k} e^{As} ds B \\ -\bar{K} & 0 \end{bmatrix}, \quad (20a)$$

$$\tilde{A}_3(h_k, \tau_k, 1) = \begin{bmatrix} e^{Ah_k} & \int_0^{h_k} e^{As} ds B \\ 0 & I \end{bmatrix}. \quad (20b)$$

Combining this with (6) and (7), we obtain the complete hybrid model, consisting of two modes in which the dynamics are described by uncertain linear systems, as depicted in Fig. 3. Loosely speaking, the semantics of this model are given as follows. After each update of ξ_k and δ_k to ξ_{k+1} and δ_{k+1} following the dynamics inside the current mode being either q_0 or q_1, one of the transitions, as indicated by the arrows, is taken. This transition can also be a transition to the same mode. A transition can only be taken if the corresponding guard is satisfied. For instance, after the update to ξ_{k+1} and δ_{k+1} to go to mode q_1, the condition $\delta_{k+1} < \bar{\delta}$ must be satisfied. To stress once more, the discrete-time hybrid automaton in Fig. 3 behaves periodically with updates according to the difference equations always followed by a transition to another or the same mode.

III. Overapproximation of Model

Direct stability analysis on the models from Section II is difficult as the infinite number of delays and sampling intervals in $[\tau_{\min}, \tau_{\max}]$ and $[h_{\min}, h_{\max}]$, respectively, and the exponential appearance of these uncertain parameters in the discretised system matrices obstruct derivation a finite number of (checkable) stability conditions. One remedy is to overapproximate systems (9), (16) and (19) by a system in which the uncertainties appear in a polytopic and/or additive

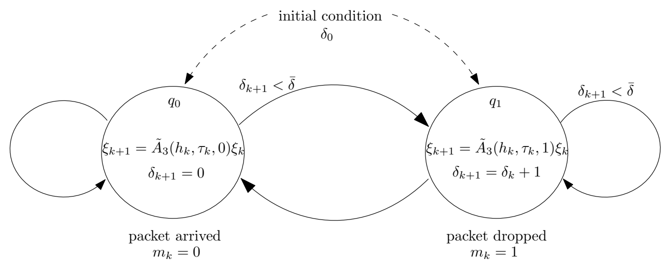

Fig. 3: A hybrid dynamical model with two subsystems.

manner. This can be achieved by using one of the available overapproximation methods (for an overview of existing methods, see, e.g., [15]). Here, we take the method from [8] that is based on the real Jordan form of the continuous-time system matrix A, although the other methods can be used as well. We apply this overapproximation method on the hybrid model from (19) and (20) and provide the corresponding stability conditions. A similar approach is applied to the models from Sections II-A and II-B as well, to obtain the numerical results in Section VI. For the sake of brevity, we only work out the details for the model presented in Section II-C. However, the results can also be applied to the models of Sections II-A and II-B in a similar fashion.

Basically, we express the matrix A, as in (1), as $A = TJT^{-1}$ with J the real Jordan form of A and T an invertible matrix that contains the (generalised) eigenvectors of A. With this real Jordan matrix, the exponential term $e^{As} = Te^{Js}T^{-1}$ has an useful general structure that can be used to obtain a model in which the uncertain parameters h_k, τ_k and m_k appear explicitly, i.e.

$$\xi_{k+1} = \left(F_0(m_k) + \sum_{i=1}^{2\nu} \alpha_i(h_k, \tau_k) F_i(m_k) \right) \xi_k, \quad (21)$$

