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On edge-magic labelings of certain disjoint unions of graphs*

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Abstract

A (p, q) graph G is called edge-magic if there exists a bijective function $f : V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ such that $f(u) + f(v) + f(uv)$ is a constant for each edge $uv \in E(G)$. Moreover, G is said to be super edge-magic if $f(V(G)) = \{1, 2, \ldots, p\}$. In this paper, it is shown that a disjoint union of multiple copies of a (super) edge-magic bipartite and tripartite graph is (super) edge-magic if the number of copies is odd. In addition to this result, the edge-magic properties of certain classes of 2-regular graphs and forests are studied, and a bound on the size of triangle-free super edge-magic graphs is provided. Finally, several open problems are stated.

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1 Introduction

The subject of edge-magic labelings of graphs had its origins three decades ago in the work of Kotzig and Rosa [9, 10] on what they called magic valuations of graphs (which are also commonly known as edge-magic total labelings; see [13]). Interest in these labelings has been lately rekindled by a paper on the subject due to Ringel and Lladó [11]. Shortly after this, Enomoto, Lladó, Nakamigawa and Ringel [2] defined a more restrictive form of edge-magic labelings, namely, super edge-magic labelings which Wallis [13] refers to as strong edge-magic total labelings. These are important since they are, in fact, felicitous, harmonious and sequential (if the size is at least as large as the order of the graph or if the graph is a tree), cordial, edge-antimagic and sometimes graceful (see [4] and [5]).

In this paper, the authors intend to make headway on the following problem: if a graph is (super) edge-magic, is the disjoint union of multiple copies of this graph (super) edge-magic as well? To this end, we prove a result that provides an affirmative answer in the case where the graph under consideration is either bipartite or tripartite and an odd number of copies of it are used. This improves a previous result by the authors [4] that applied only to linear forests. In addition to this result, we investigate the edge-magic properties of some particular classes of 2-regular graphs and forests, which leads us to propose a conjecture. We also provide a bound regarding the maximum size of triangle-free super edge-magic graphs. Finally, we state several open problems.

To achieve our goals, we first need some definitions and a couple of elementary results. We refer the reader to [1] or [8] for all other terms and notation not provided in this paper.

For a (p,q) graph G, a bijective function $f : V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ is an *edge-magic labeling* of G if $f(u)+f(v)+f(uv) = k$ is a constant, which is independent on the choice of any edge $uv \in E(G)$. If such a labeling exists, then G is said to be an *edge-magic graph*, and the constant k is called the *valence* of the labeling. Furthermore, f is a *super edge-magic labeling* of G if $f(V(G)) = \{1, 2, \ldots, p\}$. Thus, a *super edge-magic graph* is a graph that admits a super edge-magic labeling.

The following simple result found in [5] provides a characterization of super edge-magic graphs, which is often easier to use than the definition itself. Because of this result, it is only necessary to provide the labels assigned to the vertices of a graph to surmise whether this labeling induces a super edge-magic labeling of a graph or not; however, when wanting to determine if a vertex labeling induces an edge-magic labeling, one needs to be provided with the valence too.

Lemma 1.1 *A (p,q) graph G is super edge-magic if and only if there exists a bijective function $f : V(G) \to \{1, 2, \ldots, p\}$ such that the set $S = \{f(u) + f(v) | uv \in E(G)\}$ consists of q consecutive integers. In such a case, f extends to a super edge-magic labeling of G with valence $k = p + q + s$, where $s = \min(S)$ and*

$$S = \{k - (p+1), k - (p+2), \ldots, k - (p+q)\}.$$

Moreover, $\sum_{v \in V(G)} f(v) \deg v = qs + \binom{q}{2}$.

The next necessary condition due to Ringel and Lladó [11] will prove useful in this paper.

Lemma 1.2 *If G is a (p,q) graph, where q is even, $p+q \equiv 2 \pmod 4$, and every vertex of G has odd degree, then G is not edge-magic.*

2 Main Result

This section contains a tool that allows us to generate infinite classes of disconnected edge-magic and super edge-magic n-partite graphs with relative ease, where $n = 2$ or 3.

Now, recall that as mentioned in the introduction, super edge-magic graphs are often felicitous, harmonious, sequential, cordial or edge-antimagic. This makes the following theorem unexpected since previously no such a technique was available for graphs within those classes (see [6]).

Theorem 2.1 *If G is a (super) edge-magic bipartite or tripartite graph, and m is odd, then mG is (super) edge-magic.*

Proof. Without loss of generality, assume that $m \geq 3$.

Now, if G is a (super) edge-magic (p,q) bipartite or tripartite graph with partite sets U, V and W (let $W = \emptyset$ if G is bipartite), then let $E(G) = UV \cup UW \cup VW$, where the juxtaposition of two partite sets denotes the set of edges between those two sets. Also, take $f : V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ to be an arbitrary (super) edge-magic labeling of G. Then define $H \cong mG$ to be the graph with

$$V(H) = \bigcup_{i=1}^{m} (U_i \cup V_i \cup W_i) \text{ and } E(H) = \bigcup_{i=1}^{m} (U_iV_i \cup U_iW_i \cup V_iW_i),$$

where $x_i \in X_i$ for $1 \leq i \leq m$ if and only if $x \in X$ (X is one of the sets U, V, W, UV, UW or VW).

Now, consider the labeling $g : V(H) \cup E(H) \to \{1, 2, \ldots, m(p+q)\}$ such that

$$g(x_i) = \begin{cases} mf(x) - m + i, & \text{if } x \in W \cup UV \text{ and } 1 \leq i \leq m; \\ mf(x) - 2i + 1, & \text{if } x \in U \cup VW \text{ and } 1 \leq i \leq (m-1)/2; \\ mf(x) + m - 2i + 1, & \text{if } x \in U \cup VW \text{ and } (m+1)/2 \leq i \leq m; \\ mf(x) - (m-1)/2 + i, & \text{if } x \in V \cup UW \text{ and } 1 \leq i \leq (m-1)/2; \\ mf(x) - (3m-1)/2 + i, & \text{if } x \in V \cup UW \text{ and } (m+1)/2 \leq i \leq m. \end{cases}$$

Then g is a (super) edge-magic labeling of H. To verify this, notice first that $g(x) + g(y) + g(xy) = mk - 3(m-1)/2$ for every $xy \in E(H)$, where k is the valence of f. Next, to see that

$$g(V(H) \cup E(H)) = \{1, 2, \ldots, m(p+q)\},$$

notice that for every $x \in V(G) \cup E(G)$, we have that

$$\bigcup_{i=1}^{m} \{g(x_i)\} = \bigcup_{i=1}^{m} \{mf(x) - m + i\};$$

thus, the set

$$f(V(G) \cup E(G)) = \{1, 2, \ldots, p+q\}$$

is spread by the function g to the entirety of its range. □

The preceding result is the best possible in the sense that m cannot be even for Kotzig and Rosa [9] have shown that the forest mP_2 is edge-magic if and only if m is odd.

