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On Edwards' Model for Long Polymer Chains

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Abstract. An existence theorem is proved for a probability measure on continuous paths in space, proposed by Edwards as a stochastic model for the geometric properties of long polymer chains.

1. Introduction

The problem of setting up and analysing a probabilistic model for long polymer chains which takes into account the so-called "excluded volume" effect is an old one, being described already in Kac's classic survey of probabilistic methods in physics [1]. A simple discrete model is obtained by considering the "self-avoiding" random walks on a lattice, so that the problem is to determine the asymptotic behaviour of very long walks of this kind. Since the self-avoiding random walk is not a Markov process, progress has been slow indeed. Thus the survey article of Domb [2] (1969) lists no further rigorous results beyond those established by Hammersley and Kesten (by 1964)[1]. The problem has been studied by computer with results described in detail in the cited article of Domb. We do not wish to review these results here but only to call attention to Domb's conclusion that it is possible to distinguish between long and short range properties of the polymer chain, the long range properties being sensibly independent of the detail of the interaction between the links of the chain. Thus just as the asymptotics of random walks (under rather general conditions on the distribution of the individual steps) is substantially equivalent to the study of Brownian motion (the Wiener process), the long range properties of polymer chains should be studied in an appropriate continuum model.

Such a model has been proposed by Edwards [3]. In this model the chains are represented by continuous paths $\mathbf{x}(\sigma)$, $0 \leq \sigma \leq 1$, in $\mathbb{R}^3$, with $\mathbf{x}(\mathrm{o}) = \mathbf{0}$, the probability measure v on the space of paths being given in terms of Wiener measure μ by

$$\frac{dv}{d\mu} = \mathscr{Z}^{-1} \exp[-gJ], \tag{1}$$

1 See also [2a] for a more recent review

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with

$$\mathscr{Z} = \int \exp[-gJ]\, d\mu. \tag{2}$$

Here J is a functional on the space of Wiener paths which (roughly speaking) measures the time which the Brownian motion spends at its double points

$$J = \int_0^1 \int_0^1 \delta(\mathbf{x}(\sigma) - \mathbf{x}(\tau))\, d\sigma\, d\tau, \tag{3}$$

and g is a positive constant. J is intended to represent the excluded volume effect – that is the repulsive self-interaction of the polymer chain at points where it crosses itself.

In [3] Edwards simplifies the analysis of his model by introducing an uncontrolled approximation ("mean field" approximation), and on this basis obtains for the distribution of some of the physically interesting quantities results apparently in good agreement with the numerical experiments.

In [4] Abram and Edwards consider the Feynmann integral representation of the quantum mechanical motion of a non-relativistic particle moving in a random array of scattering centers, and show formally that in the limit in which the density of scattering centers $\to \infty$, and the interaction strength $\to 0$ (analog of the Grad limit in gas kinetics), the transition probabilities of the particle are given by a "measure" on path space which stands in the same relation to the polymer measure as the Feynmann "measure" does to Wiener measure. The problem of scattering in a random medium in the above limit is discussed also by Kac [5].

The continuum model appeared independently in the programm of Symanzik for the study of the ϕ_3^4 boson field theory [6]. The connection with field theory appears again in the work of de Gennes [7], and des Cloizeaux [8], who base their analysis on the heuristic ideas of Wilson; [9] contains a discussion of the excluded volume problem within the framework of Wilson's renormalisation group.

Equations (1)–(3) are not to be taken literally. (3) is naturally construed to mean

$$J = \lim_{\varepsilon \to 0} J_\varepsilon, \tag{4}$$

with

$$J = \int_0^1 \int_0^1 \delta_\varepsilon(\mathbf{x}(\sigma) - \mathbf{x}(\tau))\, d\sigma\, d\tau, \tag{5}$$

and δ_ε a suitable regularisation of the δ-function in $\mathbb{R}^3$; but the limit (4) does not exist.

In considering the status of (1)–(3), it is instructive to consider also the status of the corresponding equation for dimension $d = 1, 2, 4$; the results closely parallel those for the ϕ_d^4 boson field theory in a finite volume. For $d = 1$ (1)–(3) may be taken literally; the limit (4) exists in L^1, and in fact J may be expressed in terms of Levy's local times (cf. Example 2, § 3.1). For $d = 2$ Varadhan showed in an appendix to [6] that

$$\hat{J} = \lim_{\varepsilon \to 0} (J_\varepsilon - E[J_\varepsilon]) \tag{6}$$

exists in L^2, and that, for $g \geqq 0$, $\exp[-g\hat{J}] \in L^1$; the proof is patterned after the proof given by Nelson for the corresponding result for the ϕ_2^4 boson field theory, which may be found in Simon's lectures [10]. Thus for $d=2$ it suffices to replace J by $\hat{J}$ in (1), (2), to obtain an acceptable definition of ν formally equivalent to the original. For $d=3$ the limit (6) does not exist, and Symanzik suggested that (1)–(3) be construed to mean

$$\nu = \text{weak} \lim_{\varepsilon \to 0} \nu_\varepsilon , \tag{7}$$

with

$$\frac{d\nu_\varepsilon}{d\mu} = \mathscr{Z}_\varepsilon^{-1} \exp[-gJ_\varepsilon] , \tag{8}$$

$$\mathscr{Z}_\varepsilon = E[\exp(-gJ_\varepsilon)] . \tag{9}$$

In this paper we show that the limit (7) exists for sufficiently small g. The proof is patterned after the proof given by Glimm and Jaffe [11], and Feldman [12], for the corresponding result for the ϕ_3^4 boson field theory. The constructed measure $\nu(g)$ is, for $g>0$, not Gaussian. This may be proved by showing that the moments of $\nu(g)$ are differentiable in g as $g \to 0+$, and that their derivatives are given by the renormalized perturbation series. We have not set out this proof in detail, but it will be clear to the reader familiar with the work on the ϕ_3^4 model (the current status of this model may be obtained from [13], and references contained therein); to obtain differentiability to a given order k, it is necessary only to choose the constant L which appears in the definition of the inductive expansion (2.7.3) to be $>k$. For $d=4$ the polymer model is, like ϕ_4^4, in the sense of the classification based on the analysis of formal perturbation theory, renormalisable but not super-normalisable. This analysis suggests that for $d=4$ the problem be formulated as follows: For $c>0$ consider the transformation

$$\bar{\mathbf{x}}(\sigma) = c\mathbf{x}(\sigma) \tag{10}$$

(the analog of a field strength renormalisation in boson field theory). If $\mathbf{x}(.)$ is the stochastic process defined by the probability measure $\nu(\varepsilon, g)$ on the space of continuous paths, $\bar{\mathbf{x}}(.)$ will be a stochastic process whose defining probability measure we denote by $\nu(\varepsilon, g, c)$. For $\eta > 0$ define

$$S(\eta) = \text{weak closure } \{\nu(\varepsilon, g, c\} ; \varepsilon \leqq \eta, c>0, g \geqq 0\}$$

$$S = \bigcap_{\eta > 0} S(\eta) .$$

Determine whether or not the set S contains any non-Gaussian measures. (For $d \geqq 5$ it is to be expected that S contains only Gaussian measures.)