where 2ν is the number of time-varying functions $\alpha_i(h_k, \tau_k)$, $k \in \mathbb{N}$, with $\nu \leq n$, where n is the dimension of the state vector x. We have $\nu = n$ when each distinct eigenvalue of A corresponds to one Jordan block only and $\nu < n$ otherwise. The first ν functions $\alpha_i(h_k, \tau_k)$ are of the form $h_k^{j-1} e^{\lambda h_k}$, if λ is a real eigenvalue of A, and $h_k^{j-1} e^{a h_k} \cos(b h_k)$ or $h_k^{j-1} e^{a h_k} \sin(b h_k)$, when λ corresponds to a pair of complex conjugate eigenvalues ($\lambda = a \pm bi$) of A, where $j = 1, 2, \ldots, r$, in which r is the size of the largest Jordan block corresponding to λ. The latter ν functions $\alpha_i(h_k, \tau_k)$ are of the form $(h_k - \tau_k)^{j-1} e^{\lambda(h_k - \tau_k)}$, if λ is a real eigenvalue of A, and $(h_k - \tau_k)^{j-1} e^{a(h_k - \tau_k)} \cos(b(h_k - \tau_k))$ or $(h_k - \tau_k)^{j-1} e^{a(h_k - \tau_k)} \sin(b(h_k - \tau_k))$, when λ corresponds to a pair of complex conjugate eigenvalues of A. For more details the reader is referred to Appendix B of [16]. Note that there are some useful tricks in the literature (see, e.g., [15], [17], [18]) by which the number of exponential terms $2^{2\nu}$ can be reduced to $\nu + 1$, although the polytopic overapproximation becomes less tight. In particular, for large systems (large ν) this might be necessary to use, as otherwise the LMIs become prohibitively complex for today's LMI solvers.

By using bounds on the uncertain parameters h_k and $h_k -$

τ_k, we obtain the following sets of matrices

$$\mathcal{F}_0 = \left\{ F_0(0) + \sum_{i=1}^{\zeta} \alpha_i(h_k, \tau_k) F_i(0) \mid (h_k, \tau_k) \in \Theta \right\} \quad (22)$$

and

$$\mathcal{F}_1 = \left\{ F_0(1) + \sum_{i=1}^{\nu} \alpha_i(h_k) F_i(1) \mid (h_k, \tau_k) \in \Theta \right\}, \quad (23)$$

that contain all possible matrix combinations in (21) corresponding to $m_k = 0$ or $m_k = 1$. Note that the set in (23) only has ν uncertainty functions α_i as (20b) only depends on h_k and not on $h_k - \tau_k$. To overcome the infinite size of the sets $\mathcal{F}_0$ and $\mathcal{F}_1$, a polytopic overapproximation of the sets is pursued. Denote the maximum and minimum value of α_i, respectively, by

$$\overline{\alpha}_i = \max_{(h,\tau)\in\Theta} \alpha_i(h,\tau), \qquad \underline{\alpha}_i = \min_{(h,\tau)\in\Theta} \alpha_i(h,\tau), \quad (24)$$

with Θ as in (5). Then it is readily seen that the sets of matrices $\mathcal{F}_0$ and $\mathcal{F}_1$, are subsets of the convex hulls $\text{co}\{\mathcal{H}_{\mathcal{F}_0}\}$ and $\text{co}\{\mathcal{H}_{\mathcal{F}_1}\}$, i.e. $\mathcal{F}_0 \subseteq \text{co}\{\mathcal{H}_{\mathcal{F}_0}\}$ and $\mathcal{F}_1 \subseteq \text{co}\{\mathcal{H}_{\mathcal{F}_1}\}$, with

$$\mathcal{H}_{\mathcal{F}_0} = \Big\{ F_0(0) + \textstyle\sum_{i=1}^{\zeta} \alpha_i F_i(0) \mid \alpha_i \in \{\underline{\alpha}_i, \overline{\alpha}_i\}, i = 1, 2, \ldots, \zeta \Big\} \quad (25)$$

and

$$\mathcal{H}_{\mathcal{F}_1} = \{ F_0(1) + \textstyle\sum_{i=1}^{\nu} \alpha_i F_i(1) \mid \alpha_i \in \{\underline{\alpha}_i, \overline{\alpha}_i\}, i = 1, 2, \ldots, \nu \}. \quad (26)$$