3 Results on 2-Regular Graphs

Our main result makes it worthwhile to search for bipartite and tripartite graphs which are edge-magic or super edge-magic. In this section, we thus concentrate on the edge-magic properties of some classes of 2-regular graphs, which certainly satisfy the hypothesis of our result.

The next corollary is an example of the kind of result that follows immediately from Theorem 2.1.

Corollary 3.1 *If m is odd and $n > 1$, then the 2-regular graph mC_{2n} is edge-magic.*

Proof. In [9], Kotzig and Rosa have shown that all cycles are edge-magic (an alternative labeling of even cycles can be found in [7]). □

The authors have shown in [4] that the 2-regular graph mC_n is super edge-magic if and only if m and n are odd. Therefore, mC_n is edge-magic if m is odd. For the case in which m is even, we only know that mC_n is edge-magic if $m \equiv 2 \pmod 4$ and $n = 4$ or 6, or $n \equiv 1, 5$ or $7 \pmod{12}$. This follows from Theorem 2.1, Table 1 and the next result.

n	labeling	valence
4	$1-14-9-13-1;\ 4-6-12-5-4$	25
6	$2-8-4-11-5-9-2;\ 6-12-21-10-7-13-6$	34

Table 1: Edge-magic labelings of $2C_n$ for some small n

Theorem 3.2 *For every positive integer $n \equiv 1, 5$ or $7 \pmod{12}$, the 2-regular graph $G \cong 2C_n$ is edge-magic.*

Proof. Assume that $n \equiv 1, 5$ or $7 \pmod{12}$, and let $G \cong 2C_n$ be the 2-regular graph with

$$V(G) = \{u_i | 1 \leq i \leq n\} \cup \{v_i | 1 \leq i \leq n\}$$

and

$$E(G) = \{u_1 u_n, v_1 v_n\} \cup \{u_i u_{i+1} | 1 \leq i \leq n-1\} \cup \{v_i v_{i+1} | 1 \leq i \leq n-1\}.$$

Then there are three cases to consider.

Case 1: Let $n = 12k - 7$, where k is a positive integer, and define $f : V(G) \to \{1, 2, \ldots, 48k - 28\}$ to be the vertex labeling such that

$$f(u_j) = \begin{cases} 24k - 3i - 10, & \text{if } j = 2i - 1 \text{ and } 1 \le i \le 3k - 1; \\ 6k + 3i - 5, & \text{if } j = 2i - 1 \text{ and } 3k \le i \le 6k - 3; \\ 12k - 3i - 5, & \text{if } j = 2i \text{ and } 1 \le i \le 3k - 2; \\ 3i - 6k + 3, & \text{if } j = 2i \text{ and } 3k - 1 \le i \le 6k - 4; \end{cases}$$

$$f(v_j) = \begin{cases} 12k - 3i - 4, & \text{if } j = 2i - 1 \text{ and } 1 \le i \le 3k - 2; \\ 24k - 3i - 12, & \text{if } j = 2i \text{ and } 1 \le i \le 3k - 2; \\ 3k - 3i + 1, & \text{if } j = 6k + 6i - 9 \text{ and } 1 \le i \le k; \\ 15k - 3i - 5, & \text{if } j = 6k + 6i - 8 \text{ and } 1 \le i \le k; \\ 3k - 3i + 2, & \text{if } j = 6k + 6i - 7 \text{ and } 1 \le i \le k; \\ 15k - 3i - 7, & \text{if } j = 6k + 6i - 6 \text{ and } 1 \le i \le k - 1; \\ 3k - 3i, & \text{if } j = 6k + 6i - 5 \text{ and } 1 \le i \le k - 1; \\ 15k - 3i - 6, & \text{if } j = 6k + 6i - 4 \text{ and } 1 \le i \le k - 1. \end{cases}$$

Case 2: Let $n = 12k - 5$, where k is a positive integer, and define $f : V(G) \to \{1, 2, \ldots, 48k - 20\}$ to be the vertex labeling such that

$$f(u_j) = \begin{cases} 24k - 3i - 6, & \text{if } j = 2i - 1 \text{ and } 1 \le i \le 3k - 1; \\ 6k + 3i - 4, & \text{if } j = 2i - 1 \text{ and } 3k \le i \le 6k - 2; \\ 12k - 3i - 3, & \text{if } j = 2i \text{ and } 1 \le i \le 3k - 1; \\ 3i - 6k + 2, & \text{if } j = 2i \text{ and } 3k \le i \le 6k - 3; \end{cases}$$

$$f(v_j) = \begin{cases} 12k - 3i - 2, & \text{if } j = 2i - 1 \text{ and } 1 \le i \le 3k - 1; \\ 24k - 3i - 8, & \text{if } j = 2i \text{ and } 1 \le i \le 3k - 2; \\ 15k - 3i - 3, & \text{if } j = 6k + 6i - 8 \text{ and } 1 \le i \le k; \\ 3k - 3i + 2, & \text{if } j = 6k + 6i - 7 \text{ and } 1 \le i \le k - 1; \\ 15k - 3i - 2, & \text{if } j = 6k + 6i - 6 \text{ and } 1 \le i \le k; \\ 3k - 3i, & \text{if } j = 6k + 6i - 5 \text{ and } 1 \le i \le k - 1; \\ 15k - 3i - 4, & \text{if } j = 6k + 6i - 4 \text{ and } 1 \le i \le k - 1; \\ 3k - 3i + 1, & \text{if } j = 6k + 6i - 3 \text{ and } 1 \le i \le k - 1; \\ i, & \text{if } j = 12k + 2i - 9 \text{ and } 1 \le i \le 2. \end{cases}$$

Case 3: Let $n = 12k + 1$, where k is a positive integer, and define $f : V(G) \to \{1, 2, \ldots, 48k + 4\}$ to be the vertex labeling such that

$$f(u_j) = \begin{cases} 24k - 3i + 6, & \text{if } j = 2i - 1 \text{ and } 1 \le i \le 3k + 1; \\ 6k + 3i - 1, & \text{if } j = 2i - 1 \text{ and } 3k + 2 \le i \le 6k + 1; \\ 12k - 3i + 3, & \text{if } j = 2i \text{ and } 1 \le i \le 3k; \\ 3i - 6k - 1, & \text{if } j = 2i \text{ and } 3k + 1 \le i \le 6k; \end{cases}$$

$$f(v_j)=\begin{cases} 12k-3i+4, & \text{if } j=2i-1 \text{ and } 1\le i\le 3k;\\ 24k-3i+4, & \text{if } j=2i \text{ and } 1\le i\le 3k;\\ 3k-3i+3, & \text{if } j=6k+6i-5 \text{ and } 1\le i\le k;\\ 15k-3i+5, & \text{if } j=6k+6i-4 \text{ and } 1\le i\le k;\\ 3k-3i+4, & \text{if } j=6k+6i-3 \text{ and } 1\le i\le k;\\ 15k-3i+3, & \text{if } j=6k+6i-2 \text{ and } 1\le i\le k;\\ 3k-3i+2, & \text{if } j=6k+6i-1 \text{ and } 1\le i\le k-1;\\ 15k-3i+4, & \text{if } j=6k+6i \text{ and } 1\le i\le k;\\ i, & \text{if } j=12k+2i-3 \text{ and } 1\le i\le 2. \end{cases}$$

Therefore, f extends to an edge-magic labeling of G with valence $5n+2$. □

In addition to the above theorem, we quote a result obtained by the authors [3] on 2-regular graphs in which (a,b) and $[a,b]$ denote the greatest common divisor and the least common multiple for the integers a and b, respectively, whereas $k(G)$ denotes the number of components of a graph G.