Symanzik's proposal to link the construction of the ϕ_3^4 boson field theory to the polymer problem is now only of historical interest. Nevertheless the polymer problem is of some methodological interest for field theory, since the role of the transformation properties of the interaction under change of scale appears much more clearly here than in ϕ_3^4; in this sense our work is in the same spirit as that of Gallavotti et al. [14] on the hierachical field model.

We have separated the proof into two parts; an abstract limit theorem for certain arrays of random variables (Sect. 2) and the specific calculations for the intended application (Sect. 3). The reader will probably wish to read first 3.2 in order to be convinced that the rather elaborate situation postulated in the abstract theorem does arise quite naturally. In its essentials the proof follows the standard pattern for the existence of a Gibbs measure in the thermodynamic limit. By means of Ciesielski's representation of Brownian motion, the Wiener measure μ is identified with a product measure on a space $\mathbb{R}^\infty$ (3.2.5). The problem of constructing the polymer measure can then be viewed as the problem of constructing the Gibbs measure for a continuous spin system in which the sites are indexed by the positive integers, and the variables $\{y(v), n(v)=n\}$ specify the state of the spin at side n. Since 1-dimensional systems do not admit phase transitions unless the forces are of long range, this suggests that the limitation to small values of g is simply a limitation of the method of proof. This may well be the case but the argument is not decisive because the spin system under consideration is not one to which the standard theorems apply.

2. A Limit Theorem

2.1. Statement of the Theorem

We suppose given a probability measure space (Ω, M, μ) together with

(a) a random variable X,

(b) a σ-algebra $G \subset M$,

(c) a decomposition of (Ω, M, μ) as a product

$$(\Omega, M, \mu) = (\Omega_0, M_0, \mu_0) \times (\Omega_1, M_1, \mu_1),$$

together with isomorphisms j_0, j_1 of (Ω_0, M_0, μ_0), (Ω_1, M_1, μ_1) onto (Ω, M, μ).

We will impose a number of conditions on the above data. The formulation of these conditions demands the introduction of further notation.

Let $T = \bigcup_{n \geqq 0} T(n)$, with $T(0) = \{0\}$, and, for $n > 0$,

$$T(n) = \{(n, i) = (n, i_1, \ldots, i_n), \quad \text{with} \quad i_1, \ldots, i_n = 0 \quad \text{or} \quad 1\}.$$

If $v \in T(n)$, we say v has level n. For $v \in T$, define a map $\psi(v)$ of T into T: if $v=0$, $\psi(0)$ is the identity map; if $v=(n,i)$ with $n>0$,

$$\psi(v)(0) = v$$

$$\psi(v)(m, j_1, \ldots, j_m) = (m+n, i_1, \ldots, i_n j_1, \ldots, j_m) \qquad m > 0.$$

Write $T(v) = \psi(v)(T)$.

Denote by p_0, p_1 the projections of Ω onto Ω_0, Ω_1 given by the product decomposition, and by ζ_0, ζ_1 the maps $j_0 \circ p_0$, $j_1 \circ p_1$. For $v \in T$ define maps $\zeta(v): \Omega \to \Omega$; if $v=0$, $\zeta(0)$ is the identity map; if $v=(n, i_1, \ldots, i_n)$, with $n>0$,

$$\zeta(v) = \zeta_{i_n} \circ \zeta_{i_{n-1}} \circ \ldots \circ \zeta_{i_1}.$$

We denote by $\zeta(v)^*$ the map induced by $\zeta(v)$ on the algebra of random variables on (Ω, M, μ), and define, for $v \in T$, random variables

$$X(v) = 2^{-n/2}\,\zeta(v)^*(X),$$

n being the level of v, and σ-algebras

$$G(v) = \zeta(v)(G).$$

If S is any subset of T, we write

$$G(S) = \bigvee_{v \in S} G(v);$$

we write also $G(n) = G(T(n))$, for $n \geqq 0$.

We can now state our first conditions:

C1. $G(0) \subset G(1)$.

C2. There exists a σ-algebra F independent of $G (= G(0))$ such that

$$G(1) = G \vee F.$$

Note that *C1* implies $G(n) \subset G(n+1)$ for all $n \geqq 0$. Write

$$G(\infty) = \bigvee_{n \geqq 0} G(n).$$

Define also σ-algebras $F(v)$, $v \in T$,

$$F(v) = \zeta(v)(F),$$

and write $F(S) = \bigvee_{v \in S} F(v)$, for $S \subset T$,

$$F(n) = F(T(n)),$$

$$F^-(n) = \bigvee_{0 \leqq m < n} F(m),$$

$$F^+(n) = \bigvee_{m \geqq n} F(m),$$

and note that the σ-algebras $\{G; F(v), v \in T\}$ are independent and generate $G(\infty)$, and that, for $n \geqq 0$,

$$G(n) = G \vee F^-(n),$$

$$G(\infty) = G \vee F^+(0).$$

We can now state our remaining conditions:

C3. $X \geqq 0$.

C4. For some constants $C_1 > 0$, $\beta_1 > 0$, and all $p \geqq 1$,

$$\|X\|_p \leqq C_1 p^{\beta_1}. \tag{1}$$

C5. For some constants $C_2 > 0$, $\beta_2 > 0$, $\tau_2 < 1$, and all

$p \geqq 1$, $n \geqq 0$,

$$\|X - E[X \mid G(n)]\|_p \leqq C_2 p^{\beta_2} \tau_2^n \tag{2}$$

[so that, in particular, $X \in G(\infty)$].

C6. For some constants $C_3>0$, $\beta_3>0$, $\tau_3<1$, and all

$p\geqq 1$, $n\geqq 0$,

$$\|X(n)-E[X(n)|F^+(0)]\|_p\leqq C_3 p^{\beta_3}\tau_3^n. \tag{3}$$

Here $X(n)=\sum_{v\in T(n)} X(v)$.

For any $n\geqq 0$ set

$$S(n)=\sum_{m=0}^{n} X(m),$$

and note $S(n)\geqq 0$, so that, for any $g\geqq 0$,

$$\exp[-gS(n)]\in L^1,$$

and we may define a probability measure $\nu_n(g)$ absolutely continuous with respect to μ by

$$\frac{d\nu_n}{d\mu}=\frac{1}{\mathscr{Z}_n(g)}\exp[-gS(n)], \tag{4}$$

with $\mathscr{Z}_n(g)=E[\exp(-gS(n))]$ the appropriate normalisation constant. Write $f_{nm}=E\left[\frac{d\nu_n}{d\mu}\,\middle|\,G(m)\right]$, for $n,m\geqq 0$.

Our main result is:

Theorem 1. *With the above notations and hypotheses, there exists a constant $\bar{g}>0$ (dependent on the constants in C4, C5, C6) such that for all $m\geqq 0$, and $g\leqq\bar{g}$*

$$\lim_{n\to\infty} f_{nm}=f_m \tag{5}$$

exists in L^1. $\{f_m, m\geqq 0\}$ is a martingale relative to the increasing sequence of σ-algebras $G(m)$, and $f_m\geqq 0$, $E[f_m]=1$ for all $m\geqq 0$.