IV. Stability Analysis of NCS

Stability analysis can be performed on the overapproximated NCS with the set of vertices in (25) and (26). For enumeration purposes, we will write the set of vertices $\mathcal{H}_{\mathcal{F}_0}$ as $\mathcal{H}_{\mathcal{F}_0} = \{H_{\mathcal{F}_0,l_0} \mid l_0 = 1, 2, \ldots, 2^\zeta\}$ and $\mathcal{H}_{\mathcal{F}_1}$ as $\mathcal{H}_{\mathcal{F}_1} = \{H_{\mathcal{F}_1,l_1} \mid l_1 = 1, 2, \ldots, 2^\nu\}$. Using these finite sets of 2^ζ and 2^ν vertices, respectively, a finite number of LMI conditions to analyse stability is formulated in the following theorem. Note that the LMIs given in this theorem are only applicable for the model of Section II-C. For the LMI conditions of the other two approaches, after using the overapproximation based on the real Jordan form as above, the reader is referred to [9], [14] .

Theorem 1 *Consider the NCS model* (4) *and its discrete-time representation* (19) *with* (20) *for sequences of sampling instants, delays and packet dropouts* $\{h_k, \tau_k, m_k\}_{k\in\mathbb{N}}$*, satisfying* $(h_k, \tau_k) \in \Theta$*,* $k \in \mathbb{N}$*, with* Θ *as in* (5)*, and* (6) *for all* $k \in \mathbb{N}$*. Consider the equivalent representation* (21) *based on the real Jordan form of* A *and the set of vertices* $\mathcal{H}_{\mathcal{F}_0}$*, as in* (25) *and* $\mathcal{H}_{\mathcal{F}_1}$ *as in* (26)*.*

If there exist a set of matrices $\{P_0, \ldots, P_{\bar{\delta}}\}$ *satisfying*

$$\begin{bmatrix} P_i & H_{\mathcal{F}_0,l_0}^\top P_0 \\ P_0 H_{\mathcal{F}_0,l_0} & P_0 \end{bmatrix} \succ 0 \quad (27)$$

for all $i \in \{0, \ldots, \bar{\delta}\}$ *and* $l_0 \in \{1, 2, \ldots, 2^\zeta\}$*, and*

$$\begin{bmatrix} P_j & H_{\mathcal{F}_1,l_1}^\top P_{j+1} \\ P_{j+1} H_{\mathcal{F}_1,l_1} & P_{j+1} \end{bmatrix} \succ 0, \quad (28)$$

for all $j \in \{0, \ldots, \bar{\delta} - 1\}$ *and* $l_1 \in \{1, 2, \ldots, 2^\nu\}$*, then the closed-loop NCS* (4) *is globally asymptotically stable.*

The conditions in Theorem 1 result from showing that $V(\xi_k, \delta_k) = \xi_k^\top P_{\delta_k} \xi_k$, $\delta_k \in [0, \bar{\delta}]$, is a parameter-dependent LF for the system in (21).

V. Discussion

In this section, the advantages and drawbacks of the existing approaches are discussed and we explain why we proposed the presented extensions of the existing methods.

In Section II-A, dropouts are modelled as prolongations of the delays based on [8] and [14]. In [14] a model is given for large delays incorporating dropouts, whereas in Section II-A a model is presented that includes small delays and dropouts. Due to the general case considered in [14] (large delays), it is possible that $\bar{\delta} + 2$ different control values are active in the interval $[s_k,\ s_{k+1}]$, while in case of small delays this can be at most two. As a consequence, at most two M_j's are nonzero in the model in (9) and (10) from Section II-A. Here, we use the general result as presented in [8], [14] in which more than two M_j's can be nonzero, although it can be easily remedied of course. Moreover, in [14] the dropout counter δ_k is not used in the model. To be precise, the relation between δ_{k+1} and δ_k as in (7) is not exploited to reduce conservatism. Therefore, the number of successive dropouts at the next sampling instant s_{k+1} is not related to the number of dropouts at the current instant s_k. This may lead to spurious effects, e.g., $\delta_{k+1} = \delta_k = 1$ is possible, which means that a particular control input is considered to be dropped in one sampling interval while it is considered not to be dropped in the next. Clearly, this is not possible in reality.