Theorem 3.3 *Assume that* $G\cong\bigcup_{i=1}^{k(G)} C_{n_i}$ *is a (super) edge-magic 2-regular graph, and let* m *be odd. Then* $\bigcup_{i=1}^{k(G)}(m,n_i)C_{[m,n_i]}$ *is (super) edge-magic.*

The above knowledge about 2-regular graphs provide some progress towards answering the question posed by Kotzig and Rosa [9] pertaining to the necessary and sufficient conditions required so that a 2-regular graph is edge-magic.

4 Results on Forests

This section is devoted to the study of the edge-magic and super edge-magic properties of certain classes of forests, which complements the result in the main section nicely, since these graphs are bipartite and hence can serve as the seed for creating other infinite classes of (super) edge-magic bipartite graphs. They are also interesting since most of the forests referred to in this section have each two components and thus show that bipartite graphs with an even number of components may be edge-magic or super edge-magic.

The following theorem strengthens a previous result by the authors [4], namely, that the forest $K_{1,n}\cup K_{1,n+1}$ is super edge-magic for every positive integer n.

Theorem 4.1 *If* m *is a multiple of* $n+1$*, then the forest* $F\cong K_{1,m}\cup K_{1,n}$ *is super edge-magic.*

Proof. Let

$$V(F)=\{x,y\}\cup\{u_i|1\le i\le m\}\cup\{v_i|1\le i\le n\}$$

and

$$E(F)=\{xu_i|1\le i\le m\}\cup\{yv_i|1\le i\le n\}.$$

Then consider the vertex labeling $f:V(F)\to\{1,2,\dots,m+n+2\}$ such that $f(x)=\alpha+2$, $f(y)=1$, $f(u_i)=\alpha+2i-1$ for $i=1,2,\dots,m$, and $f(v_i)=(\alpha+1)(i+1)+1$

for $i = 1, 2, \ldots, n$, where $\alpha = m/(n+1)$. Therefore, by Lemma 1.1, f extends to a super edge-magic labeling of F with valence $\alpha + 2(m+n+3)$. □

Notice, by Lemma 1.2 and Theorem 4.1, that the forest $K_{1,1} \cup K_{1,n}$ is super edge-magic if and only if n is even. This leads us to the following two results.

Theorem 4.2 *The forest $F \cong K_{1,2} \cup K_{1,n}$ is super edge-magic if and only if n is a multiple of 3. Furthermore, there are essentially only two super edge-magic labelings of F.*

Proof. Let

$$V(F) = \{u\} \cup \{v_i | 1 \le i \le n\} \cup \{w_1, w_2, w_3\}$$

and

$$E(F) = \{uv_i | 1 \le i \le n\} \cup \{w_1w_2, w_1w_3\},$$

and let $f : V(F) \to \{1, 2, \ldots, n+4\}$ be an arbitrary super edge-magic labeling of F such that $f(u) = \alpha$ and $\{f(w_1), f(w_2), f(w_3)\} = \{i, j, k\}$. Notice then that α, i, j and k are different. Now, without loss of generality, assume that $i < j < k$. Let $S = \{f(x) + f(y) | xy \in E(F)\}$ and $L = \{\alpha+1, \alpha+2, \ldots, \alpha+n+4\}$, which are two sets of consecutive integers with $|S| = n+2$ and $|L| = n+4$. Observe then that

$$S - \{f(w_1) + f(w_2), f(w_1) + f(w_3)\} = L - \{2\alpha, \alpha+i, \alpha+j, \alpha+k\}.$$

Thus, $\{\alpha+1, \alpha+n+4\} \subset \{2\alpha, \alpha+i, \alpha+k\}$ since by removing 2α, $\alpha+i$, $\alpha+j$ and $\alpha+k$ from L, we obtain $S - \{f(w_1)+f(w_2), f(w_1)+f(w_3)\}$, which is a set of consecutive integers minus two elements and $i < j < k$. This implies that $\{1, n+4\} \subset \{\alpha, i, k\}$.

We now show that $i = 1$ and $k = n+4$. To do this, it suffices to verify that $\alpha \notin \{1, n+4\}$. Let $\beta = f(w_1)$, then since $\deg w_1 = 2$, $\deg u = n$ and $f(u) = \alpha$, it follows by Lemma 1.1 that

$$\sum_{t=1}^{n+4} t + \alpha(n-1) + \beta = (n+2)s + \binom{n+2}{2},$$

where $s = \min(S)$. Hence,

$$s = \frac{3(n+3) + \alpha(n-1) + \beta}{n+2}.$$

Now, assume, to the contrary, that $\alpha = 1$. Then $s = \beta/(n+2) + 4$, so $n+2$ divides β, which implies that $\beta = n+2$. This in turn leads us to conclude that $s = 5$. Furthermore, the vertex u which is labeled 1 cannot be adjacent to the vertices labeled 2 or 3; for otherwise $s = 3$ or 4. Therefore, $\{2, 3, n+2\} = \{i, j, k\}$, which is impossible.

Next, suppose, to the contrary, that $\alpha = n+4$. Then $s = (\beta-3)/(n+2) + n + 4$ and, consequently, $n+2$ divides $\beta - 3$, which implies that $\beta - 3 = 0$ since $n+2 \ge 3$ and $1 \le \beta \le n+3$. Thus, $\beta = 3$ and $s = n+4$. Therefore, either $f(w_2) = 1$ or

$f(w_3) = 1$, implying that $f(w_1)+f(w_2) = 4$ or $f(w_1)+f(w_3) = 4$ and $4 < s = n+4$, which is a contradiction.

Finally, since the vertices w_2 and w_3 are indistinguishable, the following three cases remain.

Case 1: If $f(w_1) = 1$, $f(w_2) = n+4$ and $f(w_3) = j$, then $\{1+j, n+5\} = \{\alpha+j, 2\alpha\}$. Thus, $1 + j = 2\alpha$ and $n + 5 = \alpha + j$, implying that $\alpha = n/3 + 2$; hence, n is a multiple of 3. Therefore, by taking $f(u) = n/3 + 2$, $f(w_1) = 1$, $f(w_2) = n + 4$ and $f(w_3) = 2n/3 + 3$, we get the same super edge-magic labeling of F as in the proof of Theorem 4.1.

