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On dynamic behavior of a hyperbolic system derived from a thermoelastic equation with memory type ☆

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Abstract

In this paper, we study the Riesz basis property of the generalized eigenfunctions of a one-dimensional hyperbolic system in the energy state space. This characterizes the dynamic behavior of the system, particularly the stability, in terms of its eigenfrequencies. This system is derived from a thermoelastic equation with memory type. The asymptotic expansions for eigenvalues and eigenfunctions are developed. It is shown that there is a sequence of generalized eigenfunctions, which forms a Riesz basis for the Hilbert state space. This deduces the spectrum-determined growth condition for the C_0-semigroup associated with the system, and as a consequence, the exponential stability of the system is concluded.
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1. Introduction

In the last two decades, much effort has been made for the following heat equation which incorporates the effect of thermomechanical coupling and the effect of inertia [1–5]:

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + \alpha\theta_x(x,t) = 0, & 0 < x < 1, \ t > 0, \\ \theta_t(x,t) - k\theta_{xx}(x,t) + \alpha u_{xt}(x,t) = 0, & 0 < x < 1, \ t > 0, \\ u(i,t) = \theta(i,t) = 0, & i = 0,1, \ t \geqslant 0, \end{cases} \tag{1.1}$$

where u represents the displacement and θ the absolute temperature. $k > 0$ is the thermal conductivity. The coupling constant $\alpha > 0$ is generally small in comparison to unity and is a measure of the mechanical–thermal coupling presenting in the system. The exponential stability of the system (1.1) was first obtained in [5] by frequency domain multiplier method. The spectral analysis in [2] shows that there are two branches of eigenvalues for the system (1.1), which have the following asymptotic expansions:

$$\begin{cases} \sigma_n = -k(n\pi)^2 + \gamma^2/k + \mathcal{O}(n^{-2}), \\ \lambda_n = -\dfrac{\gamma^2}{2k} \pm \mathrm{i}n\pi + \mathcal{O}(n^{-1}), \end{cases} \tag{1.2}$$

where n are large positive integers. It is seen from (1.2) that the first branch of eigenvalues is produced by the heat equation while the second one is associated with the elastic vibration. Later, it was shown in [3] that there is a real eigenvalue for the system (1.1) that is greater than the dominant eigenvalue of "pure" heat equation. Unfortunately, we do not know whether the temperature in Eq. (1.1) is greater than or equal to that of "pure" heat equation under the same initial values. In [4], a more profound result was proved that there is a set of generalized eigenfunctions of the system (1.1), which forms a Riesz basis for the state space. By Riesz basis property, the dynamic behavior of the system (1.1) can be expressed in terms of its eigenfrequencies. Moreover, the Riesz basis property concludes the spectrum-determined growth condition, one of the hard and important problems in the stability analysis of infinite-dimensional systems. The spectrum-determined growth condition implies automatically the exponential stability result of [5].

Eq. (1.1) was initially derived based on the Fourier's law [1]. However, it was indicated in [6] that the modeling system (1.1) does not take the memory effect into account, which may exist in some materials particularly in low temperature. Moreover, the fact that the thermal disturbance at one point affects the whole elastic body instantly, which is implied by Eq. (1.1), is not physically acceptable [6]. To overcome these shortcomings, a memory type model was derived in [6,7]:

$$\begin{cases} u_{tt}(x,t) - au_{xx}(x,t) + \alpha\theta_x(x,t) = 0, & 0 < x < 1, \ t > 0, \\ \theta_t(x,t) - (k * \theta_{xx})(x,t) + \alpha u_{xt}(x,t) = 0, & 0 < x < 1, \ t > 0, \\ u(i,t) = \theta(i,t) = 0, & i = 0,1, \ t \geqslant 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \theta(x,0) = \theta_0(x), & 0 \leqslant x \leqslant 1, \end{cases} \tag{1.3}$$

where $a > 0$ is a constant. The sign "$*$" denotes the convolution product:

$$(k * g)(x,t) = \int_0^t k(t-s)g(x,s)\,\mathrm{d}s,$$

where the kernel function k is assumed to be strongly positive-definite in the sense that $k'(t) < 0, k''(t) > 0$ for any $t > 0$ and k decays exponentially to zero as time goes to infinity [7].

Due to the appearance of the convolution product in the second equation, Eq. (1.3) becomes a system of fully hyperbolic partial differential equations (see Eq. (1.4) or (1.8) below), which is in sharp contrast to Eq. (1.1).

In this paper, we take

$$k(t) = a\mathrm{e}^{-\varepsilon t}, \quad t > 0,$$

where a is the same as that in Eq. (1.3) and $\varepsilon > 0$ is a constant. We will see that under this assumption, system (1.3) becomes a time-invariant system. Actually, let

$$v(x, t) := (k * \theta_x)(x, t).$$

Then

$$v_t(x, t) = a\theta_x(x, t) - \varepsilon v(x, t), \quad v(x, 0) = 0.$$

Therefore, Eq. (1.3) becomes

$$\begin{cases} u_{tt}(x, t) - au_{xx}(x, t) + \alpha\theta_x(x, t) = 0, & 0 < x < 1, \ t > 0, \\ \theta_t(x, t) - v_x(x, t) + \alpha u_{xt}(x, t) = 0, & 0 < x < 1, \ t > 0, \\ v_t(x, t) - a\theta_x(x, t) + \varepsilon v(x, t) = 0, & 0 < x < 1, \ t > 0, \\ u(i, t) = \theta(i, t) = 0, & i = 0, 1, \ t \geqslant 0, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad \theta(x, 0) = \theta_0(x), \quad v(x, 0) = 0, & 0 \leqslant x \leqslant 1. \end{cases} \tag{1.4}$$

Differentiate the energy function of Eq. (1.4) that is given by

$$E(t) = \frac{1}{2} \int_0^1 \left[u_t^2(x, t) + au_x^2(x, t) + \theta^2(x, t) + \frac{1}{a}v^2(x, t) \right] \mathrm{d}x, \tag{1.5}$$

to yield

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) = -\frac{\varepsilon}{a} \int_0^1 v^2(x, t)\,\mathrm{d}x \leqslant 0. \tag{1.6}$$

So system (1.4) is actually a dissipative system. However, for the general kernel function k, it was indicated in [6] that system (1.3) is weakly dissipative for the energy function

$$\widetilde{E}(t) = \frac{1}{2} \int_0^1 [u_t^2(x, t) + au_x^2(x, t) + \theta^2(x, t)]\,\mathrm{d}x. \tag{1.7}$$

It is easy to show that system (1.4) associates with a C_0-semigroup solution (see Lemma 5.1). This clearly explains, from a different point of view, the weak solution of Eq. (1.3) defined in [6] that $v(x, t) = (k * \theta_x)(x, t)$ is actually an independent state variable for system (1.3). Moreover, it is shown in Theorem 5.2 that $E(t)$ decays exponentially:

$$E(t) \leqslant M_0 \mathrm{e}^{-\omega_0 t} E(0)$$

for some positive constants M_0, ω_0, which is stronger than the result obtained in [6] for the general kernel function $k(t)$, where the exponential stability was obtained for $\widetilde{E}(t)$.

Instead of studying Eq. (1.4), we consider, in this paper, the following system of hyperbolic equations:

$$\begin{cases} w_{tt}(x,t) - aw_{xx}(x,t) + \alpha\theta_{xt}(x,t) = 0, \quad 0<x<1, \ t>0, \\ \theta_{tt}(x,t) - a\theta_{xx}(x,t) + \alpha w_{xt}(x,t) + \varepsilon\theta_t(x,t) + \varepsilon\alpha w_x(x,t) = 0, \quad 0<x<1, \ t>0, \\ w(i,t) = \theta(i,t) = 0, \quad i = 1,2, \ t \geqslant 0, \\ w(x,0) = u_1(x), \quad w_t(x,0) = au_0''(x) - \alpha\theta_0'(x), \quad 0 \leqslant x \leqslant 1, \\ \theta(x,0) = \theta_0(x), \quad \theta_t(x,0) = -\alpha u_1'(x), \quad 0 \leqslant x \leqslant 1 \end{cases}$$

(1.8)

which is obtained by setting $w = u_t$ in (1.4). Due to the lack of dissipativity, the well-posedness and stability analysis for system (1.8) is much harder than for Eq. (1.4). Notice that for system (1.8), the energy function should be

$$F(t) = \frac{1}{2}\int_0^1 [w_t^2 + aw_x^2 + \theta_t^2 + a\theta_x^2]\,dx.$$

(1.9)

To our knowledge, the study for system (1.8) is not available at all in literature.

The main objective of this paper is to study the Riesz basis property of the generalized eigenfunctions of system (1.8) in the energy state space. This characterizes the dynamic behavior of system (1.8), particularly the stability, in terms of its eigenfrequencies. The remaining parts of the paper are organized as follows: the asymptotic expansions for eigenvalues and eigenfunctions are developed in Sections 2 and 3. A remarkable feature is found in Section 2 that the heat equation part and vibration equation part in system (1.8) are symmetric under some similar transform. In Section 4, it is shown that there is a sequence of generalized eigenfunctions of system (1.8), which forms a Riesz basis for the Hilbert state space. This deduces the spectrum-determined growth condition and the exponential stability for system (1.8). The similar results for system (1.4) are presented in Section 5.