If we would resolve these two issues, which is rather straightforward for the case of small delays, the result becomes actually of a similar nature as the method of Section II-C. For comparison reasons we kept here the general method of [8], [14], without these resolutions, that also applies to the large delay case in which such simplifications are not so obvious or even impossible.

In Section II-B, dropouts are modelled as prolongations of the sampling intervals. Regarding the work in [9], it is unclear which overapproximation method is used to arrive at a polytopic embedding of the system as in (25) and (26). To implement this approach, we have proposed the real Jordan overapproximation technique discussed in Section III. Secondly, [9] uses a NCS with constant sampling intervals. We extended this to a discrete-time NCS model with varying sampling intervals, as shown in (15). The transformation of the range of sampling intervals can be taken as $h_k \in H_l := [h_{\min},\ (\bar{\delta} + 1)h_{\max}]$, which contains spurious values for h_k when $h_{\min} > 0$, because all intervals are merged into one large interval instead of considering them as separate intervals. We reduced this source of conservatism by defining the range as $\bar{h}_l \in H_u := \bigcup_{\delta=0}^{\bar{\delta}} (\delta + 1)[h_{\min},\ h_{\max}]$ such that no spurious sampling intervals occur when $h_{\min} > 0$. Considering H_u requires to perform the overapproximation for each individual (non-overlapping) interval and one can assign one Lyapunov function (LF) to each interval such that a

TABLE I: Number of LMIs for the different modelling methods for varying sampling intervals and delays.

	CQLF	δDLF
[14] small delays	$2^{(\bar{\delta}+2)\nu}$	X
[9] with H_l	$2^{2\nu}$	X
[9] with H_u	$2^{2\nu}(\bar{\delta}+1)$	$2^{2\nu}(\bar{\delta}+1)^2$
Explicit model	X	$2^{2\nu}(\bar{\delta}+1)+2^{\nu}\bar{\delta}$

dropout-dependent LF $V(\bar{x}_{l+1}, \delta_l)$ is obtained as in the proof of Theorem 1. In case subsequent intervals $\delta[h_{\min},\ h_{\max}]$ and $(\delta+1)[h_{\min},\ h_{\max}]$ overlap, one can actually unite them in one new interval $[\delta h_{\min},\ (\delta+1)h_{\max}]$ and use only one LF for this united interval, which reduces the number of LMIs (but increases conservatism). Of course, one can also consider one common LF $V(\bar{x}_l)$. With a common P and taking one large interval as in H_l, the least number of LMIs is obtained, but the conservatism is generally increased significantly. Note that in case of $h_{\min} = 0$, the interval ranges H_l and H_u are equal.

A general remark is that both [14] and [9] use a common quadratic Lyapunov function (CQLF) to perform stability analysis on the models. As shown in Section IV, we propose to use a δ(dropout)-dependent Lyapunov function (δDLF) that depends on the dropout counter δ_k. Below we will apply this δDLF to an example. Note that δ is not used in the model based on [14] and thus no δDLF can be applied to that approach. As already mentioned, although δDLFs reduce the conservatism of the stability of NCSs[1], they also lead to more LMIs than CQLFs. This higher number of LMIs results in a more complex stability analysis. The number of LMIs for all modelling methods are given in Table I for CQLFs and δDLFs. Note that, the explicit dropout modelling approach (Section II-C) is not feasible for CQLFs, because the second mode of the automaton is always unstable.