Case 2: If $f(w_1) = n + 4$, $f(w_2) = 1$ and $f(w_3) = j$, then $\{n + 5, j + n + 4\} = \{\alpha + j, 2\alpha\}$. Thus, $n + 5 = \alpha + j$ and $j + n + 4 = 2\alpha$, implying that $\alpha = 2n/3 + 3$; hence, n is a multiple of 3. Now, it is easy to verify that if we take $f(u) = 2n/3+3$, $f(w_1) = n+4$, $f(w_2) = 1$ and $f(w_3) = n/3+2$, then we attain a super edge-magic labeling of F by assigning the remaining labels to all other vertices of F.

Case 3: If $f(w_1) = j$, $f(w_2) = 1$ and $f(w_3) = n + 4$, then $\{1 + j, j + n + 4\} = \{\alpha + j, 2\alpha\}$. Now, since $\alpha > 1$, it follows that $1 + j \neq \alpha + j$. Thus, $1+j = 2\alpha$ and $j+n+4 = \alpha+j$; hence, $j = 2n+7 > n+4$, which is not possible.

The labelings provided in Cases 1 and 2 are unique (up to isomorphism), and therefore the proof is complete. □

The approach used in the previous proof can also be applied to establish the following theorem which we state without proof.

Theorem 4.3 *The forest $F \cong K_{1,3} \cup K_{1,n}$ is super edge-magic if and only if n is a multiple of 4.*

Now, in light of the two previous theorems and the remark that precedes them the authors conjectured, before submitting this paper, that the converse of Theorem 4.1 also held. The referee has since then informed us that Wimmer [14] has proved this to be the case. We state his result here for the sake of completeness.

Theorem 4.4 *The forest $K_{1,m} \cup K_{1,n}$ is super edge-magic if and only if m is a multiple of $n + 1$.*

The next characterization is the edge-magic analogue to the previous theorem.

Theorem 4.5 *For all positive integers m and n the forest $F \cong K_{1,m} \cup K_{1,n}$ is edge-magic if and only if either m or n is even.*

Proof. If F is edge-magic, then either m or n is even by Lemma 1.2.

For the converse, without loss of generality, assume that n is even, and let

$$V(F) = \{x, y\} \cup \{u_i | 1 \leq i \leq m\} \cup \{v_i | 1 \leq i \leq n\}$$

and

$$E(F) = \{xu_i | 1 \leq i \leq m\} \cup \{yv_i | 1 \leq i \leq n\}.$$

Then consider the vertex labeling $f : V(F) \to \{1, 2, \ldots, m+n+2\}$ such that

$$f(w) = \begin{cases} 3n/2 + 2m + 2, & \text{if } w = x; \\ n/2 + m + 1, & \text{if } w = y; \\ i, & \text{if } w = u_i \text{ and } 1 \leq i \leq m; \\ m + i, & \text{if } w = v_i \text{ and } 1 \leq i \leq n/2; \\ m + i + 1, & \text{if } w = v_i \text{ and } n/2 + 1 \leq i \leq n. \end{cases}$$

Therefore, f extends to an edge-magic labeling of F with valence $5n/2+4m+4$. □

In [4], the authors proved that the forest $mK_{1,n}$ is super edge-magic if m is odd. Further, in light of the previous theorem, the forest $2K_{1,n}$ is edge-magic if and only if n is even, which together with Theorem 2.1 leads us to conclude that whenever $m \equiv 2 \pmod 4$, the forest $mK_{1,n}$ is edge-magic if and only if n is even. Thus, the only instance that needs to be settled is when m is a multiple of 4. For this, we have found that the linear forest $4K_{1,2} \cong 4P_3$ is super edge-magic with valence 30 by simply labeling the four disjoint copies of P_3 as follows: $1-9-2, 4-8-5, 6-10-7$ and $11-3-12$. On the other hand, Kotzig and Rosa [9] determined that the forest $mK_{1,1} \cong mP_2$ is edge-magic if and only if m is odd (the authors recently showed in [4] that this result can be extended to state that the forest mP_2 is super edge-magic if and only if m is odd).

We studied above the forests $K_{1,1} \cup K_{1,n}$ and $K_{1,2} \cup K_{1,n}$ as members of the class of forests $K_{1,m} \cup K_{1,n}$. However, they are also in the class of forests $P_m \cup K_{1,n}$. Therefore, the next theorem fits well into our theme. This generalizes the result found in [4] that the forest $P_2 \cup P_m$ is super edge-magic for every integer $m \geq 3$.

Theorem 4.6 *For every two integers $m \geq 4$ and $n \geq 1$, the forest $F \cong P_m \cup K_{1,n}$ is super edge-magic.*

Proof. Let

$$V(F) = \{u_i | 1 \leq i \leq m\} \cup \{v_i | 1 \leq i \leq n\} \cup \{w\}$$

and

$$E(F) = \{u_i u_{i+1} | 1 \leq i \leq m-1\} \cup \{v_i w | 1 \leq i \leq n\}.$$

Then consider four cases for the vertex labeling $f : V(F) \to \{1, 2, \ldots, m+n+1\}$.

Case 1: For $m \equiv 0 \pmod 4$, let

$$f(u_j) = \begin{cases} (m+2n+2)/2, & \text{if } j = 1; \\ (m+2n+6)/2, & \text{if } j = 3; \\ n + 2i - 1, & \text{if } j = 4i \text{ and } 1 \leq i \leq m/4; \\ (m+2n+4i+6)/2, & \text{if } j = 4i+1 \text{ and } 1 \leq i \leq (m-4)/4; \\ n + 2i + 2, & \text{if } j = 4i+2 \text{ and } 0 \leq i \leq (m-4)/4; \\ (m+2n+4i+4)/2, & \text{if } j = 4i+3 \text{ and } 1 \leq i \leq (m-4)/4; \end{cases}$$

$f(v_i) = i$, if $1 \leq i \leq n$; and $f(w) = (m+2n+4)/2$.

Case 2: For $m \equiv 1 \pmod 4$, let

$$f(u_j) = \begin{cases} n+2i-1, & \text{if } j=4i \text{ and } 1 \le i \le (m-1)/4; \\ (m+2n+4i+1)/2, & \text{if } j=4i+1 \text{ and } 0 \le i \le (m-1)/4; \\ n+2i+2, & \text{if } j=4i+2 \text{ and } 0 \le i \le (m-5)/4; \\ (m+2n+4i+7)/2, & \text{if } j=4i+3 \text{ and } 0 \le i \le (m-5)/4; \end{cases}$$

$f(v_i) = i$, if $1 \le i \le n$; and $f(w) = (m+2n+3)/2$.

Case 3: For $m \equiv 2 \pmod 4$, let

$$f(u_j) = \begin{cases} m+n+1, & \text{if } j=1; \\ m+n-1, & \text{if } j=3; \\ (m+2n-4i+2)/2, & \text{if } j=4i \text{ and } 1 \le i \le (m-2)/4; \\ m+n-2i-1, & \text{if } j=4i+1 \text{ and } 1 \le i \le (m-2)/4; \\ (m+2n-4i-4)/2, & \text{if } j=4i+2 \text{ and } 0 \le i \le (m-6)/4; \\ m+n-2i, & \text{if } j=4i+3 \text{ and } 1 \le i \le (m-6)/4; \\ m+n, & \text{if } j=m; \end{cases}$$

$f(v_i) = i$, if $1 \le i \le n$; and $f(w) = (m+2n+2)/2$.