2. Asymptotic expansion of eigenvalues

To begin, we first formulate system (1.8) into an evolution equation in the state Hilbert space $\mathscr{H}$ defined by

$$\mathscr{H} := H_0^1(0,1) \times L^2(0,1) \times H_0^1(0,1) \times L^2(0,1),$$

equipped with the following inner product:

$$\langle F_1, F_2 \rangle := a\langle w_1', w_2'\rangle_{L^2} + \langle \phi_1, \phi_2\rangle_{L^2} + a\langle\theta_1', \theta_2'\rangle_{L^2} + \langle v_1, v_2\rangle_{L^2}$$
$$\forall F_i = [w_i, \phi_i, \theta_i, v_i] \in \mathscr{H}, \ i = 1,2,$$

where $\langle\cdot,\cdot\rangle_{L^2}$ denotes the usual inner product in $L^2(0,1)$. Define a linear operator $\mathscr{A}$ in $\mathscr{H}$ by

$$\mathscr{A}\begin{bmatrix} w \\ \phi \\ \theta \\ v \end{bmatrix}^{\top} = \begin{bmatrix} \phi \\ aw'' - \alpha v' \\ v \\ a\theta'' - \alpha\phi' - \varepsilon v - \varepsilon\alpha w' \end{bmatrix}^{\top} \quad \forall \begin{bmatrix} w \\ \phi \\ \theta \\ v \end{bmatrix}^{\top} \in D(\mathscr{A})$$

(2.1)

with

$$D(\mathscr{A}) = \mathscr{H} \cap (H^2(0,1) \times H_0^1(0,1))^2. \tag{2.2}$$

Then Eq. (1.8) can be formulated as an evolution equation in $\mathscr{H}$:

$$\frac{\mathrm{d}}{\mathrm{d}t} Y(t) = \mathscr{A} Y(t), \quad Y(0) = Y_0 \tag{2.3}$$

with $Y(\cdot, t) = [w(\cdot, t), w_t(\cdot, t), \theta(\cdot, t), \theta_t(\cdot, t)]$ and $Y_0 = [u_1, au_0'' - \alpha\theta_0', \theta_0, -\alpha u_1']$.

Lemma 2.1. *Let $\mathscr{A}$ be given in Eqs. (2.1) and (2.2). Then $\mathscr{A}^{-1}$ exists and is compact on $\mathscr{H}$. Therefore, $\sigma(\mathscr{A})$, the spectral set of $\mathscr{A}$, consists of only isolated eigenvalues with finite algebraic multiplicity.*

Proof. Let $g = [g_1, g_2, g_3, g_4] \in \mathscr{H}$. Solve $\mathscr{A}F = G$ for $F = [w, \phi, \theta, v] \in D(\mathscr{A})$, that is,

$$\begin{cases} \phi = g_1, \quad v = g_3, \\ aw'' - \alpha g_3' = g_2, \\ a\theta'' - \alpha g_1' - \varepsilon g_3 - \varepsilon\alpha w' = g_4, \\ w(0) = w(1) = \theta(0) = \theta(1) = 0, \end{cases}$$

to obtain, after a direct computation, that

$$\begin{cases} \phi = g_1, \quad v = g_3, \\ w(x) = w'(0)x + \dfrac{1}{a} \int_0^x \int_0^t [\alpha g_3'(s) + g_2(s)]\,\mathrm{d}s\,\mathrm{d}t, \\ w'(0) = -\dfrac{1}{a} \int_0^1 \int_0^1 [\alpha g_3'(s) + g_2(s)]\,\mathrm{d}s\,\mathrm{d}t, \\ \theta(x) = \theta'(0)x + \dfrac{1}{a} \int_0^x \int_0^t [\alpha g_1'(s) + \varepsilon g_3(s) + \varepsilon\alpha w'(s) + g_4(s)]\,\mathrm{d}s\,\mathrm{d}t, \\ \theta'(0) = -\dfrac{1}{a} \int_0^1 \int_0^1 [\alpha g_1'(s) + \varepsilon g_3(s) + \varepsilon\alpha w'(s) + g_4(s)]\,\mathrm{d}s\,\mathrm{d}t. \end{cases}$$

Hence, $F \in D(\mathscr{A})$ and $\mathscr{A}^{-1}$ exists. Moreover, the Sobolev embedding theorem implies that $\mathscr{A}^{-1}$ is compact on $\mathscr{H}$, proving the required result. $\quad\square$

Since $\sigma(\mathscr{A})$ consists of only eigenvalues, it is easily seen that $\mathscr{A}(w, \phi, \theta, v) = \lambda(w, \phi, \theta, v)$ if and only if $\phi = \lambda w, v = \lambda\theta$ and $(w, \theta) \neq 0$ satisfies the following system of ordinary differential equations:

$$\begin{cases} \lambda^2 w(x) - aw''(x) + \alpha\lambda\theta'(x) = 0, \\ \lambda^2 \theta(x) + \alpha\lambda w'(x) - a\theta''(x) + \varepsilon\lambda\theta(x) + \varepsilon\alpha w'(x) = 0, \\ w(0) = \theta(0) = \theta(1) = w(1) = 0. \end{cases} \tag{2.4}$$

For brevity in notation, we set

$$r := \sqrt{\frac{1}{a}}, \quad a_1 := \frac{\alpha}{2ar} > 0, \quad a_2 := \frac{\varepsilon}{ar}, \quad a_3 := \frac{\varepsilon\alpha}{a}. \tag{2.5}$$

Then Eq. (2.4) becomes

$$\begin{cases} r^2\lambda^2 w(x) - w''(x) + 2a_1 r\lambda\theta'(x) = 0, \\ r^2\lambda^2\theta(x) - \theta''(x) + 2a_1 r\lambda w'(x) + a_2 r\lambda\theta(x) + a_3 w'(x) = 0, \\ w(0) = \theta(0) = \theta(1) = w(1) = 0. \end{cases} \quad (2.6)$$

Now, using the operator pencil method, we show that eigenvalue problem (2.6) has some symmetric feature, which explains at least that when $\varepsilon = 0$, each eigenvalue is of geometrical multiplicity two. Indeed, define a positive definite operator A in $L^2(0,1)$ by

$$Af = f'', \quad D(A) = \{f \in H^2(0,1) \,|\, f(0) = f(1) = 0\}.$$

Then Eq. (2.6) with its boundary conditions can be written as of the operator form:

$$\mathscr{L}(\lambda)(w,\theta)^\top = 0 \quad (2.7)$$

with

$$\mathscr{L}(\lambda) := r^2\lambda^2 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + 2a_1 r\lambda \begin{pmatrix} 0 & A^{1/2} \\ A^{1/2} & 0 \end{pmatrix}$$
$$- \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} + a_2 r\lambda \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} + a_3 \begin{pmatrix} 0 & 0 \\ A^{1/2} & 0 \end{pmatrix}.$$

By making a transform to the above pencil $\mathscr{L}$ via $\widetilde{S}$ given by (see [8])

$$\widetilde{S} = \begin{pmatrix} I & -I \\ I & I \end{pmatrix},$$

we obtain a new pencil $\mathscr{L}_s(\lambda) = \widetilde{S}^{-1}\mathscr{L}(\lambda)\widetilde{S}$:

$$\mathscr{L}_s(\lambda) := r^2\lambda^2 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + 2a_1 r\lambda \begin{pmatrix} A^{1/2} & 0 \\ 0 & -A^{1/2} \end{pmatrix}$$
$$- \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} + \frac{1}{2}a_2 r\lambda \begin{pmatrix} I & I \\ I & I \end{pmatrix} + \frac{1}{2}a_3 \begin{pmatrix} A^{1/2} & -A^{1/2} \\ A^{1/2} & -A^{1/2} \end{pmatrix}.$$

Let $(f,g)^\top = \widetilde{S}^{-1}(w,\theta)^\top$. Then

$$(w,\theta) = (f - g, f + g), \quad (f,g) = \tfrac{1}{2}(\theta + w, \ \theta - w). \quad (2.8)$$

Obviously, $\mathscr{L}_s(\lambda)(f,g)^\top = 0$ if and only if $\mathscr{L}(\lambda)(w,\theta)^\top = 0$. So, Eq. (2.6) is equivalent to the problem of following:

$$\begin{cases} r^2\lambda^2 f(x) - f''(x) + 2a_1 r\lambda f'(x) + \frac{1}{2}a_2 r\lambda(f(x) + g(x)) + \frac{1}{2}a_3(f'(x) - g'(x)) = 0, \\ r^2\lambda^2 g(x) - g''(x) - 2a_1 r\lambda g'(x) + \frac{1}{2}a_2 r\lambda(f(x) + g(x)) + \frac{1}{2}a_3(f'(x) - g'(x)) = 0, \\ f(0) = f(1) = g(0) = g(1) = 0. \end{cases} \quad (2.9)$$

It is apparent that if $(f(x), g(x))$ is a solution to (2.9) with respect to λ, so is $(g(1-x), f(1-x))$, which reflects the symmetry of the eigenvalue problem.