VI. Illustrative Example

In this section, we show the results of the stability analysis for the double integrator example, given by $\dot{x} = Ax + Bu$, where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{29}$$

Here, we limit ourselves to a constant sampling interval $h = h_{\min} = h_{\max} = 10^{-3}$, where a different number of subsequent packet dropouts $\bar{\delta}$ can occur, in combination with a time-varying delay that is upper bounded by $\tau_{\max}$, which is equal or smaller than the fixed sampling interval ($0 = \tau_{\min} \leq \tau_{\max} \leq h$). The gain of the static state feedback controller $u_k = -\bar{K}x_k$ is given by $\bar{K} = \begin{bmatrix} 6000 & K_b \end{bmatrix}$. We are interested in the stability region, i.e., all values of K_b that guarantee stability for each combination of delays, satisfying

[1]One can reduce the conservatism of NCSs further by using parameter dependent Lyapunov functions that also depend on the delay and sampling (or actually on the vertices of the polytopic overapproximation in (25) and (26)) at the cost of more LMIs. For ease of exposition, we use dependency on δ only.

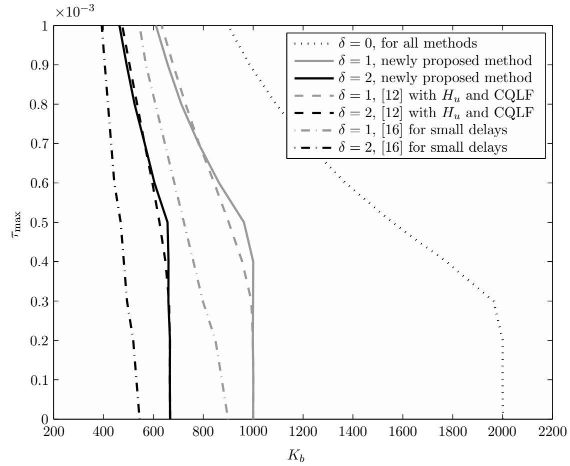

Fig. 4: Upper bounds of the stability region for $\bar{K} = [6000\ \ K_b]$ for different values of $\bar{\delta}$ and $\tau \in [0, \tau_{\max}]$.

$\tau_k \in [0, \tau_{\max}]$ and sequences of packet dropouts, satisfying (6). Based on the conditions in [19] and the ones in Theorem 1, we obtain the stability regions in Fig. 4 and Fig. 5, given by an upper bound for K_b, for $\bar{\delta} = 0$, i.e. no packet dropouts, $\bar{\delta} = 1$ and $\bar{\delta} = 2$. Note that K_b also has lower bounds, which are not shown in Fig. 4 and Fig. 5. The reason for this is that these are small ($K_b < 20$ for all approaches and for all values of $\tau_{\max}$ and $\bar{\delta}$) in comparison with the upper bounds on K_b.

Clearly, packet dropouts decrease the allowable controller gains that stabilise the NCS. To illustrate that the presented explicit dropout modelling approach reduces conservatism with respect to the approach presented in [14], their results are also shown in Fig. 4. Note that, in the approach of [14] the maximal controller gain K_b for which stability is guaranteed, for a maximal delay equal to the sampling interval ($\tau_{\max} = h$) without dropouts ($\bar{\delta} = 0$), is the same as the gain for one dropout with no delay ($\bar{\delta} = 1$, $\tau_{\max} = 0$), denoted by the dash-dotted line in Fig. 4. The same holds for more dropouts, e.g. K_b corresponding to ($\tau_{\max} = h,\ \bar{\delta} = 1$) for which stability can be proven, equals K_b corresponding to ($\tau_{\max} = 0,\ \bar{\delta} = 2$). This is a consequence of the modelling technique as discussed in Section II-A, i.e. a dropout without delay is modelled as a delay that is equal to the sampling interval $\tau_{\max} = h$. Essentially, this is caused by the computation of the bounds $t_{j,\min}$ and $t_{j,\max}$ or the control update instants.