Case 4: For $m \equiv 3 \pmod 4$, let

$$f(u_j) = \begin{cases} (m+2n+1)/2, & \text{if } j=1; \\ (m+2n+5)/2, & \text{if } j=3; \\ n+2i-1, & \text{if } j=4i \text{ and } 1 \le i \le (m-3)/4; \\ (m+2n+4i+5)/2, & \text{if } j=4i+1 \text{ and } 1 \le i \le (m-3)/4; \\ n+2i+2, & \text{if } j=4i+2 \text{ and } 0 \le i \le (m-7)/4; \\ (m+2n+4i+3)/2, & \text{if } j=4i+3 \text{ and } 1 \le i \le (m-3)/4; \\ (m+2n-1)/2, & \text{if } j=m-1; \end{cases}$$

$f(v_i) = i$, if $1 \le i \le n$; and $f(w) = (m+2n+3)/2$.

Therefore, by Lemma 1.1, f extends to a super edge-magic labeling of F with valence

$$k = \begin{cases} 5m/2+3n+2, & \text{if } m \equiv 2 \pmod 4; \\ \lfloor m/2 \rfloor + 2m + 3n + 3, & \text{otherwise.} \end{cases}$$

□

The next class of forest that we study is $2P_n$. To do this, notice that we have shown that the forest $K_{1,2} \cup K_{1,n}$ is super edge-magic if and only if n is a multiple of 3; hence, the forest $2P_3$ is not super edge-magic. However, it is edge-magic by labeling the vertices of one P_3 with $1-9-2$ and the ones of the other P_3 with $3-4-5$, and letting the valence be 17. Finally, notice that Kotzig and Rosa [9] proved that the forest nP_2 is edge-magic if and only if n is odd.

Theorem 4.7 *The forest $F \cong 2P_n$ $(n>1)$ is super edge-magic if and only if $n \ne 2$ or 3.*

Proof. Assume that $n \geq 4$, and define the forest $F \cong 2P_n$ with

$$V(F) = \{u_i | 1 \leq i \leq n\} \cup \{v_i | 1 \leq i \leq n\}$$

and

$$E(F) = \{u_i u_{i+1} | 1 \leq i \leq n-1\} \cup \{v_i v_{i+1} | 1 \leq i \leq n-1\}.$$

We proceed by cases according to the possible values of the integer n.

Case 1: For $n = 9$, let $f : V(F) \to \{1, 2, \ldots, 18\}$ be the vertex labeling such that

$$(f(u_i))_{i=1}^{9} = (10, 17, 7, 14, 4, 13, 6, 16, 9)$$

and

$$(f(v_i))_{i=1}^{9} = (8, 18, 5, 15, 1, 11, 2, 12, 3).$$

Case 2: For $n = 4k$, where k is a positive integer, let $f : V(F) \to \{1, 2, \ldots, 8k\}$ be the vertex labeling such that

$$f(u_j) = \begin{cases} 1, & \text{if } j = 1; \\ 2k + i - 1, & \text{if } j = 2i - 1 \text{ and } 2 \leq i \leq k; \\ 2k - i + 2, & \text{if } j = 2i - 1 \text{ and } k + 1 \leq i \leq 2k; \\ 6k + i, & \text{if } j = 2i \text{ and } 1 \leq i \leq k; \\ 6k - i + 1, & \text{if } j = 2i \text{ and } k + 1 \leq i \leq 2k; \end{cases}$$

$$f(v_j) = \begin{cases} 3k + i - 1, & \text{if } j = 2i - 1 \text{ and } 1 \leq i \leq k + 1; \\ 3k - i + 2, & \text{if } j = 2i - 1 \text{ and } k + 2 \leq i \leq 2k; \\ 7k + i, & \text{if } j = 2i \text{ and } 1 \leq i \leq k; \\ 7k - i + 1, & \text{if } j = 2i \text{ and } k + 1 \leq i \leq 2k. \end{cases}$$

Case 3: For $n = 12k - 7$, where k is a positive integer, let $f : V(F) \to \{1, 2, \ldots, 24k - 14\}$ be the vertex labeling such that

$$f(u_j) = \begin{cases} 12k - 3i - 4, & \text{if } j = 2i - 1 \text{ and } 1 \leq i \leq 3k - 1; \\ 3i - 6k + 1, & \text{if } j = 2i - 1 \text{ and } 3k \leq i \leq 6k - 3; \\ 24k - 3i - 12, & \text{if } j = 2i \text{ and } 1 \leq i \leq 3k - 2; \\ 6k + 3i - 4, & \text{if } j = 2i \text{ and } 3k - 1 \leq i \leq 6k - 4; \end{cases}$$

$$f(v_j) = \begin{cases} 24k - 3i - 11, & \text{if } j = 2i - 1 \text{ and } 1 \leq i \leq 3k - 2; \\ 12k - 3i - 6, & \text{if } j = 2i \text{ and } 1 \leq i \leq 3k - 2; \\ 15k - 3i - 6, & \text{if } j = 6k + 6i - 9 \text{ and } 1 \leq i \leq k; \\ 3k - 3i + 1, & \text{if } j = 6k + 6i - 8 \text{ and } 1 \leq i \leq k; \\ 15k - 3i - 5, & \text{if } j = 6k + 6i - 7 \text{ and } 1 \leq i \leq k; \\ 3k - 3i - 1, & \text{if } j = 6k + 6i - 6 \text{ and } 1 \leq i \leq k - 1; \\ 15k - 3i - 7, & \text{if } j = 6k + 6i - 5 \text{ and } 1 \leq i \leq k - 1; \\ 3k - 3i, & \text{if } j = 6k + 6i - 4 \text{ and } 1 \leq i \leq k - 1. \end{cases}$$

Case 4: For $n = 12k - 6$, where k is a positive integer, let $f : V(F) \to \{1, 2, \ldots, 24k - 12\}$ be the vertex labeling such that

$$f(u_j) = \begin{cases} 12k - 3i - 3, & \text{if } j = 2i - 1 \text{ and } 1 \le i \le 3k - 1; \\ 3i - 6k + 2, & \text{if } j = 2i - 1 \text{ and } 3k \le i \le 6k - 3; \\ 24k - 3i - 11, & \text{if } j = 2i \text{ and } 1 \le i \le 3k - 2; \\ 6k + 3i - 3, & \text{if } j = 2i \text{ and } 3k - 1 \le i \le 6k - 3; \end{cases}$$

$$f(v_j) = \begin{cases} 24k - 3i - 10, & \text{if } j = 2i - 1 \text{ and } 1 \le i \le 3k - 2; \\ 12k - 3i - 5, & \text{if } j = 2i \text{ and } 1 \le i \le 3k - 2; \\ 15k - 3i - 5, & \text{if } j = 6k + 6i - 9 \text{ and } 1 \le i \le k; \\ 3k - 3i + 2, & \text{if } j = 6k + 6i - 8 \text{ and } 1 \le i \le k - 1; \\ 15k - 3i - 4, & \text{if } j = 6k + 6i - 7 \text{ and } 1 \le i \le k; \\ 3k - 3i, & \text{if } j = 6k + 6i - 6 \text{ and } 1 \le i \le k - 1; \\ 15k - 3i - 6, & \text{if } j = 6k + 6i - 5 \text{ and } 1 \le i \le k - 1; \\ 3k - 3i + 1, & \text{if } j = 6k + 6i - 4 \text{ and } 1 \le i \le k - 1; \\ i, & \text{if } j = 12k + 2i - 10 \text{ and } 1 \le i \le 2. \end{cases}$$