2.2. Motivation of the Proof

We begin with some preliminary reductions. We will actually prove a stronger statement than (2.1.5). Namely, we will prove that for any p, $1<p<\infty$, there exists a constant $\bar{g}(p)$ such that for all $m\geqq 0$, and $g\leqq\bar{g}(p)$

$$\lim_{n\to\infty} f_{nm}=f_m \tag{1}$$

exists in L^p. Let q be conjugate to p; (1) is equivalent to an estimate

$$|E[(f_{nm}-f_{km})R]|\leqq K(m,n,k)\,\|R\|_q \tag{2}$$

for all $R\in L^q(\Omega.\,G(m),\mu)$, with $K(m,n,k)\to 0$ as $n,k\to\infty$ for fixed m. The left side of (2) may be written

$$|E[R\exp(-gS(n))]\,\mathscr{Z}_n(g)^{-1}-E[R\exp(-gS(k))]\,\mathscr{Z}_k(g)^{-1}|. \tag{3}$$

If $S(n)$ were a Gaussian random variable, we should have $\mathscr{Z}_n(g)=C_n(g)$, with

$$C_n(g)=\exp\{-gE[S(n)]+1/2g^2\,\mathrm{Var}[S(n)]\}. \tag{4}$$

We will prove that $S(n)$ is approximately Gaussian in the sense that

$$\lim_{n\to\infty} \mathscr{Z}_n(g)C_n(g)^{-1} \tag{5}$$

exists, and is non-zero for sufficiently small g. It will then suffice to prove

$$|E[R\exp(-gS(n))]\,C_n(g)^{-1} - E[R\exp(-gS(k))]\,C_k(g)^{-1}| \leqq K(m,n,k)\,\|R\|_q\,, \tag{6}$$

in place of (2).

To motivate the proof of Theorem 1 consider the special case in which $X\in F$ (despite the fact that the conclusion of the theorem in this case is trivial). The random variables $X(v)$, $v\in T$, are then independent, with $X(v)$ identical in law with $2^{-n/2}X$ [for $v\in T(n)$]. If X is not constant both

$$E[S(n)] = 2^{n/2}\left(1-2^{-\frac{(n+1)}{2}}\right)(1-2^{-1/2})^{-1}E[X]$$

and

$$\mathrm{Var}[S(n)] = (n+1)\,\mathrm{Var}[X]\,,$$

are divergent as $n\to\infty$. Thus the existence of the limit (5) requires cancellation, neither of the factors $\mathscr{Z}_n(g)$, $C_n(g)$ having a non-zero limit as $n\to\infty$. To show this cancellation, write, for $g\geqq 0$,

$$F(g) = E[\exp(-gX)]\,.$$

Then

$$\mathscr{Z}_n(g) = \prod_{m=0}^{n} [F(g2^{-m/2})]^{2^m}\,.$$

For some $\varepsilon>0$ we may estimate, for $g\leqq\varepsilon$

$$\log F(g) = -gE[X] + 1/2g^2\,\mathrm{Var}[X] + R(g)\,,$$

with $|R|\leqq Cg^3$. Choose k sufficiently large that $g2^{-k/2}\leqq\varepsilon$. Then

$$\log(\mathscr{Z}_n(g)C_n^{-1}(g)) = \log(\mathscr{Z}_k(g)C_k^{-1}(g)) + \sum_{m=k+1}^{n} 2^m R(g2^{-m/2})\,,$$

and $|2^m R(g2^{-m/2})|\leqq Cg^3 2^{-m/2}$, so the limit (5) exists.

The proof of (5) in the general case will folow the proof just given for independent $X(v)$ in that a partial Taylor expansion of $\mathscr{Z}_n(g)$, $C_n(g)$ about $g=0$ will be used to show the cancellation leading to the existence of the limit.

Conditions $C5$, $C6$ limit the dependence of the $X(v)$ in the general case. For each $n\geqq 0$ define a sequence of σ-algebras $\{F(n,m),\, m\geqq 0\}$ by

$$F(n,m) = G(n+m)\cap F^{+}(n-m) \quad \text{if} \quad m\leqq n$$

$$F(n,m) = G(n+m) \quad \text{if} \quad m>n\,,$$

so that $F(n,0)$ is the trivial σ-algebra, and $F(n,m)\uparrow G(\infty)$ as $m\to\infty$. Write $X(n,m) = E[X(n)\,|\,F(n,m)] - E[X(n)\,|\,F(n,m-1)]$, for $m\geqq 1$. We will write

$$X(n) = E[X(n)] + \sum_{m=1}^{\infty} X(n,m)\,; \tag{7}$$

estimates for the rate of convergence will be obtained in 2.5. We will refer to the martingale differences $X(n,m)$ as localised terms. We use (7) to write $S(n)$ as a sum of localised terms.

A difficulty arises in combining the splitting of $S(n)$ into localised terms with the Taylor expansion method: the difference $X(n,m)$ need not be bounded below, so $\exp\{-gX(n,m)\}$ need not be in L^1 for $g \geqq 0$. This difficulty forces us to use the Taylor expansion method only for truncated random variables. Introduce a sequence $\{b_n, n \geqq 0\} = B$ of positive real numbers, and define

$$X(v; B) = \min\{X(v), 2^{-n/2} b_n\}$$

for $v \in T$ of level n. Quantities constructed from the truncated random variables $X(v; B)$ will be denoted by the same symbols as the corresponding quantities constructed from the $X(v)$ with the addition of a further label B – thus $S(n; B)$ etc. In order to be able to deduce (5), (6) from the corresponding results for the truncated random variables we will have to chose truncation levels $b_n \to \infty$ as $n \to \infty$. This choice results in a loss of scale invariance; for $w \in T(v)$, $v \in T(n)$,

$$\zeta(v)_*^{-1}[X(w; B)] = 2^{-n/2} X(\psi(v)^{-1} w; B_n), \tag{8}$$

with B_n the truncation sequence $\{b_{n+m}, m \geqq 0\}$. However, consideration of the case in which the random variables $X(v)$, $v \in T$, are assumed independent, with $X(v)$ identical in law with $\theta^n X$ for $v \in T(n)$, for some $\theta > 0$, suggests that if the truncation levels b_n do not rise too rapidly this loss of scale invariance will not cause difficulty – in the special case cited the proof of the existence of the limit (5) given for the case $\theta = 2^{-1/2}$ remains valid provided $\theta < 2^{-1/3}$.

2.3. Counterterms

Write

$$F_n(g) = \mathscr{Z}_n(g) C_n(g)^{-1}. \tag{1}$$

Following the usage of quantum field theory, we will refer to $C_n(g)$ in (1), and to any terms resulting from expansion of $C_n(g)$, as counterterms. The purpose of this section is to introduce some notations which will ensure that when $S(n)$ is split into a sum of localised terms and an expansion is made of $\mathscr{Z}_n(g)$ that a parallel expansion is made of $C_n(g)^{-1}$, in such a way that the counterterms are properly matched with the terms they are supposed to cancel.