Proposition 2.1. *Suppose $(w(x), \theta(x))$ is a solution to Eq. (2.6). Then $(-w(1-x), \theta(1-x))$ is also a solution to Eq. (2.6). In particular when $\varepsilon = 0$, λ is exactly of geometrically two and the*

corresponding linearly independent solutions to Eq. (2.6) *are* $(w, \theta) = (f(x), f(x))$ *and* $(w, \theta) = (-f(1-x), f(1-x))$, *where* f *is the solution of*

$$\begin{cases} r^2\lambda^2 f(x) - f''(x) + 2a_1 r\lambda f'(x) = 0, \\ f(0) = f(1) = 0. \end{cases} \tag{2.10}$$

Proof. The first part is obvious from Eq. (2.9). For the second part, notice that when $\varepsilon = 0$, Eq. (2.9) becomes

$$\begin{cases} r^2\lambda^2 f(x) - f''(x) + 2a_1 r\lambda f'(x) = 0, \\ r^2\lambda^2 g(x) - g''(x) - 2a_1 r\lambda g'(x) = 0, \\ f(0) = f(1) = g(0) = g(1) = 0. \end{cases} \tag{2.11}$$

Obviously, Eq. (2.11) admits only two linearly independent solutions $(f(x), 0)$ and $(0, f(1-x))$, where f is a linearly independent solution to the equation of Eq. (2.10). The proof is completed by noticing Eq. (2.8). $\square$

With this preparation, we go back to Eq. (2.6). Set

$$w_1 := w, \quad w_2 := w', \quad \theta_1 := \theta, \quad \theta_2 := \theta', \quad \Phi(x) := [w_1, w_2, \theta_1, \theta_2]^\top. \tag{2.12}$$

Then Eq. (2.6) becomes

$$\begin{cases} T^D(x, \lambda)\Phi(x) = \Phi'(x) + A(\lambda)\Phi(x) = 0, \\ T^R\Phi := W^0\Phi(0) + W^1\Phi(1) = 0, \end{cases} \tag{2.13}$$

where

$$A(\lambda) := A_0 - \lambda A_1 - \lambda^2 A_2, \tag{2.14}$$

with

$$A_0 := \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -a_3 & 0 & 0 \end{bmatrix}, \quad A_1 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2a_1 r \\ 0 & 0 & 0 & 0 \\ 0 & 2a_1 r & a_2 r & 0 \end{bmatrix},$$

$$A_2 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ r^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \end{bmatrix}, \quad W^0 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad W^1 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{2.15}$$

Summarizing, we have proved the following Theorem 2.1.

Theorem 2.1. *Eq.* (2.6) *is equivalent to the boundary-value problem of the first order linear system* (2.13). *Moreover,* $\lambda \in \sigma(\mathscr{A})$ *if and only if Eq.* (2.13) *admits a nonzero solution.*

The following technique due to Birkhoff and Langer [9] and Tretter [10,11] is standard for the asymptotic expansion of characteristic determinant of Eq. (2.13), which has been successfully used to the analysis of the system of coupled partial differential equations

(see [12,13]). First, diagonalize the leading term $\lambda^2 A_2$ in Eq. (2.14). To this purpose, let

$$\theta_1 = a_1 - \sqrt{a_1^2 + 1}, \quad \theta_2 = a_1 + \sqrt{a_1^2 + 1}, \quad \theta_3 = -\theta_2, \quad \theta_4 = -\theta_1. \tag{2.16}$$

These are roots of the quadratic equations

$$\theta^2 - 2a_1\theta - 1 = 0 \quad \text{and} \quad \theta^2 + 2a_1\theta - 1 = 0$$

and all $\theta_i, i = 1, 2, 3, 4$ are distinct: $\theta_1 \neq \theta_2 \neq \theta_3 \neq \theta_4$. Next, define an invertible matrix $P(\lambda)$:

$$P(\lambda) := S \begin{bmatrix} P_1(\lambda) & \\ & P_2(\lambda) \end{bmatrix}, \quad \lambda \in \mathbb{C}, \ \lambda \neq 0, \tag{2.17}$$

where

$$P_1(\lambda) := \begin{bmatrix} r\lambda & r\lambda \\ \theta_1 r^2 \lambda^2 & \theta_2 r^2 \lambda^2 \end{bmatrix}, \quad P_1^{-1}(\lambda) := \frac{1}{\theta_2 - \theta_1} \begin{bmatrix} \dfrac{\theta_2}{r\lambda} & \dfrac{-1}{r^2\lambda^2} \\ \dfrac{-\theta_1}{r\lambda} & \dfrac{1}{r^2\lambda^2} \end{bmatrix}, \tag{2.18}$$

$$P_2(\lambda) := \begin{bmatrix} r\lambda & r\lambda \\ -\theta_2 r^2 \lambda^2 & -\theta_1 r^2 \lambda^2 \end{bmatrix}, \quad P_2^{-1}(\lambda) := \frac{1}{\theta_2 - \theta_1} \begin{bmatrix} \dfrac{-\theta_1}{r\lambda} & \dfrac{-1}{r^2\lambda^2} \\ \dfrac{\theta_2}{r\lambda} & \dfrac{1}{r^2\lambda^2} \end{bmatrix}, \tag{2.19}$$

and

$$S := \begin{bmatrix} I_2 & -I_2 \\ I_2 & I_2 \end{bmatrix}, \quad S^{-1} := \frac{1}{2} \begin{bmatrix} I_2 & I_2 \\ -I_2 & I_2 \end{bmatrix}. \tag{2.20}$$

Here I_2 is a 2×2 identity matrix. It is easy to see that

$$P(\lambda)^{-1} = \begin{bmatrix} P_1^{-1}(\lambda) & \\ & P_2^{-1}(\lambda) \end{bmatrix} S^{-1} \quad \forall \lambda \neq 0.$$

So the matrix $P(\lambda)$ is a polynomial of degree 2 in λ. Thirdly, define

$$\Psi(x) := P^{-1}(\lambda)\Phi(x) \quad \text{(that is } \Phi(x) = P(\lambda)\Psi(x)\text{)} \tag{2.21}$$

and $\widehat{T}^D(x, \lambda) := P(\lambda)^{-1} T^D(x, \lambda) P(\lambda)$. Then we have

$$\widehat{T}^D(\lambda)\Psi(x) = \Psi'(x) + \widehat{A}(\lambda)\Psi(x) = 0, \tag{2.22}$$

where

$$\widehat{A}(\lambda) := P(\lambda)^{-1} A(\lambda) P(\lambda).$$

Since

$$S^{-1}A(\lambda)S = S^{-1} \begin{bmatrix} 0 & -1 & 0 & 0 \\ -r^2\lambda^2 & 0 & 0 & -2a_1r\lambda \\ 0 & 0 & 0 & -1 \\ 0 & -a_3 - 2a_1r\lambda & -a_2r\lambda - r^2\lambda^2 & 0 \end{bmatrix} S$$

$$= S^{-1} \begin{bmatrix} 0 & -1 & 0 & 1 \\ -r^2\lambda^2 & -2a_1r\lambda & r^2\lambda^2 & -2a_1r\lambda \\ 0 & -1 & 0 & -1 \\ -a_2r\lambda - r^2\lambda^2 & -a_3 - 2a_1r\lambda & -a_2r\lambda - r^2\lambda^2 & a_3 + 2a_1r\lambda \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 & 0 \\ -r^2\lambda^2 - \frac{1}{2}a_2r\lambda & -2a_1r\lambda - \frac{1}{2}a_3 & -\frac{1}{2}a_2r\lambda & \frac{1}{2}a_3 \\ 0 & 0 & 0 & -1 \\ -\frac{1}{2}a_2r\lambda & -\frac{1}{2}a_3 & -\frac{1}{2}a_2r\lambda - r^2\lambda^2 & 2a_1r\lambda + \frac{1}{2}a_3 \end{bmatrix},$$

it follows that

$$\widehat{A}(\lambda) = \begin{bmatrix} P_1^{-1}(\lambda) & \\ & P_2^{-1}(\lambda) \end{bmatrix}$$

$$\times (r^2\lambda^2) \begin{bmatrix} -\theta_1 & -\theta_2 & 0 & 0 \\ a_4 + a_5r\lambda & a_6 + a_7r\lambda & a_6 & a_4 \\ 0 & 0 & \theta_2 & \theta_1 \\ a_4 & a_6 & a_6 + a_7r\lambda & a_4 + a_5r\lambda \end{bmatrix}$$

$$= \frac{1}{\theta_2 - \theta_1} \begin{bmatrix} \theta_2r\lambda & -1 & & \\ -\theta_1r\lambda & 1 & & \\ & & -\theta_1r\lambda & -1 \\ & & \theta_2r\lambda & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} -\theta_1 & -\theta_2 & 0 & 0 \\ a_4 + a_5r\lambda & a_6 + a_7r\lambda & a_6 & a_4 \\ 0 & 0 & \theta_2 & \theta_1 \\ a_4 & a_6 & a_6 + a_7r\lambda & a_4 + a_5r\lambda \end{bmatrix}$$

$$= \begin{bmatrix} -\theta_1r\lambda - b_1 & -b_2 & -b_2 & -b_1 \\ b_1 & -\theta_2r\lambda + b_2 & b_2 & b_1 \\ -b_1 & -b_2 & \theta_2r\lambda - b_2 & -b_1 \\ b_1 & b_2 & b_2 & \theta_1r\lambda + b_1 \end{bmatrix}$$

with

$$\begin{cases} a_4 := -\frac{1}{2}a_2 - \frac{1}{2}a_3\theta_1, \quad a_5 := -1 - 2a_1\theta_1, \quad a_6 := -\frac{1}{2}a_2 - \frac{1}{2}a_3\theta_2, \\ a_7 := -1 - 2a_1\theta_2, \quad b_1 := \frac{a_4}{\theta_2 - \theta_1}, \quad b_2 := \frac{a_6}{\theta_2 - \theta_1}. \end{cases} \tag{2.23}$$

Furthermore, we decompose $\widehat{A}(\lambda)$ into a sum of dominant term and a low term as following:

$$\widehat{A}(\lambda) := -\lambda\widehat{A}_1 - \widehat{A}_0 \tag{2.24}$$

with

$$\widehat{A}_1 := \begin{bmatrix} r\theta_1 & & & \\ & r\theta_2 & & \\ & & -r\theta_2 & \\ & & & -r\theta_1 \end{bmatrix}, \quad \widehat{A}_0 := \begin{bmatrix} b_1 & b_2 & b_2 & b_1 \\ -b_1 & -b_2 & -b_2 & -b_1 \\ b_1 & b_2 & b_2 & b_1 \\ -b_1 & -b_2 & -b_2 & -b_1 \end{bmatrix}. \tag{2.25}$$