Case 5: For $n = 12k - 5$, where k is a positive integer, let $f : V(F) \to \{1, 2, \ldots, 24k - 10\}$ be the vertex labeling such that

$$f(u_j) = \begin{cases} 24k - 3i - 7, & \text{if } j = 2i - 1 \text{ and } 1 \le i \le 3k - 1; \\ 6k + 3i - 5, & \text{if } j = 2i - 1 \text{ and } 3k \le i \le 6k - 2; \\ 12k - 3i - 3, & \text{if } j = 2i \text{ and } 1 \le i \le 3k - 1; \\ 3i - 6k + 2, & \text{if } j = 2i \text{ and } 3k \le i \le 6k - 3; \end{cases}$$

$$f(v_j) = \begin{cases} 12k - 3i - 2, & \text{if } j = 2i - 1 \text{ and } 1 \le i \le 3k - 1; \\ 24k - 3i - 9, & \text{if } j = 2i \text{ and } 1 \le i \le 3k - 2; \\ 15k - 3i - 4, & \text{if } j = 6k + 6i - 8 \text{ and } 1 \le i \le k; \\ 3k - 3i + 2, & \text{if } j = 6k + 6i - 7 \text{ and } 1 \le i \le k - 1; \\ 15k - 3i - 3, & \text{if } j = 6k + 6i - 6 \text{ and } 1 \le i \le k; \\ 3k - 3i, & \text{if } j = 6k + 6i - 5 \text{ and } 1 \le i \le k - 1; \\ 15k - 3i - 5, & \text{if } j = 6k + 6i - 4 \text{ and } 1 \le i \le k - 1; \\ 3k - 3i + 1, & \text{if } j = 6k + 6i - 3 \text{ and } 1 \le i \le k - 1; \\ i, & \text{if } j = 12k + 2i - 9 \text{ and } 1 \le i \le 2. \end{cases}$$

Case 6: For $n = 12k - 2$, where k is a positive integer, let $f : V(F) \to \{1, 2, \ldots, 24k - 4\}$ be the vertex labeling such that

$$f(u_j) = \begin{cases} 12k - 3i + 1, & \text{if } j = 2i - 1 \text{ and } 1 \le i \le 3k; \\ 3i - 6k - 1, & \text{if } j = 2i - 1 \text{ and } 3k + 1 \le i \le 6k - 1; \\ 24k - 3i - 2, & \text{if } j = 2i \text{ and } 1 \le i \le 3k - 1; \\ 6k + 3i - 1, & \text{if } j = 2i \text{ and } 3k \le i \le 6k - 1; \end{cases}$$

$$f(v_j) = \begin{cases} 12k-3i, & \text{if } j = 2i-1 \text{ and } 1 \le i \le 3k-1; \\ 24k-3i-3, & \text{if } j = 2i \text{ and } 1 \le i \le 3k-1; \\ 3k-3i+2, & \text{if } j = 6k+6i-7 \text{ and } 1 \le i \le k; \\ 15k-3i+1, & \text{if } j = 6k+6i-6 \text{ and } 1 \le i \le k; \\ 3k-3i+3, & \text{if } j = 6k+6i-5 \text{ and } 1 \le i \le k; \\ 15k-3i-1, & \text{if } j = 6k+6i-4 \text{ and } 1 \le i \le k; \\ 3k-3i+1, & \text{if } j = 6k+6i-3 \text{ and } 1 \le i \le k; \\ 15k-3i, & \text{if } j = 6k+6i-2 \text{ and } 1 \le i \le k. \end{cases}$$

Case 7: For $n = 12k-1$, where k is a positive integer, let $f : V(F) \to \{1, 2, \ldots, 24k-2\}$ be the vertex labeling such that

$$f(u_j) = \begin{cases} 12k-3i+2, & \text{if } j = 2i-1 \text{ and } 1 \le i \le 3k; \\ 3i-6k-2, & \text{if } j = 2i-1 \text{ and } 3k+1 \le i \le 6k; \\ 24k-3i, & \text{if } j = 2i \text{ and } 1 \le i \le 3k; \\ 6k+3i-1, & \text{if } j = 2i \text{ and } 3k+1 \le i \le 6k-1; \end{cases}$$

$$f(v_j) = \begin{cases} 24k-3i+1, & \text{if } j = 2i-1 \text{ and } 1 \le i \le 3k; \\ 12k-3i, & \text{if } j = 2i \text{ and } 1 \le i \le 3k-1; \\ 3k-1, & \text{if } j = 6k; \\ 15k-3i+2, & \text{if } j = 6k+6i-5 \text{ and } 1 \le i \le k; \\ 3k-3i+3, & \text{if } j = 6k+6i-4 \text{ and } 1 \le i \le k; \\ 15k-3i, & \text{if } j = 6k+6i-3 \text{ and } 1 \le i \le k; \\ 3k-3i+1, & \text{if } j = 6k+6i-2 \text{ and } 1 \le i \le k; \\ 15k-3i+1, & \text{if } j = 6k+6i-1 \text{ and } 1 \le i \le k; \\ 3k-3i-1, & \text{if } j = 6k+6i \text{ and } 1 \le i \le k-1. \end{cases}$$

Case 8: For $n = 12k+1$, where k is a positive integer, let $f : V(F) \to \{1, 2, \ldots, 24k+2\}$ be the vertex labeling such that

$$f(u_j) = \begin{cases} 24k-3i+5, & \text{if } j = 2i-1 \text{ and } 1 \le i \le 3k+1; \\ 6k+3i-2, & \text{if } j = 2i-1 \text{ and } 3k+2 \le i \le 6k+1; \\ 12k-3i+3, & \text{if } j = 2i \text{ and } 1 \le i \le 3k; \\ 3i-6k-1, & \text{if } j = 2i \text{ and } 3k+1 \le i \le 6k; \end{cases}$$

$$f(v_j) = \begin{cases} 12k-3i+4, & \text{if } j = 2i-1 \text{ and } 1 \le i \le 3k; \\ 24k-3i+3, & \text{if } j = 2i \text{ and } 1 \le i \le 3k; \\ 3k-3i+3, & \text{if } j = 6k+6i-5 \text{ and } 1 \le i \le k; \\ 15k-3i+4, & \text{if } j = 6k+6i-4 \text{ and } 1 \le i \le k; \\ 3k-3i+4, & \text{if } j = 6k+6i-3 \text{ and } 1 \le i \le k; \\ 15k-3i+2, & \text{if } j = 6k+6i-2 \text{ and } 1 \le i \le k; \\ 3k-3i+2, & \text{if } j = 6k+6i-1 \text{ and } 1 \le i \le k-1; \\ 15k-3i+3, & \text{if } j = 6k+6i \text{ and } 1 \le i \le k; \\ i, & \text{if } j = 12k+2i-3 \text{ and } 1 \le i \le 2. \end{cases}$$