Denote by $L_0^2(\Omega, M, \mu) \subset L^2(\Omega, M, \mu)$ the Hilbert space of random variables f on (Ω, M, μ) with finite variance and zero mean, and by $\Gamma(\cdot)$ the Gaussian process indexed by $L_0^2(\Omega, M, \mu)$. We will regard this process as independent of the random variables on (Ω, M, μ). If $A \subset M$ is a sub σ-algebra of M, we write $\Gamma(A) = \sigma\{\Gamma(f); f \in L_0^2(\Omega, A, \mu)\}$. Define a complex valued process $\psi(\cdot)$ indexed by $L_0^2(\Omega, M, \mu)$

$$\psi(f) = f + \sqrt{-1}\,\Gamma(f), \tag{2}$$

and note that the map $\psi: f \to \psi(f)$ is linear, and that

$$E[\psi(f_1)\psi(f_2)] = 0, \tag{3}$$

for all $f_1, f_2 \in L^2_0(\Omega, M, \mu)$. If $f \in L^2(\Omega, M, \mu)$ is of the form $X(\xi)$, with ξ some label, we write

$$Y(\xi) = f - E[f]\,; \qquad Z(\xi) = \psi(Y(\xi))\,.$$

Note that for any $m \geqq 0$ the sets of random variables

$$\{Z(\psi(1,0)v\,; B_m), v \in T\}, \quad \{Z(\psi(1,1)v\,; B_m), v \in T\}$$

are independent, each being identical in law with

$$\{2^{-1/2} Z(v\,; B_{m+1}), v \in T\}\,. \tag{4}$$

Since $E[\exp(\sqrt{-1}\,\Gamma(f))] = \exp\{-1/2\,\mathrm{Var}[f]\}$, for $f \in L^2_0(\Omega, M, \mu)$, we have the basic identity of this section

$$F_n(g) = E[\exp\{-g W(n)\}]\,, \tag{5}$$

with $W(n) = \psi(S(n) - E[S(n)])$.

2.4. *Taylor Expansions*

For any integer $N \geqq 1$, we define a random variable t_N, with $0 \leqq t_N \leqq 1$, by specifying the probability density of t_N as

$$p_N(u) = N(1-u)^{N-1}\,, \tag{1}$$

$0 \leqq u \leqq 1$.

If $f(x)$ is a C^∞ function of the real variable x, and X a random variable, we then have, for any $N \geqq 1$, the Taylor expansion formula

$$E[f(X)] = E\left[\sum_{j=0}^{n-1} \frac{X^j}{j!} f^{(j)}(0) + \frac{X^N}{N!} f^{(N)}(t_N X)\right], \tag{2}$$

with t_N independent of X (the existence of the expectations being assumed).

This notation, apparently the result of pursuing the probabilitists' abhorrence of integral signs to the point of mania, will enable us to write compactly the Taylor expansions of the inductive expansion (2.6).

2.5. *Localisation Estimates*

In this section $B = \{b_n, n \geqq 0\}$ will denote a sequence of truncation levels satisfying the following condition

B1. For some constants $C_4, \beta_4 > 0$, with $\beta_4 > \beta_1$ (cf. *C4*), and all $n \geqq 0$,

$$b_n \geqq C_4 n^{\beta_4}\,. \tag{1}$$

The estimates obtained will be uniform over the set of sequences satisfying *B1*. The importance of this uniformity for the proof of Theorem 1 lies in the observation that if B is a sequence satisfying *B1*, then the shifted sequences $B_m = \{b_{n+m}, n \geqq 0\}$, $m \geqq 0$, also satisfy *B1*, so that the estimates of this section will be uniform over the set of sequences $\{B_m, m \geqq 0\}$.

Lemma 1. *For all* $p \geqq 1$, *and* $n \geqq 0$

$$\|X(B)-E[X(B)\mid G(n)]\|_p \leqq 2C_2 p^{\beta_2}\tau_2^n. \tag{2}$$

Proof. The proof of (2) relies on two general remarks. First, if X is a random variable in L^p, and F a σ-algebra then

$$\|X-E[X\mid F]\|_p \leqq 2\inf_Y \|X-Y\|_p, \tag{3}$$

the infimum being taken over all F-measurable $Y \in L^p$. Second, if X and Y are random variables, and b a real number then

$$|\min(X,b)-\min(Y,b)| \leqq |X-Y|. \tag{4}$$

Thus

$$\begin{aligned}\|X(B)-E[X(B)\mid G(n)]\|_p &\leqq 2\|\min(X,b_0)-\min(E[X\mid G(n)],b_0)\|_p \quad &[\text{by } (3)]\\ &\leqq 2\|X-E[X\mid G(n)]\|_p &[\text{by } (4)]\\ &\leqq 2C_2 p^{\beta_2}\tau_2^n &(\text{by } C5). \quad \square\end{aligned}$$

Lemma 2. *The limits*

$$\lim_{n\to\infty}(E[S(n)]-E[S(n;B)])$$

$$\lim_{n\to\infty}(\operatorname{Var}[S(n)]-\operatorname{Var}[S(n;B)])$$

exist.

Proof. From $C4$ we obtain a bound on the tail of the distribution of X

$$\begin{aligned}\Pr\{X \geqq b\} &\leqq \inf_{k\geqq 1} b^{-k}\|X\|_k^k\\ &\leqq \exp\{-C_5 b^{\beta_5}\},\end{aligned} \tag{5}$$

for $b \geqq C_6$. Here $\beta_5=\beta_1^{-1}$, $C_6=C_1\exp(\beta_1)$, $C_5=\beta_1\exp(-1-\beta_5\log C_1)$.

Now $E[S(n)]-E[S(n;B)]=\sum_{m=0}^{n} E[X(m)-X(m;B)]$, and

$$\begin{aligned}E[X(m)-X(m;B)] &= 2^{m/2}E[X-\min(X,b_m)] \leqq 2^{m/2}\|X\|_2(\Pr\{X\geqq b_m\})^{1/2}\\ &\leqq 2^{m/2}\|X\|_2\exp\left(-\frac{C_5}{2}b_m^{\beta_5}\right),\end{aligned}$$

(for m sufficiently large that $b_m \geqq C_6$)

$$\leqq 2^{m/2}\|X\|_2\exp(-C_7 m^{\beta_7}), \tag{6}$$

with $C_7=\dfrac{C_4^{\beta_5}C_5}{2}>0$, and $\beta_7=\beta_4\beta_1^{-1}>1$, by B1.

The series whose general term is (6) converges, so the first assertion of the lemma is established.

$$\operatorname{Var}[S(n)]=\sum_{m=0}^{n} H(m),$$

with $H(m)=E[Y(m)^2]+2\sum_{k=m+1}^{n} E[Y(k)Y(m)]$, so that to establish the second assertion of the lemma it suffices to bound, uniformly in n, $|H(m)-H(m;B)|$ by the general term of a convergent series. We will obtain the desired majorant by estimating

$$|E[Y(k)Y(m)-Y(k;B)Y(m;B)]| \\ \leqq |E[Y(k)(Y(m)-Y(m;B))]|+|E[Y(m;B)(Y(k)-Y(k;B))]|, \tag{7}$$

and bounding each term in (7) by a corresponding term of a series summable over k, m with $k \geqq m$.