Theorem 2.2. *Let $0 \neq \lambda \in \mathbb{C}$, and let $\widehat{A}(\lambda)$ be defined by Eqs. (2.24)–(2.25). Let*

$$E(x, \lambda) := \begin{bmatrix} e^{r\theta_1\lambda x} & & & \\ & e^{r\theta_2\lambda x} & & \\ & & e^{-r\theta_2\lambda x} & \\ & & & e^{-r\theta_1\lambda x} \end{bmatrix}, \quad x \in [0, 1]. \tag{2.26}$$

Then there exists a fundamental matrix solution $\widehat{\Psi}(x, \lambda)$ to system (2.22) such that for all λ with sufficiently large modulus, it has

$$\widehat{\Psi}(x, \lambda) = [\widehat{\Psi}_0(x) + \mathcal{O}(\lambda^{-1})]E(x, \lambda), \tag{2.27}$$

where

$$\widehat{\Psi}_0(x) := \mathrm{diag}(e^{b_1 x}, e^{-b_2 x}, e^{b_2 x}, e^{-b_1 x}). \tag{2.28}$$

Proof. By Eqs. (2.24) and (2.25), Assumption 2.1 of [10] on p. 135 is satisfied and hence Theorem 2.2 of [10] on p.134 can be directly applied to our problem (see also [9]), that is to say, there is a fundamental matrix solution to Eq. (2.22) which is of the following form

$$\widehat{\Psi}(x, \lambda) = (\widehat{\Psi}_0(x) + \lambda^{-1}\widehat{\Psi}_1(x) + \lambda^{-2}\widetilde{\Theta}(x, \lambda))E(x, \lambda),$$

where $\widetilde{\Theta}(x, \lambda)$ is uniformly bounded in λ and $x \in [0, 1]$. Since $\widehat{A}_1$ is a diagonal matrix, $E(x, \lambda)$ is a fundamental matrix solution to the dominant term of Eq. (2.22), in other words,

$$E'(x, \lambda) = \lambda\widehat{A}_1 E(x, \lambda).$$

Next, compute $\widehat{\Psi}'(x, \lambda)$ and $-\widehat{A}(\lambda)\widehat{\Psi}(x, \lambda)$ to yield

$$\begin{aligned} \widehat{\Psi}'(x, \lambda) &= (\widehat{\Psi}_0'(x) + \lambda^{-1}\widehat{\Psi}_1'(x) + \lambda^{-2}\widetilde{\Theta}_x(x, \lambda))E(x, \lambda) \\ &\quad + \lambda(\widehat{\Psi}_0(x) + \lambda^{-1}\widehat{\Psi}_1(x) + \lambda^{-2}\widetilde{\Theta}(x, \lambda))\widehat{A}_1 E(x, \lambda) \end{aligned}$$

and

$$-\widehat{A}(\lambda)\widehat{\Psi}(x, \lambda) = (\lambda\widehat{A}_1 + \widehat{A}_0)(\widehat{\Psi}_0(x) + \lambda^{-1}\widehat{\Psi}_1(x) + \lambda^{-2}\widetilde{\Theta}(x, \lambda))E(x, \lambda).$$

Inserting the above two equations into Eq. (2.22) and equating the corresponding coefficients of $\lambda^i, i = 1, 0, -1$, we obtain

$$\widehat{\Psi}_0(x)\widehat{A}_1 - \widehat{A}_1\widehat{\Psi}_0(x) = 0, \tag{2.29}$$

$$\widehat{\Psi}_0'(x) - \widehat{A}_0\widehat{\Psi}_0(x) + \widehat{\Psi}_1(x)\widehat{A}_1 - \widehat{A}_1\widehat{\Psi}_1(x) = 0. \tag{2.30}$$

The proof will be accomplished if the leading order term $\widehat{\Psi}_0(x)$ is given by Eq. (2.28). Indeed, from Eq. (2.29) and the fact that $\theta_i, i = 1, 2, 3, 4$ are distinct each other, we can conclude that the matrix function $\widehat{\Psi}_0(x)$ is of the diagonal form

$$\widehat{\Psi}_0(x) := \text{diag}[\psi_{11}(x), \ \psi_{22}(x), \ \psi_{33}(x), \ \psi_{44}(x)]$$

and its entries can be obtained by substituting them into Eq. (2.30) as

$$\begin{cases} \psi_{11}' = b_1\psi_{11}, \quad \psi_{22}' = -b_2\psi_{22}, \quad \psi_{33}' = b_2\psi_{33}, \quad \psi_{44}' = -b_1\psi_{44}, \\ \widehat{\Psi}_0(0) = I. \end{cases} \tag{2.31}$$

Eq. (2.28) then follows. $\quad\square$

Corollary 2.1. *Let $\widehat{\Psi}(x, \lambda)$ given by Eq. (2.27) be a fundamental matrix solution to system (2.22). Then*

$$\widehat{\Phi}(x, \lambda) := P(\lambda)\widehat{\Psi}(x, \lambda) \tag{2.32}$$

is a fundamental matrix solution to the first order linear system (2.13).

We are now in a position to estimate the asymptotics of the eigenvalues of $\mathscr{A}$. Note that the eigenvalues of $\mathscr{A}$ are the zeros of the characteristic determinant:

$$\Delta(\lambda) := \det(T^R\widehat{\Phi}(x, \lambda)), \quad \lambda \in \mathbb{C}, \tag{2.33}$$

where the operator T^R is given by Eq. (2.13) and $\widehat{\Phi}(x, \lambda)$ is any fundamental matrix to the equation $T^D(x, \lambda)\Phi(x) = 0$ [10]. The basic idea to get the asymptotic expansion of eigenvalues is to substitute Eqs. (2.27) and (2.32) into Eq. (2.33) by taking the boundary conditions of Eq. (2.13) into account. Actually, since

$$T^R\widehat{\Phi} = W^0 P(\lambda)\widehat{\Psi}(0, \lambda) + W^1 P(\lambda)\widehat{\Psi}(1, \lambda), \tag{2.34}$$

a simple computation, using Eqs. (2.15) and (2.17), gives

$$W^0 P(\lambda) = W^0 \begin{bmatrix} P_1(\lambda) & -P_2(\lambda) \\ P_1(\lambda) & P_2(\lambda) \end{bmatrix} = \begin{bmatrix} r\lambda & r\lambda & -r\lambda & -r\lambda \\ r\lambda & r\lambda & r\lambda & r\lambda \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$W^1 P(\lambda) = W^1 \begin{bmatrix} P_1(\lambda) & -P_2(\lambda) \\ P_1(\lambda) & P_2(\lambda) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ r\lambda & r\lambda & -r\lambda & -r\lambda \\ r\lambda & r\lambda & r\lambda & r\lambda \end{bmatrix}.$$

We shall carry out our estimations by using the following notation

$$[a]_1 := a + \mathcal{O}(\lambda^{-1}).$$

Since $\widehat{\Psi}_0(0) = I$ and $E(0, \lambda) = I$, a direct computation yields

$$W^1 P(\lambda)\widehat{\Psi}(1, \lambda) = r\lambda \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ [1]_1 e^{\theta_1 r\lambda + b_1} & [1]_1 e^{\theta_2 r\lambda - b_2} & -[1]_1 e^{-\theta_2 r\lambda + b_2} & -[1]_1 e^{-\theta_1 r\lambda - b_1} \\ [1]_1 e^{\theta_1 r\lambda + b_1} & [1]_1 e^{\theta_2 r\lambda - b_2} & [1]_1 e^{-\theta_2 r\lambda + b_2} & [1]_1 e^{-\theta_1 r\lambda - b_1} \end{bmatrix},$$

$$W^0 P(\lambda)\widehat{\Psi}(0, \lambda) = r\lambda \begin{bmatrix} [1]_1 & [1]_1 & -[1]_1 & -[1]_1 \\ [1]_1 & [1]_1 & [1]_1 & [1]_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and hence

$$T^R\widehat{\Phi}(x, \lambda) = r\lambda \begin{bmatrix} [1]_1 & [1]_1 & -[1]_1 & -[1]_1 \\ [1]_1 & [1]_1 & [1]_1 & [1]_1 \\ [1]_1 e^{\theta_1 r\lambda + b_1} & [1]_1 e^{\theta_2 r\lambda - b_2} & -[1]_1 e^{-\theta_2 r\lambda + b_2} & -[1]_1 e^{-\theta_1 r\lambda - b_1} \\ [1]_1 e^{\theta_1 r\lambda + b_1} & [1]_1 e^{\theta_2 r\lambda - b_2} & [1]_1 e^{-\theta_2 r\lambda + b_2} & [1]_1 e^{-\theta_1 r\lambda - b_1} \end{bmatrix}.$$