Fig. 4 also shows that the approach from [9] (extended with H_u) gives similar stability regions as the newly proposed explicit modelling approach, i.e. for some $\tau_{\max}$ the extension of [9] is less conservative and for some $\tau_{\max}$ the newly proposed explicit modelling method is less conservative. Note that, as mentioned in Section V, García-Rivera and Barreiro use a common quadratic LF and the newly proposed method uses a δDLF. Introducing a δDLF to the extended approach from [9], results in slightly less conservative stability regions than the CQLF results. Though, the results are still comparable with the results of the newly proposed method, as shown in Fig. 5. This observation turns

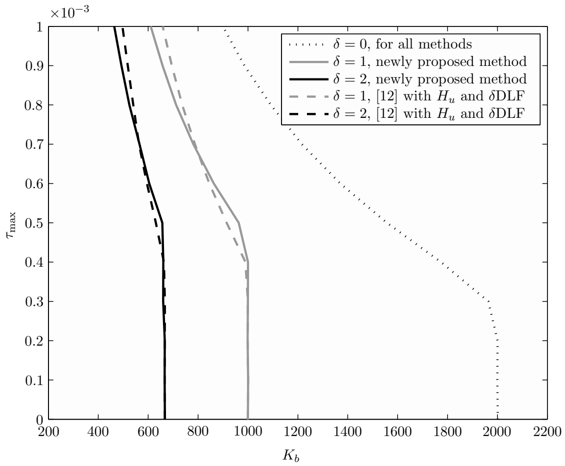

Fig. 5: Upper bounds of the stability region for $\bar{K} = [6000 \;\; K_b]$ for different values of $\bar{\delta}$ and $\tau \in [0, \tau_{\max}]$.

out to be true for various examples.

As such, from a conservatism point of view, the newly proposed method and the extended method based on [9] seem to be comparable. Therefore, to decide which method you will use can be best be determined by the numerical complexity (number of LMIs) as given in Table I. In case you are interested in the least conservative approaches, (either [9] with H_u and δDLF or the newly proposed method) it is better to use the newly proposed method when dropouts are present as the comparable stability regions can be obtain with less computationally demanding computations. However, if a more conservative approach is acceptable, e.g., [9] with H_u and CQLF or even [9] with H_l, then the computational burden for these methods is much lower than for the newly proposed method. This might be particularly relevant when the size of the problems becomes too large (large ν). However, in general, this is a natural tradeoff between numerical complexity and desired conservatism of the results and the choice for the method will depend on the particular problem at hand.

VII. Conclusions

In this paper, we studied the stability of networked control systems (NCSs) using a discrete-time model that incorporates time-varying sampling intervals, time-varying delays and packet dropouts. Existing approaches modelled packet dropouts as prolongations of the delay or prolongations of the sampling interval. Here, we proposed a third method based on modelling packet dropouts explicitly leading to a discrete-time hybrid automaton-based NCS model. By doing so, the phenomenon of packet dropouts is truly modelled and no spurious effects are introduced as in the case of modelling the dropouts as prolongation of the delays or sampling intervals. Based on this new model, in which hybrid modelling plays an important role, constructive LMI conditions for analysing stability of the closed-loop system are derived. In addition, we extended the method from [9] that is based on prolongation of the sampling interval in three ways, namely (i) sampling intervals are now time-varying in a continuous range, (ii) the model is overapproximated using the real Jordan form and (iii) a dropout-dependent Lyapunov function is introduced.

After reviewing the three modelling methods based on their perspectives, we compared them numerically using an double integrator example. We showed that the newly proposed approach reduces conservatism when compared to the delay prolongation approach from [14]. We also showed that the newly proposed method, in combination with a parameter-dependent LF, gives similar stability regions as the extended sampling interval prolongation technique from [9], in combination with a common quadratic or parameter-dependent LF. Numerical complexity of the methods (as computed in Table I) can be used to choose between the methods.

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