The second term in (7) is bounded, for k sufficiently large, by

$$\begin{aligned} &\|Y(m;B)\|_2 \, \|Y(k)-Y(k;B)\|_2 \\ &\quad = \|Y(0;B_m)\|_2 \, \|Y(0)-Y(0;B_k)\|_2 \\ &\quad = \|X\|_2 \, \|X\|_4 (\Pr\{X \geqq b_k\})^{1/4} \\ &\quad \leqq \|X\|_2 \, \|X\|_4 \exp\left(-\frac{C_7}{2} k^{\beta_7}\right), \end{aligned} \tag{8}$$

where we have used scaling and independence at the first step, and (1), (5) at the last. Note

$$\sum_{k,m:\,0 \leqq m \leqq k < \infty} \exp\left(\frac{-C_7}{2} k^{\beta_7}\right) = \sum_{k=0}^{\infty} (k+1)\exp\left(\frac{-C_7}{2} k^{\beta_7}\right) < \infty .$$

To estimate the first term in (7) set $s=\left[\frac{k-m-1}{2}\right]$. If $k=m$ we may bound the term by (8) as in the estimation of the second term. So we suppose $s \geqq 0$; then the σ-algebras $G(m+s)=G \vee F^-(m+s)$ and $F^+(k-s)$ are independent. Hence $E[E[Y(k)|F^+(k-s)]\,E[Y(m)-Y(m;B)|G(m+s)]]=0$, and

$$\begin{aligned} &|E[Y(k)(Y(m)-Y(m;B))]| \\ &\quad = |E[\{Y(k)-E[Y(k)|F^+(k-s)]\}\{Y(m)-Y(m;B)\}] \\ &\quad + E[E[Y(k)|F^+(k-s)]\{Y(m)-Y(m;B)-E[Y(m)-Y(m;B)|G(m+s)]\}]| \\ &\quad \leqq \|Y(k)-E[Y(k)|F^+(k-s)]\|_2 \, \|Y(m)-Y(m;B)\|_2 \\ &\qquad + \|E[Y(k)|F^+(k-s)]\|_2 \, \|Y(m)-Y(m;B)-E[Y(m)-Y(m;B)|G(m+s)]\|_2 \end{aligned} \tag{9}$$

$$\|Y(m)-Y(m;B)\|_2 \leqq \|X\|_4 \exp\left(\frac{-C_7}{2} m^{\beta_7}\right),$$

as in the proof of (8), and (again using independence and scaling)

$$\begin{aligned} \|Y(k)-E[Y(k)|F^+(k-s)]\|_2 &= \|X(s)-E[X(s)|F^+(0)]\|_2 \\ &\leqq C_3 2^{\beta_3} \tau_3^s, \quad \text{by } C6 . \end{aligned}$$

These two bounds give a satisfactory majorant for the first term of (9). As for the second term, its first factor is bounded

$$\|E[Y(k)|F^+(k-s)]\|_2 \leqq \|Y(k)\|_2 = \|Y(0)\|_2 ,$$

while the second factor can be estimated in either of two ways:

$$\begin{aligned}&\|Y(m)-Y(m;B)-E[Y(m)-Y(m;B)|G(m+s)]\|_2\\&\quad=\|X-X(B_m)-E[X-X(B_m)|G(s)]\|_2\\&\quad\leqq\|X-E[X|G(s)]\|_2+\|X(B_m)-E[X(B_m)|G(s)]\|_2\\&\quad\leqq 3C_2 2^{\beta_2}\tau_2^s,\qquad\text{by } C5 \text{ and Lemma 1,}\end{aligned}$$

or

$$\begin{aligned}&\leqq\|X-X(B_m)\|_2\\&\leqq\|X\|_4\exp\left(\frac{-C_7}{2}m^{\beta_7}\right),\end{aligned}$$

as in the proof of (8). By taking the geometric mean of these two bounds as our bound for the second factor, we obtain a satisfactory majorant for the second term of (9), and this completes the proof of the second assertion of Lemma 2. □

Lemma 2 has the immediate

Corollary. *$\lim_{n\to\infty} C_n(g)C_n^{-1}(g;B)$ exists, and is non-zero.*

To proceed further we will need the following Lemma on attraction to the Gaussian Law.

Lemma 3. (a) *Let X be a random variable with $E[X]=0$, N a positive integer, and $X_1,\ldots,X_n$ independent random variables having the same distribution as X, with normalised sum $S_N=N^{-1/2}[X_1+\ldots+X_N]$. Suppose that for some $C>0$, $\beta>0$, and all $p\geqq 1$, $\|X\|_p\leqq Cp^\beta$. Then there exists a constant $\alpha>0$ such that, for any choice of $A>0$, we have, for some $K=K(C,\beta,A)$,*

$$\|S_N\|_p\leqq Kp^{1/2}$$

for all $N\geqq 1$, $p\leqq AN^\alpha$.

(b) *Let $\{X(N), N\geqq 1\}$ be a sequence of random variables with zero mean. Let $X_1(N),\ldots,X_N(N)$ be independent random variables having the same distribution as $X(N)$. Suppose that for some $C>0$, $\beta>0$ and all $N\geqq 1$, $p\geqq 1$*

$$\|X(N)\|_p\leqq Cp^\beta,$$

and that for some $D>0$, $\delta>0$ and all $N\geqq 1$,

$$\|X(N)\|_2\leqq DN^{-\delta}.$$

Then there exists a constant $\alpha=\alpha(C,D,\beta,\delta)>0$, such that, for any choice of $A>0$, we have, for some $K=K(C,D,\beta,\delta,A)$,

$$\|S_N\|_p\leqq KN^{-\alpha}$$

for all $N\geqq 1$, $p\leqq AN^\alpha$.

Proof. We will prove (a), (b) at the same time, suppressing the dependence of X on N in (b) in the notation. If Z is any random variable, and $Z^s=Z_1-Z_2$ (Z_1, Z_2 independent, and identical in law with Z) its symmetrisation, we have

$$\|Z\|_p\leqq\|Z^s\|_p\leqq 2\|Z\|_p,$$

for any $p \geqq 1$ such that $Z \in L^p$, and, for any $N \geqq 1$,

$$[S_N(Z)]^s = S_N(Z^s).$$

It will therefore suffice to prove the lemma for symmetric X.

Choose $\alpha > 0$ with $\alpha \leqq (1+4\beta)^{-1}$ in (a), and $\alpha \leqq \min\left\{\frac{2\delta}{3}, [4+4\beta]^{-1}\right\}$ in (b). For any $M>0$ denote by $X^M = \min\{|X|, M\}\,\mathrm{sgn}(X)$ the truncation of X to the range $[-M, M]$, and note that X^M also is symmetric. As in the proof of Lemma 2, the bound on $\|X\|_p$, $p \geqq 1$, implies, for some $E = E(C, \beta)$,

$$\Pr\{|X| \geqq M\} \leqq \exp\{-EM^{1/\beta}\},$$

for $M \geqq C\exp(\beta)$. Take $M = C\exp(\beta) N^{2\alpha\beta}$. Then

$$\begin{aligned}
\|S_N(X - X^M)\|_p &\leqq N^{1/2}\|X - X^M\|_p \\
&\leqq N^{1/2}\|X\|_{2p}\{\Pr[|X| \geqq M]\}^{1/2p} \\
&\leqq N^{1/2} C(2p)^\beta \exp\left\{-\frac{EM^{1/\beta}}{2p}\right\} \\
&\leqq N^{1/2+\alpha\beta} C(2A)^\beta \exp\{-FN^\alpha\},
\end{aligned}$$

for $p \leqq AN^\alpha$ [with $F = E(2A)^{-1}(C\exp(\beta))^{1/\beta}$]. We may bound this by $K_1 p^{1/2}$ in (a), or $K_1 N^{-\alpha}$ in (b), for some K_1, since, for any exponent μ, $N^\mu \exp\{-FN^\alpha\}$ is bounded in N. Now

$$\|S_N(X)\|_p \leqq \|S_N(X - X^M)\|_p + \|S_N(X^M)\|_p,$$

so it remains to estimate $\|S_N(X^M)\|_p$.