Therefore,

$$\begin{aligned}
\Delta(\lambda) &= \det(T^R\widehat{\Phi}(x, \lambda)) \\
&= r^4 \lambda^4 \left\{ \det \begin{bmatrix} 0 & 0 & -2 & -2 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -2e^{-\theta_2 r\lambda + b_2} & -2e^{-\theta_1 r\lambda - b_1} \\ e^{\theta_1 r\lambda + b_1} & e^{\theta_2 r\lambda - b_2} & e^{-\theta_2 r\lambda + b_2} & e^{-\theta_1 r\lambda - b_1} \end{bmatrix} + \mathcal{O}(\lambda^{-1}) \right\} \\
&= -4r^4 \lambda^4 \left\{ \det \begin{bmatrix} 1 & 1 \\ e^{\theta_1 r\lambda + b_1} & e^{\theta_2 r\lambda - b_2} \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ e^{-\theta_2 r\lambda + b_2} & e^{-\theta_1 r\lambda - b_1} \end{bmatrix} + \mathcal{O}(\lambda^{-1}) \right\} \\
&= -4r^4 \lambda^4 \{ (e^{\theta_2 r\lambda - b_2} - e^{\theta_1 r\lambda + b_1}) \times (e^{-\theta_1 r\lambda - b_1} - e^{-\theta_2 r\lambda + b_2}) + \mathcal{O}(\lambda^{-1}) \} \\
&= -4r^4 \lambda^4 e^{-(\theta_1 + \theta_2)r\lambda + (b_2 - b_1)} \{ (e^{\theta_2 r\lambda - b_2} - e^{\theta_1 r\lambda + b_1})^2 + \mathcal{O}(\lambda^{-1}) \}.
\end{aligned}$$

Theorem 2.3. *Let $\Delta(\lambda)$ be the characteristic determinant (2.33). Then $\Delta(\lambda)$ has the following asymptotic expansion*:

$$\Delta(\lambda) = -4r^4 \lambda^4 e^{-(\theta_1 + \theta_2)r\lambda + (b_2 - b_1)} \{ \Delta_1^2(\lambda) + \mathcal{O}(\lambda^{-1}) \}, \tag{2.35}$$

where

$$\Delta_1(\lambda) := e^{\theta_2 r\lambda - b_2} - e^{\theta_1 r\lambda + b_1}. \tag{2.36}$$

Theorem 2.4. *Let $\mathscr{A}$ be defined by Eqs. (2.1)–(2.2). Then the eigenvalues λ_k of $\mathscr{A}$ has the following asymptotic expansions*:

$$\lambda_k = \frac{1}{(\theta_2 - \theta_1)r}(b_1 + b_2 + \pi i + 2k\pi i) + \mathcal{O}(k^{-1}), \quad k \in \mathbb{Z}, \tag{2.37}$$

for $|k| \geqslant N$, where N is a large enough positive integer. Furthermore, by Eqs. (2.5), (2.16), and (2.23), it follows that

$$\frac{b_1 + b_2}{(\theta_2 - \theta_1)r} = \frac{-a_2 - \frac{1}{2}a_3(\theta_1 + \theta_2)}{(\theta_2 - \theta_1)^2 r} = \frac{-a_2 - a_1 a_3}{4(1 + a_1^2)r} = -\frac{2a + \alpha^2}{8a + 2\alpha^2}\varepsilon < 0. \tag{2.38}$$

Therefore,

$$\operatorname{Re}\lambda_k \to -\frac{2a + \alpha^2}{8a + 2\alpha^2}\varepsilon < 0 \quad \text{as } k \to \infty. \tag{2.39}$$

Proof. By the asymptotic expansion of $\Delta(\lambda)$ in Theorem 2.3, we only need to find the solution of the form

$$\Delta_1(\lambda) + \mathcal{O}(\lambda^{-1}) = 0,$$

that is

$$e^{\theta_2 r\lambda - b_2} - e^{\theta_1 r\lambda + b_1} + \mathcal{O}(\lambda^{-1}) = 0. \tag{2.40}$$

Using the Rouché's theorem, the roots of Eq. (2.40) can be estimated by those of

$$e^{\theta_2 r\lambda - b_2} - e^{\theta_1 r\lambda + b_1} = 0,$$

which are found explicitly as following:

$$\tilde{\lambda}_k = \frac{1}{(\theta_2 - \theta_1)r}(b_1 + b_2 + \pi i + 2k\pi i), \quad k \in \mathbb{Z}. \tag{2.41}$$

Thus, the roots of Eq. (2.40) satisfy

$$\lambda_k = \frac{1}{(\theta_2 - \theta_1)r}(b_1 + b_2 + \pi i + 2k\pi i) + \mathcal{O}(k^{-1}), \quad |k| \geqslant N, \ k \in \mathbb{Z}, \tag{2.42}$$

where N is a sufficiently large positive integer. The proof is complete. $\quad\square$

Remark 2.1. Compared with Eq. (1.2), the asymptotics of eigenvalues of $\mathscr{A}$ expressed in Eq. (2.37) is completely different with that of usual thermoelastic system (1.1) in Eq. (1.2). It is a typical property for hyperbolic systems.

3. The asymptotic expansion of eigenfunctions

Theorem 3.1. Let $\{\lambda_k, k \in \mathbb{Z}\}$ be the eigenvalues of $\mathscr{A}$ with λ_k being given in Eq. (2.37). Then there are at least two families of the corresponding eigenfunctions

$$\begin{cases} \{\mathscr{F}_k = [w_k(x), \lambda_k w_k(x), \theta_k(x), \lambda_k \theta_k(x)]; \ k \in \mathbb{Z}\}, \\ \{\widetilde{\mathscr{F}}_k = [-w_k(1-x), -\lambda_k w_k(1-x), \theta_k(1-x), \lambda_k \theta_k(1-x)]; \ k \in \mathbb{Z}\}, \end{cases} \tag{3.1}$$

with the following asymptotic expressions:

$$
\begin{cases}
w'_k(x) = \theta_1 e^{(\theta_1 r\lambda_k + b_1)x} - \theta_2 e^{(\theta_2 r\lambda_k - b_2)x} + \mathcal{O}(k^{-1}), \\[2mm]
\lambda_k w_k(x) = \dfrac{1}{r} e^{(\theta_1 r\lambda_k + b_1)x} - \dfrac{1}{r} e^{(\theta_2 r\lambda_k - b_2)x} + \mathcal{O}(k^{-1}), \\[2mm]
\theta'_k(x) = \theta_1 e^{(\theta_1 r\lambda_k + b_1)x} - \theta_2 e^{(\theta_2 r\lambda_k - b_2)x} + \mathcal{O}(k^{-1}), \\[2mm]
\lambda_k \theta_k(x) = \dfrac{1}{r} e^{(\theta_1 r\lambda_k + b_1)x} - \dfrac{1}{r} e^{(\theta_2 r\lambda_k - b_2)x} + \mathcal{O}(k^{-1}).
\end{cases}
\tag{3.2}
$$

Moreover, $\{[w_k, \lambda_k w_k, \theta_k, \lambda_k \theta_k], \ k \in \mathbb{Z}\}$ *are approximately normalized in* $\mathcal{H}$ *in the sense that there exist positive constants* c_1 *and* c_2, *independent of* k, *such that for* $k \in \mathbb{Z}$,

$$
c_1 \leqslant \|w'_k\|_{L^2}, \|\lambda_k w_k\|_{L^2}, \|\theta'_k\|_{L^2}, \|\lambda_k \theta_k\|_{L^2} \leqslant c_2.
\tag{3.3}
$$

Proof. A nontrivial solution $\Phi(x) = [w_1(x), w_2(x), \theta_1(x), \theta_2(x)]^\top$ in Eq. (2.12) corresponding to eigenvalue λ can be obtained as follows: its jth component is the determinant of the matrix determined by replacing one of the rows of $T^R\widehat{\Phi}$ in (2.34) with $e_j^\top(\widehat{\Phi}(x, \lambda))$, and the symmetric row with $e_i^\top$ so that its determinant is not identical to zero, where e_j is the jth column of the identity matrix.

Now let us to find the first nontrivial solution $\Phi_1(x)$ in which each component is determined as the determinant by replacing the third row of $T^R\widehat{\Phi}$ in Eq. (2.34) with $e_j^\top(\widehat{\Phi}(x, \lambda))$, and the fourth row with $e_4^\top$. From Eq. (2.32), $\widehat{\Phi}(x, \lambda) = P(\lambda)\widehat{\Psi}(x, \lambda)$ and hence by Eqs. (2.17)–(2.20)

$$
\widehat{\Phi}(x, \lambda) = \begin{bmatrix} P_1(\lambda) & -P_2(\lambda) \\ P_1(\lambda) & P_2(\lambda) \end{bmatrix} [\widehat{\Psi}_0(x) + \mathcal{O}(\lambda^{-1})] E(x, \lambda),
\tag{3.4}
$$

where $P_1(\lambda)$ and $P_2(\lambda)$ are given in Eqs. (2.18) and (2.19), respectively, and $\widehat{\Psi}_0(x)$ and $E(x, \lambda)$ are given in Eqs. (2.28) and (2.26), respectively.