Case 9: For $n = 12k + 2$, where k is a positive integer, let $f : V(F) \to \{1, 2, \ldots, 24k + 4\}$ be the vertex labeling such that

$$f(u_j) = \begin{cases} 12k - 3i + 5, & \text{if } j = 2i - 1 \text{ and } 1 \le i \le 3k + 1; \\ 3i - 6k - 3, & \text{if } j = 2i - 1 \text{ and } 3k + 2 \le i \le 6k + 1; \\ 24k - 3i + 6, & \text{if } j = 2i \text{ and } 1 \le i \le 3k; \\ 6k + 3i + 1, & \text{if } j = 2i \text{ and } 3k + 1 \le i \le 6k + 1; \end{cases}$$

$$f(v_j) = \begin{cases} 12k - 3i + 4, & \text{if } j = 2i - 1 \text{ and } 1 \le i \le 3k; \\ 24k - 3i + 5, & \text{if } j = 2i \text{ and } 1 \le i \le 3k + 1; \\ 3k + i - 1, & \text{if } j = 6k + 2i - 1 \text{ and } 1 \le i \le 2; \\ 15k - 3i + 6, & \text{if } j = 6k + 6i - 2 \text{ and } 1 \le i \le k; \\ 3k - 3i + 1, & \text{if } j = 6k + 6i - 1 \text{ and } 1 \le i \le k; \\ 15k - 3i + 4, & \text{if } j = 6k + 6i \text{ and } 1 \le i \le k - 1; \\ 3k - 3i + 2, & \text{if } j = 6k + 6i + 1 \text{ and } 1 \le i \le k; \\ 15k - 3i + 2, & \text{if } j = 6k + 6i + 2 \text{ and } 1 \le i \le k - 1; \\ 3k - 3i, & \text{if } j = 6k + 6i + 3 \text{ and } 1 \le i \le k - 1; \\ 12k + i + 2, & \text{if } j = 12k + 2i - 2 \text{ and } 1 \le i \le 2. \end{cases}$$

Case 10: For $n = 12k + 3$, where k is a positive integer, let $f : V(F) \to \{1, 2, \ldots, 24k + 6\}$ be the vertex labeling such that

$$f(u_j) = \begin{cases} 12k - 3i + 6, & \text{if } j = 2i - 1 \text{ and } 1 \le i \le 3k + 1; \\ 3i - 6k - 4, & \text{if } j = 2i - 1 \text{ and } 3k + 2 \le i \le 6k + 2; \\ 24k - 3i + 8, & \text{if } j = 2i \text{ and } 1 \le i \le 3k + 1; \\ 6k + 3i + 1, & \text{if } j = 2i \text{ and } 3k + 2 \le i \le 6k + 1; \end{cases}$$

$$f(v_j) = \begin{cases} 24k - 3i + 9, & \text{if } j = 2i - 1 \text{ and } 1 \le i \le 3k + 2; \\ 12k - 3i + 4, & \text{if } j = 2i \text{ and } 1 \le i \le 3k; \\ 3k + i - 1, & \text{if } j = 6k + 2i \text{ and } 1 \le i \le 2; \\ 15k - 3i + 7, & \text{if } j = 6k + 6i - 1 \text{ and } 1 \le i \le k; \\ 3k - 3i + 1, & \text{if } j = 6k + 6i \text{ and } 1 \le i \le k; \\ 15k - 3i + 5, & \text{if } j = 6k + 6i + 1 \text{ and } 1 \le i \le k - 1; \\ 3k - 3i + 2, & \text{if } j = 6k + 6i + 2 \text{ and } 1 \le i \le k; \\ 15k - 3i + 3, & \text{if } j = 6k + 6i + 3 \text{ and } 1 \le i \le k - 1; \\ 3k - 3i, & \text{if } j = 6k + 6i + 4 \text{ and } 1 \le i \le k - 1; \\ 12k + i + 3, & \text{if } j = 12k + 2i - 1 \text{ and } 1 \le i \le 2. \end{cases}$$

Case 11: For $n = 12k + 9$, where k is a positive integer, let $f : V(F) \to \{1, 2, \ldots, 24k + 18\}$ be the vertex labeling such that

$$f(u_j) = \begin{cases} 12k - 3i + 12, & \text{if } j = 2i - 1 \text{ and } 1 \le i \le 3k + 3; \\ 3i - 6k - 7, & \text{if } j = 2i - 1 \text{ and } 3k + 4 \le i \le 6k + 5; \\ 24k - 3i + 20, & \text{if } j = 2i \text{ and } 1 \le i \le 3k + 2; \\ 6k + 3i + 4, & \text{if } j = 2i \text{ and } 3k + 3 \le i \le 6k + 4; \end{cases}$$

$$f(v_j) = \begin{cases} 24k-3i+21, & \text{if } j=2i-1 \text{ and } 1 \le i \le 3k+2; \\ 12k-3i+10, & \text{if } j=2i \text{ and } 1 \le i \le 3k+3; \\ 15k+i+10, & \text{if } j=6k+2i+3 \text{ and } 1 \le i \le 2; \\ 3k-3i+5, & \text{if } j=6k+6i+2 \text{ and } 1 \le i \le k+1; \\ 15k-3i+12, & \text{if } j=6k+6i+3 \text{ and } 1 \le i \le k; \\ 3k-3i+3, & \text{if } j=6k+6i+4 \text{ and } 1 \le i \le k; \\ 15k-3i+13, & \text{if } j=6k+6i+5 \text{ and } 1 \le i \le k; \\ 3k-3i+1, & \text{if } j=6k+6i+6 \text{ and } 1 \le i \le k; \\ 15k-3i+11, & \text{if } j=6k+6i+7 \text{ and } 1 \le i \le k-1; \\ 12k+i+9, & \text{if } j=12k+2i+5 \text{ and } 1 \le i \le 2. \end{cases}$$

Therefore, by Lemma 1.1, f extends to a super edge-magic labeling of F with valence $5n$ when $n = 4k$ and $5n+1$, otherwise. □

Now, we state the edge-magic analogue to the previous theorem.

Theorem 4.8 *The forest $2P_n$ is edge-magic if and only if $n \neq 2$.*

The next corollary to Theorems 2.1 and 4.8, and the comments immediately following Theorem 4.5 partially settles Yegnanarayanan's conjecture stated in [12] that the forest nP_3 is edge-magic for every positive integer n.

Corollary 4.9 *If n is odd, $n \equiv 2 \pmod 4$ or $n \equiv 4 \pmod 8$, then the forest nP_3 is edge-magic.*

Our final result on forests concerns $K_{1,m} \cup 2nP_2$.