It suffices to obtain for $\|S_N(X^M)\|_p$ a bound of the stated form in case p is an even integer. In that case we have for any $u>0$, since X^M is symmetric,

$$\begin{aligned}
\|S_N(X^M)\|_p^p &= E[\{S_N(X^M)\}^p] \\
&\leqq u^{-p} p!\, E\left[\sum_{k=0}^{\infty} \left\{\frac{S_N(X^M)}{2k!}\right\}^{2k} u^{2k}\right] \\
&= u^{-p} p!\, E\left[\sum_{m=0}^{\infty} \left\{S_N \frac{(X^M)}{m!}\right\}^{m} u^{m}\right],
\end{aligned}$$

so

$$\begin{aligned}
\|S_N(X^M)\|_p &\leqq u^{-1} p (E[\exp\{uS_N(X^M)\}])^{1/p} \\
&= u^{-1} p (E[\exp\{uN^{-1/2} X^M\}])^{N/p}. \qquad (10)
\end{aligned}$$

We will choose $u = p^{1/2}$ in (a), $u = N^{2\alpha}$ in (b), so that $u^{-1}p = p^{1/2}$ in (a), and $u^{-1}p \leqq AN^{-\alpha}$ for $p \leqq AN^\alpha$ in (b), and then verify that, for this choice of u, and $p \leqq AN^\alpha$, the remaining factor in (10) is bounded in N

$$E[\exp\{uN^{-1/2}X^M\}] = 1 + \tfrac{1}{2} E[u^2 N^{-1} (X^M)^2 \exp\{t_2 u N^{-1/2} X^M\}],$$

by Taylor expansion (2.4.2), and symmetry of X^M. The argument of the exponential is bounded by GN^{γ}, with G a constant, and $\gamma = 2\alpha\beta + \frac{\alpha}{2} - \frac{1}{2}$ in (a), $\gamma = 2\alpha\beta + 2\alpha - \frac{1}{2}$ in (b). Our choice of α ensures $\gamma \leqq 0$, so

$$E[\exp\{uN^{-1/2}X^M\}] \leqq 1 + u^2(2N)^{-1}\exp(G)E[(X^M)^2]$$
$$\leqq 1 + \frac{p}{N}\left\{\frac{u^2p^{-1}}{2}\exp(G)\,\|X\|_2^2\right\}.$$

$u^2p^{-1} = 1$ in (a), and in (b) $u^2p^{-1}\|X\|_2^2$ is bounded by HN^{ε}, with H a constant, and $\varepsilon = 3\alpha - 2\delta \leqq 0$, by choice of α. Finally we note that for any $B > 0$ the function $f(x) = (1 + Bx^{-1})^x$ is bounded on $(0, \infty)$; this completes the proof that the remaining factor in (10) is bounded in N, and so of the lemma. □

We use Lemma 3(b) to transfer (2.1.3) to the array of truncated random variables.

Lemma 4. *For some constants* $C_8 > 0$, $\beta_8 > 0$, $\tau_8 < 1$, *and all* $p \geqq 1$, $n \geqq 0$,

$$\|X(n; B) - E[X(n; B) | F^+(0)]\|_p \leqq C_8 p^{\beta_8} \tau_8^n. \tag{11}$$

Proof.

$$\|X(n; B) - E[X(n; B) | F^+(0)]\|_p$$
$$\leqq \|X(n) - E[X(n)|F^+(0)]\|_p + \|Y(n) - Y(n; B)\|_p + \|E[Y(n) - Y(n; B)|F^+(0)]\|_p$$
$$\leqq C_3 p^{\beta_3}\tau_3^n + 2\|Y(n) - Y(n; B)\|_p. \tag{12}$$

Set $N = 2^n$, $X_N = Y(0) - Y(0; B_n)$, so that, with $S_N(\cdot)$ as in Lemma 3, $Y(n) - Y(n; B) = S_N(X_N)$. We have

$$\|X_N\|_p \leqq \|X\|_p \leqq C_1 p^{\beta_1},$$

and

$$\|X_N\|_2 \leqq \|X(B_n)\|_2$$
$$\leqq \|X\|_4 (\Pr\{X \geqq b_n\})^{1/4}$$
$$\leqq \|X\|_4 \exp\left(-\frac{C_7}{2} n^{\beta_7}\right)$$
$$\leqq DN^{-\delta},$$

for any choice of $\delta > 0$, and some $D = D(\delta)$ (*recall* $\beta_7 > 1$). For N not a power of 2 set $X_N = 0$; the sequence $\{X_N, N \geqq 1\}$ then satisfies the conditions of Lemma 3(b), and, taking $A = 1$ in that Lemma, we conclude

$$\|Y(n) - Y(n; B)\|_p \leqq K2^{-n\alpha}, \tag{13}$$

for some $K > 0$, $\alpha > 0$ and all $n \geqq 0$, $p \leqq 2^{n\alpha}$. For $p \geqq 2^{n\alpha}$ we estimate

$$\|Y(n) - Y(n; B)\|_p \leqq 2^{n/2}\|X\|_p$$
$$\leqq C_1 p^{\beta_9} 2^{-n\alpha}, \tag{14}$$

with $\beta_9 = \beta_1 + 1 + (2\alpha)^{-1}$. (12)–(14) combine to give the bound (11), with $\beta_8 = \max(\beta_3, \beta_9)$ and $\tau_8 = \max(\tau_3, 2^{-\alpha})$. □

Remark. Inspection of the proofs of Lemma 3(b), 4 shows that if $\tau_3 > 2^{-1/2}$ we could have insisted on $\tau_8 = \tau_3$ in (11).

The proof of our main localisation estimate (Lemma 6 below) relies on Lemma 3(a), together with the following complement to Lemma 3(a).