With the above interpretation, the first component of $\Phi_1(x)$ is thus given by

$$
w_1(x, \lambda) = \det \begin{bmatrix}
r\lambda[1]_1 & r\lambda[1]_1 \\
r\lambda[1]_1 & r\lambda[1]_1 \\
r\lambda[1]_1 e^{\theta_1 rx\lambda + b_1 x} & r\lambda[1]_1 e^{\theta_2 rx\lambda - b_2 x} \\
0 & 0 \\
-r\lambda[1]_1 & -r\lambda[1]_1 \\
r\lambda[1]_1 & r\lambda[1]_1 \\
-r\lambda[1]_1 e^{-\theta_2 rx\lambda + b_2 x} & -r\lambda[1]_1 e^{-\theta_1 rx\lambda - b_1 x} \\
0 & 1
\end{bmatrix}
$$

$$
= r^3 \lambda^3 \det \left\{ \begin{bmatrix}
1 & 1 & -1 & -1 \\
1 & 1 & 1 & 1 \\
e^{\theta_1 rx\lambda + b_1 x} & e^{\theta_2 rx\lambda - b_2 x} & -e^{-\theta_2 rx\lambda + b_2 x} & -e^{-\theta_1 rx\lambda - b_1 x} \\
0 & 0 & 0 & 1
\end{bmatrix} + \mathcal{O}(\lambda^{-1}) \right\}
$$

$$
= 2r^3\lambda^3 \det\left\{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ e^{\theta_1 rx\lambda+b_1 x} & e^{\theta_2 rx\lambda-b_2 x} & -e^{-\theta_2 rx\lambda+b_2 x} \end{bmatrix} + \mathcal{O}(\lambda^{-1})\right\}
$$

$$
= 2r^3\lambda^3\left\{-\det\begin{bmatrix} 1 & 1 \\ e^{\theta_1 rx\lambda+b_1 x} & e^{\theta_2 rx\lambda-b_2 x} \end{bmatrix} + \mathcal{O}(\lambda^{-1})\right\}
$$

$$
= 2r^3\lambda^3\{e^{\theta_1 rx\lambda+b_1 x} - e^{\theta_2 rx\lambda-b_2 x} + \mathcal{O}(\lambda^{-1})\},
$$

or

$$
2^{-1}r^{-3}\lambda^{-3}w_1(x,\lambda) = e^{\theta_1 rx\lambda+b_1 x} - e^{\theta_2 rx\lambda-b_2 x} + \mathcal{O}(\lambda^{-1}). \tag{3.5}
$$

Similarly, the second component of $\Phi_1(x)$ is given by

$$
w_2(x,\lambda) = \det\begin{bmatrix} r\lambda[1]_1 & r\lambda[1]_1 \\ r\lambda[1]_1 & r\lambda[1]_1 \\ \theta_1 r^2\lambda^2[1]_1 e^{\theta_1 rx\lambda+b_1 x} & \theta_2 r^2\lambda^2[1]_1 e^{\theta_2 rx\lambda-b_2 x} \\ 0 & 0 \\[6pt] -r\lambda[1]_1 & -r\lambda[1]_1 \\ r\lambda[1]_1 & r\lambda[1]_1 \\ \theta_2 r^2\lambda^2[1]_1 e^{-\theta_2 rx\lambda+b_2 x} & \theta_1 r^2\lambda^2[1]_1 e^{-\theta_1 rx\lambda-b_1 x} \\ 0 & 1 \end{bmatrix}
$$

$$
= r^4\lambda^4 \det\left\{\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ \theta_1 e^{\theta_1 rx\lambda+b_1 x} & \theta_2 e^{\theta_2 rx\lambda-b_2 x} & \theta_2 e^{-\theta_2 rx\lambda+b_2 x} \end{bmatrix} + \mathcal{O}(\lambda^{-1})\right\}
$$

$$
= 2r^4\lambda^4\left\{-\det\begin{bmatrix} 1 & 1 \\ \theta_1 e^{\theta_1 rx\lambda+b_1 x} & \theta_2 e^{\theta_2 rx\lambda-b_2 x} \end{bmatrix} + \mathcal{O}(\lambda^{-1})\right\}
$$

$$
= 2r^4\lambda^4\{\theta_1 e^{\theta_1 rx\lambda+b_1 x} - \theta_2 e^{\theta_2 rx\lambda-b_2 x} + \mathcal{O}(\lambda^{-1})\}.
$$

Hence

$$
2^{-1}r^{-4}\lambda^{-4}w_2(x,\lambda) = \theta_1 e^{\theta_1 rx\lambda+b_1 x} - \theta_2 e^{\theta_2 rx\lambda-b_2 x}. \tag{3.6}
$$

Furthermore,

$$
w_3(x,\lambda) = \det\begin{bmatrix} r\lambda[1]_1 & r\lambda[1]_1 & -r\lambda[1]_1 & -r\lambda[1]_1 \\ r\lambda[1]_1 & r\lambda[1]_1 & r\lambda[1]_1 & r\lambda[1]_1 \\ r\lambda[1]_1 e^{\theta_1 rx\lambda+b_1 x} & r\lambda[1]_1 e^{\theta_2 rx\lambda-b_2 x} & r\lambda[1]_1 e^{-\theta_2 rx\lambda+b_2 x} & r\lambda[1]_1 e^{-\theta_1 rx\lambda-b_1 x} \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$= r^3 \lambda^3 \det \left\{ \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ e^{\theta_1 r \lambda x + b_1 x} & e^{\theta_2 r \lambda x - b_2 x} & e^{-\theta_2 r \lambda x + b_2 x} \end{bmatrix} + \mathcal{O}(\lambda^{-1}) \right\}$$

$$= 2 r^3 \lambda^3 \left\{ -\det \begin{bmatrix} 1 & 1 \\ e^{\theta_1 r x \lambda + b_1 x} & e^{\theta_2 r x \lambda - b_2 x} \end{bmatrix} + \mathcal{O}(\lambda^{-1}) \right\}$$

$$= 2 r^3 \lambda^3 \{ e^{\theta_1 r x \lambda + b_1 x} - e^{\theta_2 r x \lambda - b_2 x} + \mathcal{O}(\lambda^{-1}) \},$$

or

$$2^{-1} r^{-3} \lambda^{-3} w_3(x, \lambda) = e^{\theta_1 r x \lambda + b_1 x} - e^{\theta_2 r x \lambda - b_2 x} + \mathcal{O}(\lambda^{-1}). \tag{3.7}$$

Finally,

$$w_4(x, \lambda) = \det \begin{bmatrix} r\lambda[1]_1 & r\lambda[1]_1 \\ r\lambda[1]_1 & r\lambda[1]_1 \\ \theta_1 r^2 \lambda^2 [1]_1 e^{\theta_1 r x \lambda + b_1 x} & \theta_2 r^2 \lambda^2 [1]_1 e^{\theta_2 r x \lambda - b_2 x} \\ 0 & 0 \\ -r\lambda[1]_1 & -r\lambda[1]_1 \\ r\lambda[1]_1 & r\lambda[1]_1 \\ -\theta_2 r^2 \lambda^2 [1]_1 e^{-\theta_2 r x \lambda + b_2 x} & -\theta_1 r^2 \lambda^2 [1]_1 e^{-\theta_1 r x \lambda - b_1 x} \\ 0 & 1 \end{bmatrix}$$

$$= r^4 \lambda^4 \det \left\{ \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ \theta_1 e^{\theta_1 r x \lambda + b_1 x} & \theta_2 e^{\theta_2 r x \lambda - b_2 x} & -\theta_2 e^{-\theta_2 r x \lambda + b_2 x} \end{bmatrix} + \mathcal{O}(\lambda^{-1}) \right\}$$

$$= 2 r^4 \lambda^4 \left\{ -\det \begin{bmatrix} 1 & 1 \\ \theta_1 e^{\theta_1 r x \lambda + b_1 x} & \theta_2 e^{\theta_2 r x \lambda - b_2 x} \end{bmatrix} + \mathcal{O}(\lambda^{-1}) \right\}$$

$$= 2 r^4 \lambda^4 \{ \theta_1 e^{\theta_1 r x \lambda + b_1 x} - \theta_2 e^{\theta_2 r x \lambda - b_2 x} + \mathcal{O}(\lambda^{-1}) \}.$$

Hence

$$2^{-1} r^{-4} \lambda^{-4} w_4(x, \lambda) = \theta_1 e^{\theta_1 r x \lambda + b_1 x} - \theta_2 e^{\theta_2 r x \lambda - b_2 x} + \mathcal{O}(\lambda^{-1}). \tag{3.8}$$

Based on above computations, Eq. (3.2) can then be deduced from Eqs. (3.5)–(3.8) by setting

$$w(x) = \frac{w_1(x, \lambda)}{2 r^4 \lambda^4}, \quad \theta(x) = \frac{w_3(x, \lambda)}{2 r^4 \lambda^4}. \tag{3.9}$$

Finally, it follows from Eqs. (2.37)–(2.38) that

$$\| e^{\theta_i r x \lambda_k} \|_{L^2} = \frac{1 - e^{-2\theta_i \mu}}{2 \theta_i \mu} + \mathcal{O}(k^{-1}) \quad \text{for } i = 1, 2, \tag{3.10}$$

where

$$\mu := \frac{a_2 + a_1 a_3}{4(1 + a_1^2)}.$$

These together with Eq. (3.2) yield Eq. (3.3). The proof is complete. $\quad\square$

4. Riesz basis property

In order to establish the Riesz basis property, we need the following modified classical Bari's Theorem [14].

Theorem 4.1. *Let* **A** *be a densely defined discrete operator* (*that is,* $(\lambda - \mathbf{A})^{-1}$ *is compact for some* λ) *in a Hilbert space* **H**. *Let* $\{z_n\}_1^\infty$ *be a Riesz basis for* **H**. *If there are an integer* $N \geqslant 0$ *and a sequence of generalized eigenvectors* $\{x_n\}_{N+1}^\infty$ *of* **A** *such that*

$$\sum_{N+1}^\infty \|x_n - z_n\|^2 < \infty$$

then

(i) *There are an* $M > N$ *and generalized eigenvectors* $\{x_{n0}\}_1^M$ *of* **A** *such that* $\{x_{n0}\}_1^M \cup \{x_n\}_{M+1}^\infty$ *forms a Riesz basis for* **H**.
(ii) *Let* $\{x_{n0}\}_1^M \cup \{x_n\}_{M+1}^\infty$ *correspond to eigenvalues* $\{\sigma_n\}_1^\infty$ *of* **A**. *Then* $\sigma(\mathbf{A}) = \{\sigma_n\}_1^\infty$, *where* σ_n *is counted according to its algebraic multiplicity.*

In order to apply Theorem 4.1 to $\mathscr{A}$, we need a reference Riesz basis. This can be obtained by collecting the eigenfunctions of $\mathscr{A}_0$, a skew-adjoint operator in $\mathscr{H}$

$$\mathscr{A}_0 \begin{bmatrix} w \\ \phi \\ \theta \\ v \end{bmatrix}^\top = \begin{bmatrix} \phi \\ aw'' - \alpha v' \\ v \\ a\theta'' - \alpha\phi' \end{bmatrix}^\top \quad \forall \begin{bmatrix} w \\ \phi \\ \theta \\ v \end{bmatrix}^\top \in D(\mathscr{A}_0) = D(\mathscr{A}). \tag{4.1}$$

It is seen that $\mathscr{A}_0$ is just the operator $\mathscr{A}$ with $\varepsilon = 0$. Lemma 2.1 tells us that $\mathscr{A}_0^{-1}$ exists and is compact on $\mathscr{H}$.