Theorem 4.10 *The forest $F \cong K_{1,m} \cup 2nP_2$, where m and n are positive integers, is super edge-magic. Furthermore, if $m+2n$ and $2n+1$ are relatively prime, then only the valences $2m+9n+4$ and $3m+9n+3$ are attained by the super edge-magic labelings of F.*

Proof. Let $F \cong K_{1,m} \cup 2nP_2$ be a (p,q) forest such that

$$V(F) = \{u\} \cup \{v_i | 1 \le i \le m\} \cup \{w_i | 1 \le i \le 4n\}$$

and

$$E(F) = \{uv_i | 1 \le i \le m\} \cup \{w_i w_{2n+i} | 1 \le i \le 2n\}.$$

Then let $f, g : V(F) \to \{1, 2, \ldots, p\}$ be the vertex labelings of F with

$$f(x) = \begin{cases} n+1, & \text{if } x=u; \\ 2n+i+1, & \text{if } x=v_i \text{ and } 1 \le i \le m; \\ i, & \text{if } x=w_i \text{ and } 1 \le i \le n; \\ i+1, & \text{if } x=w_i \text{ and } n+1 \le i \le 2n; \\ m+n+i+1, & \text{if } x=w_i \text{ and } 2n+1 \le i \le 3n; \\ m-n+i+1, & \text{if } x=w_i \text{ and } 3n+1 \le i \le 4n; \end{cases}$$

and

$$g(x) = \begin{cases} m+3n+1, & \text{if } x = u; \\ i, & \text{if } x = v_i \text{ and } 1 \le i \le m; \\ m+2i, & \text{if } x = w_i \text{ and } 1 \le i \le n; \\ m-2n+2i-1, & \text{if } x = w_i \text{ and } n+1 \le i \le 2n; \\ m+5n-i+1, & \text{if } x = w_i \text{ and } 2n+1 \le i \le 3n; \\ m+7n-i+2, & \text{if } x = w_i \text{ and } 3n+1 \le i \le 4n. \end{cases}$$

Thus, by Lemma 1.1, f and g extend to super edge-magic labelings of F with valences $2m+9n+4$ and $3m+9n+3$, respectively.

To see that the above two valences are the only possible ones when $m+2n$ and $2n+1$ are relatively prime, let k be the valence of a super edge-magic labeling h of F. Then

$$k = \frac{(m-1)h(u) + \sum_{i=1}^{p+q} i}{q} = 2m+8n+3+h(u) + \frac{(2n+1)(n+1-h(v))}{m+2n}.$$

This implies that there exists an integer α such that $\alpha(m+2n) = 1+n-h(v)$. Now, since $1 \le h(v) \le p$, it follows that α is 0 or -1, values that lead to the valences $2m+9n+4$ and $3m+9n+3$, respectively. □

Notice that if we relax the hypotheses of the previous theorem to refer to just edge-magic labelings, then we have that another valence occurs as stated in the following corollary.

Corollary 4.11 *Let $F \cong K_{1,m} \cup 2nP_2$, where m and n are positive integers such that $m+2n$ and $2n+1$ are relatively prime. Then only the valences $2m+9n+4$, $3m+9n+3$ and $4m+9n+2$ are attained by the edge-magic labelings of F.*

Proof. To prove this, we use the facts and notation of the proof of the previous theorem. First, notice that the vertex labeling $h : V(F) \to \{1, 2, \ldots p\}$ such that $h(v) = p+q+1-f(v)$ extends to an edge-magic labeling of F with valence $4m+9n+2$.

Next, if we allow edge-magic labelings of F, then the value of α in the proof can also be -2, and thus only one further valence is attained. □

5 An Improved Bound

Enomoto, Lladó, Nakamigawa and Ringel [2] proved that the inequality $q \le 2p-3$ holds for every super edge-magic (p,q) graph. The final result of this paper improves this bound for certain graphs.

Theorem 5.1 *If G is a super edge-magic (p,q) graph with $p \ge 4$ and $q \ge 2p-4$, then G contains triangles.*

Proof. Assume, to the contrary, that G is triangle-free. Furthermore, let $V(G) = \{v_1, v_2, \dots, v_p\}$, and let $f : V(G) \cup E(G) \to \{1, 2, \dots, p+q\}$ be a super edge-magic labeling of G so that $f(v_i) = i$ for every integer i with $1 \leq i \leq p$. Observe first that since $q \geq 2p - 4$, it follows that either v_1 and v_2 or v_p and v_{p-1} are adjacent as the numbers 3 and $2p - 1$ can be expressed uniquely as the sums of integers in the range 1 through p; so suppose, without loss of generality, that v_1 and v_2 are adjacent. Then v_1 and v_3 are adjacent also since the sum 4 can be expressed uniquely with integers in the permitted range. This in turn implies that v_2 and v_3 cannot be adjacent since G is triangle-free, and thus v_1 and v_4 are adjacent. Continuing to avoid triangles in this manner, we conclude that v_1 is adjacent to the vertices v_2 through v_{d+1}, where $d = \deg(v_1)$, and none of these vertices are adjacent to one another. We have thus accounted for the sums 3 through $d + 2$.

Now, if $d = p - 1$, then we are done since there is no way for us to obtain the sum $d + 3$ avoiding triangles. Otherwise, if $d < p - 1$, then, with the remaining options, the smallest sum possible is $d + 4$ (joining v_2 with v_{d+2}), and we would have no way of obtaining the sum $d + 3$. Therefore, in either case, we have arrived to a contradiction. □

The contrapositive to the previous theorem provides the desired bound.

Corollary 5.2 *If G is a triangle-free super edge-magic (p, q) graph of order $p \geq 4$, then $q \leq 2p - 5$.*

The authors believe this bound is sharp for all possible values of p. Indeed, it is easy to find, through *ad hoc* methods, super edge-magic (p, q) bipartite graphs with $q = 2p - 5$ for small values of $p \leq 8$. For instance, if $p = 8$, then consider the graph G with $V(G) = \{1, 2, 3, 4, 5, 6, 7, 8\}$, eulerian trail $1-5-2-8-4-7-1-3-2-7-6-8$ and super edge-magic labeling $f : V(G) \to V(G)$ such that $f(v) = v$ when $v \in V(G)$.

6 Conclusions

The relevance of this paper is twofold.

First, it provides a construction method for (super) edge-magic bipartite and tripartite graphs. This enlarges significantly the classes of graphs heretofore known to be felicitous, harmonious or sequential. In particular, very few 2-regular graphs were known to be in those classes (see [6]). Moreover, because our methods have the insurmountable limitation of only working for an odd number of copies, the following question is of great interest to the authors: for which bipartite or tripartite graphs G is $2^n G$ (super) edge-magic for some positive integer n.

Second, the results on linear forests in this paper lead to the open problems of whether or not do linear forests which are not super edge-magic aside from $2P_2$ and $2P_3$ exist, and, analogously, the existence or non-existence of linear forests which are not edge-magic other than $2P_2$.

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