Lemma 5. *Let X be a random variable satisfying the conditions of Lemma 3(a). Then for some $K>0$, and all $N \geqq 1$, $p \geqq 1$,*

$$\|S_N(X)\|_p \leqq K p^{\beta + 1/2}.$$

Proof. As in the proof of Lemma 3 we may suppose X symmetric, and p even. Then

$$\|S_N(X)\|_p^p = E[S_N(X)^p] = N^{-p/2} \Sigma(p_1 \overset{p}{\ldots} p_N)\, E[X_1^{p_1} \ldots X_N^{p_N}],$$

the sum being taken over N-tuples $(p_1, \ldots, p_N)$ of *even* integers with sum N,

$$\leqq N^{-p/2} C^p p^{\beta p} \Sigma(p_1 \overset{p}{\ldots} p_N)$$

$$\leqq N^{-p/2} C^p p^{\beta p} \Sigma(p_1 \ldots p_N)\, E[Y_1^{p_1} \ldots Y_N^{p_N}],$$

with $Y_1, \ldots, Y_N$ independent Gaussian random variables of mean 0, variance 1

$$= C^p p^{\beta p} \|S_N(Y)\|_p^p,$$

with Y Gaussian of mean 0, variance 1. But $S_N(Y) \simeq Y$ in law, so

$$\|S_N(Y)\|_p^p = \|Y\|_p^p \leqq p^{p/2},$$

and the stated bound on $\|S_N(X)\|_p$ follows. □

Lemma 6. *There are constants $\tau_{10}, \tau_{11} < 1$, such that for any choice of $C_{10} > 0$, there exists a constant C_{11} such that for all $n \geqq 0$, $m \geqq 0$, and $p \leqq C_{10} \tau_{10}^{-(n+m)}$*

$$\|X(n;B) - E[X(n;B) \,|\, F(n,m)]\|_p \leqq C_{11} p^{1/2} \tau_{11}^m. \tag{15}$$

Proof. Note that for all $n \geqq 0$, $m \geqq 0$

$$X(n;B) - E[X(n;B) \,|\, G(n+m)] \simeq S_N\{X(B_n) - E[X(B_n) \,|\, G(m)]\}. \tag{16}$$

Here $N = 2^n$, $S_N(\cdot)$ is as in Lemma 3, and $\simeq$ denotes identity in law. Similarly, for $0 \leqq m \leqq n$,

$$X(n;B) - E[X(n;B) \,|\, F^+(n-m)] \simeq S_N\{X(m;B_k) - E[X(m;B_k) \,|\, F^+(0)]\}, \tag{17}$$

with $k = n-m$, $N = 2^k$.

Lemma 1 shows that $\tau_2^{-m}\{X(B_n) - E[X(B_n) \,|\, G(m)]\}$ satisfies the hypotheses of Lemma 3(a) (with $C = 2C_2$, $\beta = \beta_2$ independent of m, n), and we conclude that for any $A > 0$, and some $\alpha_1 > 0$, $K_1 = K_1(A) > 0$, we have for all $n \geqq 0$, $m \geqq 0$ and $p \leqq A 2^{n\alpha_1}$

$$\|X(n;B) - E[X(n;B) \,|\, G(n+m)]\|_p \leqq K_1 p^{1/2} \tau_2^m. \tag{18}$$

Furthermore Lemma 5 gives the alternative bound

$$\|X(n;B) - E[X(n;B) \,|\, G(n+m)]\|_p \leqq K_2 p^{\beta_2 + 1/2} \tau_2^m, \tag{19}$$

for some $K_2>0$, valid without restriction on p. Choose $\alpha_2>0$ sufficiently small that $\tau_{12}=2^{\alpha_2\beta_2}\tau_2<1$, and for $p\leqq A2^{m\alpha_2}$ obtain from (19)

$$\|X(n;B)-E[X(n;B)|G(n+m)]\|_p\leqq K_3p^{1/2}\tau_{12}^m, \tag{20}$$

with $K_3=K_2A^{\beta_2}$. Now set $K_4=\max(K_1,K_3)$, $\tau_{13}=2^{-1/2\min(\alpha_1,\alpha_2)}<1$, and note $p\leqq A\tau_{13}^{-(m+n)}$ implies $p\leqq A2^{n\alpha_1}$ or $p\leqq A2^{m\alpha_2}$, so (18), (20) combine to give

$$\|X(n;B)-E[X(n;B)|G(n+m)]\|_p\leqq K_4p^{1/2}\tau_{12}^m, \tag{21}$$

for all $n,m\geqq 0$, $p\leqq A\tau_{13}^{-(m+n)}$.

Lemma 4 shows that $\tau_8^{-m}\{X(m;B_k)-E[X(m;B_k)|F^+(0)]\}$ satisfies the hypothesis of Lemma 3(a) (with $C=C_8$, $\beta=\beta_8$ independent of m, n), and we conclude that for any $A>0$, and some $\alpha_3>0$, $K_5=K_5(A)>0$, we have for all $n\geqq m\geqq 0$ and $p\leqq A2^{(n-m)\alpha_3}$

$$\|X(n;B)-E[X(n;B)|F^+(n-m)]\|_p\leqq K_5p^{1/2}\tau_8^m. \tag{22}$$

Lemma 5 gives also the alternative bound

$$\|X(n;B)-E[X(n;B)|F^+(n-m)]\|_p\leqq K_6p^{\beta_8+1/2}\tau_8^m, \tag{23}$$

for some $K_6>0$, valid without restriction on p. Choose $\alpha_4>0$ sufficiently small that $\tau_{14}=2^{\alpha_4\beta_4}\tau_8<1$, and for $p\leqq A2^{m\alpha_2}$ obtain from (23)

$$\|X(n;B)-E[X(n;B)|F^+(n-m)]\|_p\leqq K_7p^{1/2}\tau_{14}^m, \tag{24}$$

with $K_7=K_6A^{\beta_8}$. Now set $K_8=\max(K_5,K_7)$, $\tau_{15}=2^{-1/3\min(\alpha_3,\alpha_4)}<1$, and note $p\leqq A\tau_{15}^{-(m+n)}$ implies $p\leqq A2^{(n-m)\alpha_3}$ or $p\leqq A2^{m\alpha_4}$, so (22), (24) combine to give, for all $n\geqq m\geqq 0$, and $p\leqq A\tau_{15}^{-(m+n)}$,

$$\|X(n;B)-E[X(n;B)|F^+(n-m)]\|_p\leqq K_8p^{1/2}\tau_{14}^m. \tag{25}$$

Suppose $X\in L^p$ is a random variable, and F_1, F_2 σ-algebras such that $F_1=(F_1\cap F_2)\vee F_3$, with F_3 independent of F_2. Then

$$\begin{aligned}E[E[X|F_2]|F_1]&=E[E[X|F_2]|(F_1\cap F_2)\vee F_3]\\&=E[E[X|F_2]|F_1\cap F_2]\\&=E[X|F_1\cap F_2],\end{aligned}$$

and hence

$$\begin{aligned}\|X-E[X|F_1\cap F_2]\|_p&\leqq\|X-E[X|F_1]\|_p+\|E[X|F_1]-E[X|F_1\cap F_2]\|_p\\&=\|X-E[X|F_1]\|_p+\|E[X-E[X|F_2]|F_1]\|_p\\&\leqq\|X-E[X|F_1]\|_p+\|X-E[X|F_2]\|_p,\end{aligned}$$

since $E[\cdot|F_1]$ is a contraction on L^p. This remark is applicable to $X=X(n;B)$, $F_1=G(n+m)$, $F_2=F^+(n-m)$ for $0\leqq m\leqq n$ [take $F_3=G(n-m)$], and allows us to combine (21), (25) in the form (15) [with $C_{10}=A$, $\tau_{10}=\max(\tau_{13},\tau_{15})$, $C_{11}=K_4+K_8$, $\tau_{11}=\max(\tau_{12},\tau_{14})$]. □

Lemma 6 yields immediately the following estimate for the martingale differences $X(n,m;B)$

$$\|X(n,m;B)\|_p\leqq C_{12}p^{1/2}\tau_{11}^m, \tag{26}$$

with $C_{12}=C_{11}\tau_{11}^{-1}$; (26) is valid for $n\geqq 0$, $m\geqq 1$, $p\leqq C_{10}\tau_{10}^{-(n+m)}$.