By the definition of $\mathscr{A}_0$, the following result is immediate.

Lemma 4.1. *Let* $\mathscr{A}_0$ *be defined by* (4.1). *Then* $\mathscr{A}_0$ *is a skew-adjoint operator in* $\mathscr{H}$ *with compact resolvents. Hence there exists a sequence of eigenfunctions of* $\mathscr{A}_0$, *which forms an orthogonal basis for* $\mathscr{H}$. *The asymptotic expressions for the eigenvalues and eigenfunctions of* $\mathscr{A}_0$ *can be obtained directly by Eqs.* (2.37) *and* (3.2) *with* $\varepsilon \equiv 0$.

Theorem 4.2. *Let* $\mathscr{A}$ *be defined by Eqs.* (2.1) *and* (2.2). *Then there exists a sequence of generalized eigenfunctions of* $\mathscr{A}$, *which forms a Riesz basis for* $\mathscr{H}$. *Moreover, any* $\lambda \in \sigma(\mathscr{A})$ *with* $|\lambda|$ *sufficiently large,* λ *is semi-simple, that is, it has the same algebraic and geometric multiplicities and the multiplicity is two.*

Proof. Let $\{\mathscr{F}_k, \widetilde{\mathscr{F}}_k\}_{k\in\mathbb{Z}}$ and $\{\mathscr{G}_k, \widetilde{\mathscr{G}}_k\}_{k\in\mathbb{Z}}$ be the eigenfunctions of $\mathscr{A}$ and $\mathscr{A}_0$, respectively. $\mathscr{F}_k$ has the asymptotic expansion (3.2) and $\mathscr{G}_k$ has the asymptotic expansion (3.2) with

$\varepsilon \equiv 0$. Obviously, there is a positive number N such that

$$\sum_{|k|>N} \|\mathscr{F}_k - \mathscr{G}_k\|^2 = \sum_{|k|>N} \mathcal{O}(k^{-2}) < \infty.$$

The same thing is true for $\{\widetilde{\mathscr{F}}_k\}_{k\in\mathbb{Z}}$ and $\{\widetilde{\mathscr{G}}_k\}_{k\in\mathbb{Z}}$.

Since $\{\mathscr{G}_k, \widetilde{\mathscr{G}}_k\}_{k\in\mathbb{Z}}$ form an orthogonal basis for $\mathscr{H}$, we conclude, from Theorem 4.1, that $\{\mathscr{F}_k, \widetilde{\mathscr{F}}_k\}_{k\in\mathbb{Z}}$ form a Riesz basis for $\mathscr{H}$ too. Moreover, since any eigenvalue of $\mathscr{A}_0$ is semi-simple with multiplicity 2 claimed by Proposition 2.1, so is for eigenvalue of $\mathscr{A}$ with large module. $\quad\square$

The following fundamental result for system (1.8) is a direct consequence of Theorem 4.2 [15].

Corollary 4.1. *$\mathscr{A}$ generates a C_0-semigroup $\mathrm{e}^{\mathscr{A}t}$ on $\mathscr{H}$ and the spectrum-determined growth condition holds true for $\mathrm{e}^{\mathscr{A}t}$, that is, $s(\mathscr{A}) = \omega(\mathscr{A})$, where*

$$s(\mathscr{A}) := \sup\{\operatorname{Re}\lambda \mid \lambda \in \sigma(\mathscr{A})\}$$

is the spectral bound of $\mathscr{A}$ and $\omega(\mathscr{A})$ stands for the growth bound of $\mathrm{e}^{\mathscr{A}t}$.

Theorem 4.3. *The $\mathrm{e}^{\mathscr{A}t}$ is exponentially stable, that is to say, there exist constants $M>1, \omega>0$ such that*

$$\|\mathrm{e}^{\mathscr{A}t}\| \leqslant M\mathrm{e}^{-\omega t}$$

or equivalently

$$F(t) \leqslant M\mathrm{e}^{-\omega t} F(0),$$

where F is the energy function (1.9).

Proof. By the spectrum-determined growth condition claimed by Corollary 4.1 and the asymptote for eigenvalues of (2.39), $\mathrm{e}^{\mathscr{A}t}$ is exponentially stable if and only if

$$\operatorname{Re}\lambda < 0 \quad \forall \lambda \in \sigma(\mathscr{A}). \tag{4.2}$$

Now suppose $\lambda \neq -\varepsilon$ and let $u(x) = w(x)/\lambda, v(x) = a/(\lambda+\varepsilon)\theta'(x)$. Then Eq. (2.4) becomes

$$\begin{cases} \lambda^2 u(x) - au''(x) + \alpha\theta'(x) = 0, \\ \lambda\theta(x) - v'(x) + \alpha\lambda u'(x) = 0, \\ \lambda v(x) - a\theta'(x) + \varepsilon v(x) = 0, \\ u(0) = u(1) = \theta(0) = \theta(1) = 0. \end{cases} \tag{4.3}$$

Hence Eq. (4.3) is just the eigenvalue problem of Eq. (1.4). By Eq. (1.6), we get immediately Eq. (4.2). $\quad\square$

5. Results for system (1.4)

In this section, we give the parallel results for system (1.4). As before, we first formulate system (1.4) into an evolution equation in the state Hilbert space $\widetilde{\mathscr{H}}$ defined by

$$\widetilde{\mathscr{H}} := H_0^1(0,1) \times L^2(0,1) \times L^2(0,1) \times L^2(0,1),$$

equipped with the inner product: for any $F_i = [u_i, \phi_i, \theta_i, v_i] \in \mathscr{H}, i = 1, 2$:

$$\langle F_1, F_2 \rangle_{\widetilde{\mathscr{H}}} := a\langle u'_1, u'_2 \rangle_{L^2} + \langle \phi_1, \phi_2 \rangle_{L^2} + \langle \theta_1, \theta_2 \rangle_{L^2} + \frac{1}{a}\langle v_1, v_2 \rangle_{L^2}.$$

Define a linear operator $\widetilde{\mathscr{A}}$ in $\widetilde{\mathscr{H}}$ by

$$\widetilde{\mathscr{A}}\begin{bmatrix} u \\ \phi \\ \theta \\ v \end{bmatrix}^\top = \begin{bmatrix} \phi \\ au'' - \alpha\theta' \\ v' - \alpha\phi' \\ a\theta' - \varepsilon v \end{bmatrix}^\top \quad \forall \begin{bmatrix} u \\ \phi \\ \theta \\ v \end{bmatrix}^\top \in D(\widetilde{\mathscr{A}}) \tag{5.1}$$

with

$$D(\widetilde{\mathscr{A}}) = \widetilde{\mathscr{H}} \cap (H^2(0,1) \times H^1_0(0,1) \times H^1_0(0,1) \times H(0,1)). \tag{5.2}$$

Then Eq. (1.4) can be formulated as an evolution equation in $\widetilde{\mathscr{H}}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}Z(t) = \widetilde{\mathscr{A}}Z(t), \quad Z(0) = Z_0 \tag{5.3}$$

with $Z(\cdot, t) := [u(\cdot, t), u_t(\cdot, t), \theta(\cdot, t), v(\cdot, t)]$ and $Z_0 = [u_0, u_1, \theta_0, 0]$.

Lemma 5.1. *Let $\widetilde{\mathscr{A}}$ be given in Eqs. (5.1) and (5.2). Then $\widetilde{\mathscr{A}}^{-1}$ exists and is compact on $\widetilde{\mathscr{H}}$. Therefore $\sigma(\widetilde{\mathscr{A}})$, the spectral set of $\widetilde{\mathscr{A}}$, consists of only isolated eigenvalues with finite algebraic multiplicity. Moreover, $\widetilde{\mathscr{A}}$ is dissipative in $\widetilde{\mathscr{H}}$ and thus $\widetilde{\mathscr{A}}$ generates a C_0-semigroup of contractions $\mathrm{e}^{\mathscr{A}t}$ on $\widetilde{\mathscr{H}}$.*

Proof. This is similar to the proof of Lemma 2.1 with

$$\widetilde{\mathscr{A}}^{-1}G = F \quad \text{for any } G = [g_1, g_2, g_3, g_4] \in \widetilde{\mathscr{H}},$$

where $F = [u, \phi, \theta, v] \in D(\widetilde{\mathscr{A}})$,

$$\begin{cases} \phi = g_1, \quad v = v(0) + \alpha g_1 + \int_0^x g_3(s)\,\mathrm{d}s, \\ \theta(x) = \dfrac{\varepsilon}{a}v(0)x + \dfrac{1}{a}\int_0^x \left[\varepsilon\alpha g_1 + \int_0^s \varepsilon g_3(\xi)\,\mathrm{d}\xi + g_4(s)\right]\mathrm{d}s, \\ u(x) = u'(0)x + \dfrac{1}{a}\int_0^x \left[\alpha\theta(s) + \int_0^s g_2(\xi)\,\mathrm{d}\xi\right]\mathrm{d}s, \\ v(0) = -\dfrac{1}{\varepsilon}\int_0^1 \left[\varepsilon\alpha g_1 + \int_0^s \varepsilon g_3(\xi)\,\mathrm{d}\xi + g_4(s)\right]\mathrm{d}s, \\ u'(0) = -\dfrac{1}{a}\int_0^1 \left[\alpha\theta(s) + \int_0^s g_2(\xi)\,\mathrm{d}\xi\right]\mathrm{d}s. \end{cases}$$

And it follows from Eq. (1.6) that for each $F = [u, \phi, \theta, v] \in D(\widetilde{\mathscr{A}})$,

$$\mathrm{Re}\langle \widetilde{\mathscr{A}}F, F \rangle_{\widetilde{H}} = -\frac{\varepsilon}{a}\|v\|^2_{L^2} \leqslant 0,$$

proving the result. $\square$

Proposition 5.1. $\sigma(\mathscr{A}) = \sigma(\widetilde{\mathscr{A}}) + \{-\varepsilon\}$.

Proof. Since $0 \in \rho(\mathscr{A}) \cap \rho(\widetilde{\mathscr{A}})$ and for $\lambda \neq -\varepsilon$, the eigenvalue problem (4.3) of $\widetilde{\mathscr{A}}$ is equivalent to the eigenvalue problem (2.4) of $\mathscr{A}$. So the proof will be accomplished if we can show that $-\varepsilon \in \sigma(\widetilde{\mathscr{A}})$ but $-\varepsilon \notin \sigma(\mathscr{A})$. Firstly, we show that $-\varepsilon \in \sigma(\widetilde{\mathscr{A}})$. Let $\lambda = -\varepsilon$.

Then Eq. (4.3) becomes

$$\begin{cases} \varepsilon^2 u(x) - a u''(x) + \alpha \theta'(x) = 0, \\ \varepsilon \theta(x) + v'(x) + \alpha \varepsilon u'(x) = 0, \\ a\theta'(x) = 0, \quad u(0) = u(1) = \theta(0) = \theta(1) = 0. \end{cases}$$

The third equation above gives that $\theta \equiv 0$ and hence

$$\begin{cases} \varepsilon^2 u(x) - a u''(x) = 0, \quad u(0) = u(1) = 0, \\ v'(x) + \alpha \varepsilon u'(x) = 0, \end{cases}$$

which further yields that the above equations have nonzero solution

$$u \equiv 0, \quad v = \text{constant}.$$

Hence $-\varepsilon \in \sigma(\widetilde{\mathscr{A}})$. Next, we show that $-\varepsilon \notin \sigma(\mathscr{A})$. Similarly, plugging $\lambda = -\varepsilon$ into Eq. (2.4) will lead

$$\begin{cases} \varepsilon^2 w(x) - a w''(x) - \alpha \varepsilon \theta'(x) = 0, \\ \varepsilon^2 \theta(x) - \alpha \varepsilon w'(x) - a\theta''(x) - \varepsilon^2 \theta(x) + \varepsilon \alpha w'(x) = 0, \\ w(0) = \theta(0) = \theta(1) = w(1) = 0, \end{cases} \qquad (5.4)$$

which leads

$$\begin{cases} \varepsilon^2 w(x) - a w''(x) - \alpha \varepsilon \theta'(x) = 0, \\ a\theta''(x) = 0, \quad w(0) = \theta(0) = \theta(1) = w(1) = 0. \end{cases}$$

The second equation with boundary conditions yields $\theta \equiv 0$ and hence

$$\varepsilon^2 w(x) - a w''(x) = 0, \quad w(0) = w(1) = 0.$$

A simple computation shows that the above equation admits only zero solution. So does for Eq. (5.4). Therefore, $-\varepsilon \notin \sigma(\mathscr{A})$. The proof is complete. $\quad\square$

Theorem 5.1. *Let $\widetilde{\mathscr{A}}$ be defined by Eqs. (5.1)–(5.2). Then the eigenvalues of $\widetilde{\mathscr{A}}$ have the asymptotic expansions (2.37). And the corresponding eigenfunctions*

$$\begin{cases} \{[u_k(x), \lambda_k u_k(x), \theta_k(x), v_k(x)], k \in \mathbb{Z}\} \\ \{[-u_k(1-x), -\lambda_k u_k(1-x), \theta_k(1-x), v_k(1-x)], k \in \mathbb{Z}\} \end{cases}$$

have the following asymptotic expansions with $j = 1, 2$,

$$\begin{cases} u_k'(x) = r\theta_1 e^{(\theta_1 r\lambda_k + b_1)x} - r\theta_2 e^{(\theta_2 r\lambda_k - b_2)x} + \mathcal{O}(k^{-1}), \\ \lambda_k u_k(x) = e^{(\theta_1 r\lambda_k + b_1)x} - e^{(\theta_2 r\lambda_k - b_2)x} + \mathcal{O}(k^{-1}), \\ \theta_k(x) = e^{(\theta_1 r\lambda_k + b_1)x} - e^{(\theta_2 r\lambda_k - b_2)x} + \mathcal{O}(k^{-1}), \\ v_k(x) = r\theta_1 e^{(\theta_1 r\lambda_k + b_1)x} - r\theta_2 e^{(\theta_2 r\lambda_k - b_2)x} + \mathcal{O}(k^{-1}). \end{cases} \qquad (5.5)$$

Moreover, $\{[u_k, \lambda_k u_k, \theta_k, v_k], k \in \mathbb{Z}\}$ are approximately normalized in $\widetilde{\mathscr{H}}$ in the sense that there exist positive constants $\tilde{c}_1$ and $\tilde{c}_2$, independent of k, such that

$$\tilde{c}_1 \leqslant \|u_k'\|_{L^2}, \|\lambda_k u_k\|_{L^2}, \|\theta_k\|_{L^2}, \|v_k\|_{L^2} \leqslant \tilde{c}_2, \quad \forall k \in \mathbb{Z}. \qquad (5.6)$$

Proof. Since from Proposition 5.1, $\sigma(\mathscr{A}) = \sigma(\widetilde{\mathscr{A}}) + \{-\varepsilon\}$, the eigenvalues of $\widetilde{\mathscr{A}}$ have the same asymptotic expansions (2.37). As for the second part, notice Eq. (3.2) and the relationship between (λ, u, θ, v) and (λ, w, θ) that

$$w = \lambda u, \quad v = \frac{a\theta'}{\lambda + \varepsilon},$$

Eq. (5.5) can then be deduced from Eqs. (3.5)–(3.8) by setting, respectively,

$$u(x) = \frac{w_1(x, \lambda)}{2r^3\lambda^4}, \quad \theta(x) = \frac{w_3(x, \lambda)}{2r^3\lambda^3}, \quad v(x) = \frac{w_4(x, \lambda)}{2r^3(\lambda + \varepsilon)\lambda^3}. \tag{5.7}$$

Finally Eq. (5.6) is a direct consequence of Eqs. (5.5) and (3.10). The proof is complete. $\square$

Theorem 5.2. *Let $\widetilde{\mathscr{A}}$ be defined by Eqs. (5.1)–(5.2). Then*

 (i) *There exists a sequence of generalized eigenfunctions of $\widetilde{\mathscr{A}}$, which forms a Riesz basis for $\mathscr{H}$. Moreover, for any $\lambda \in \sigma(\widetilde{\mathscr{A}})$ with $|\lambda|$ sufficiently large, λ is semi-simple, that is, it has the same algebraic and geometric multiplicities and the multiplicity is two.*

 (ii) *The C_0-semigroup $\mathrm{e}^{\widetilde{\mathscr{A}}t}$ satisfies the spectrum-determined growth condition: $s(\widetilde{\mathscr{A}}) = \omega(\widetilde{\mathscr{A}})$.*

(iii) *$\mathrm{e}^{\widetilde{\mathscr{A}}t}$ is exponentially stable, that is to say, there exist constants $M_0 > 1, \omega_0 > 0$ such that*

$$\|\mathrm{e}^{\widetilde{\mathscr{A}}t}\| \leqslant M_0 \mathrm{e}^{-\omega_0 t}$$

or equivalently

$$E(t) \leqslant M_0 \mathrm{e}^{-\omega_0 t} E(0)$$

where E is the energy function (1.5).

Proof. The proof of (i) can follow exactly the same way as that for Theorem 4.2 by taking

$$\widetilde{\mathscr{A}}_0 \begin{bmatrix} w \\ \phi \\ \theta \\ v \end{bmatrix}^\top = \begin{bmatrix} \phi \\ au'' - \alpha\theta' \\ v' - \alpha\phi' \\ a\theta' \end{bmatrix}^\top \quad \forall \begin{bmatrix} w \\ \phi \\ \theta \\ v \end{bmatrix}^\top \in D(\widetilde{\mathscr{A}}_0) = D(\widetilde{\mathscr{A}}). \tag{5.8}$$

The $\widetilde{\mathscr{A}}_0$ is just the operator $\widetilde{\mathscr{A}}$ with $\varepsilon = 0$. By noting the fact that the relationship between $\widetilde{\mathscr{A}}$ and $\widetilde{\mathscr{A}}_0$ is the same as that $\mathscr{A}$ and $\mathscr{A}_0$, we get through the proof. The details are omitted.

The proofs for (ii) and (iii) are similar to Corollary 4.1 and Theorem 4.3. The details are omitted also. $\square$

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