Since $\Gamma(X(n,m;B))$ is Gaussian with variance $\|X(n,m;B)\|_2^2$,

$$\begin{aligned}\|\Gamma(X(n,m;B))\|_p &\leqq p^{1/2}\|X(n,m;B)\|_2\\ &\leqq p^{1/2}C_{12}2^{1/2}\tau_{11}^m .\end{aligned} \tag{27}$$

(26), (27) give

$$\|Z(n,m;B)\|_p\leqq C_{13}p^{1/2}\tau_{11}^m, \tag{28}$$

with $C_{13}=C_{12}(1+2^{1/2})$, under the same conditions on n, m, p as in (26).

2.6. *Inductive Expansion*

Write $G=F(-1)$, so that $G(\infty)=\bigvee_{n\geqq -1} F(n)$, the σ-algebras $F(n)$, $n\geqq -1$, being independent. For f a random variable measurable with respect to the σ-algebra $G(\infty)\vee\Gamma(G(\infty))$ we define the *location* of f, $\mathrm{loc}(f)$, a subset of the integers $n\geqq -1$, as the smallest subset S for which $f\in G(S)\vee\Gamma(G(S))$. If $\mathrm{loc}(f)$ is finite, f may be said to be *localised*. Note that if f, h have disjoint locations, then f, h are independent. The martingale differences $X(n,m)$, $Z(n,m)$ are localised, with

$$\mathrm{loc}(X(n,m))=\mathrm{loc}(Z(n,m))\subset I(n,m)=[\max(n-m,-1),n+m-1].$$

The inductive expansion will be defined by a choice of a set $J=\{j(n,m);\ n\geqq 0, m\geqq 1\}$ of positive integers. We will eventually make a specific choice of J, but for the moment the choice of J will be left open subject only to the condition

J1. $j(n,m)\geqq 3$ for all $n\geqq 0, m\geqq 1$.

For each (n,m) denote by $t(n,m)$ a random variable identical in law with $t_{j(n,m)}$ (cf. 2.4); the random variables $t(n,m)$ are to be independent of each other, and of all the random variables considered hitherto. If D is a subset of $(n,m): n\geqq 0$, $m\geqq 1$, $s: D\to\mathbb{Z}^+$ will be called J-admissible if $s(n,m)\leqq j(n,m)$ for all $(n,m)\in D$. For such a map s, we define $D_1=\{(n,m): s(n,m)<j(n,m)\}$, $D_2=\{(n,m): s(n,m)=j(n,m)\}$, and the random variable $P(s,g;B)$

$$\begin{aligned}P(s,g;B)=&\prod_{(n,m)\in D_1}\frac{[-gZ(n,m;B)]^{s(n,m)}}{s(n,m)!}\\ &\cdot\prod_{(n,m)\in D_2}\frac{[-\mathrm{g}Z(\mathrm{n},\mathrm{m};B)]^{j(n,m)}}{j(n,m)!}\exp\{-gt(n,m)Z(n,m;B)\}.\end{aligned} \tag{1}$$

For $k\geqq 0$ write $g_k=g2^{-k/2}$. The inductive expansion is an expansion procedure defined on expressions of the form

$$E\left[R\exp\left\{-g_k\sum_{(n,m)\in D}Z(n,m;B_k)\right\}\right], \tag{2}$$

with R a localised random variable, and $D\subset\mathbb{Z}^+\times\mathbb{Z}^+$ such that

$$\max\{n:(n,m)\in D\}<\infty. \tag{3}$$

We will use the letter J also to denote the expansion operation itself; thus $J\{\ldots\}$ indicates that the expression within parentheses is to be expanded according to the rules we state below (so the operation J does not change the value of an expression, but only the form in which it is expressed).

The definition of the inductive expansion is by induction on

$$d = 2\max\{n : (n,m)\in D\} - \min\{\max(n-m, -1) : (n,m)\in D\}.$$

By (3) d is initially finite. Given an expression of the form (2) write

$$D_1 = \{(n,m)\in D : I(n,m)\cap(\mathrm{loc}(R)\cup\{-1,0\}) \neq \emptyset\},$$
$$D_2 = \{(n,m)\in D : I(n,m)\cap(\mathrm{loc}(R)\cup\{-1,0\}) = \emptyset\}.$$

If both D_1, $D_2 = \emptyset$, so $D=\emptyset$, the expansion *terminates*

$$J\{E[R]\} = E[R]. \tag{4}$$

If $D_1 = \emptyset$ the next expansion step is a *factorisation*

$$J\left\{E\left[R\exp\left\{-g_k \sum_{(n,m)\in D} Z(n,m;B_k)\right\}\right]\right\}$$
$$= E[R]\left[J\left\{E\left[\exp\left\{-g_{k+1} \sum_{(n,m)\in D'} Z(n,m\colon B_{k+1})\right\}\right]\right\}\right]^2, \tag{5}$$

with $D' = \{(n,m) : (n+1,m)\in D\}$.

If $D_1 \neq \emptyset$ the next expansion step is a *Taylor expansion*

$$J\left\{E\left[R\exp\left\{-g_k \sum_{(n,m)\in D} Z(n,m;B_k)\right\}\right]\right\}$$
$$= \sum_s J\left\{E\left[RP(s,g_k;B_k)\exp\left\{-g_k \sum_{(n,m)\in D'} Z(n,m;B_k)\right\}\right]\right\}, \tag{6}$$

with $D' = D_2$. The sum in (6) is over all J-admissible maps s defined on D_1.

Note that in both (5), (6), $d(D') \leqq d(D) - 1$, so that the expansion process terminates in a finite number of steps.

In 2.7 we obtain a majorant for the series given by the inductive expansion; the interchange of the integration operation E with $\sum_s$ in the Taylor expansion step will then be seen to be justified by dominated convergence.

We define the *weight* of term $P(s,g;B)$ appearing in the inductive expansion by

$$W = \sum_{n,m} s(n,m).$$

Condition J1, and the construction of the random variables $Z(n,m;B)$, ensure that $E[P]=0$ if $w=1$ or 2; terms in the expansion having $E[P]$ as a factor therefore cancel. This cancellation is the raison d'être of the expansion.

2.7. *Majorisation*

In the majorisation argument of this section we are concerned with sums $\sum a_r$ of terms which are not known at the outset to converge. If $\sum a_r$, $\sum b_s$ are two such sums, we write $\sum a_r \subset \sum b_s$ if each term of the sum on the left appears precisely once among the terms in the sum on the right. If $\sum a_r$ is a sum, and $C \in [0, +\infty]$ a positive extended real number, we write $\sum a_r \prec C$ if $\sum |a_r| \leqq C$. Note that if $\sum a_r \subset \sum b_s$, and $\sum b_s \prec C$ then $\sum a_r \prec C$. If A^n, $n \geqq 1$, is a sequence of sums with $A^n \subset A^{n+1}$ for all n, we write $A = \lim_{n\to\infty} A^n$ for the sum whose set of terms is the union of the sets of terms of the A^n; the notation is justified by the observation that if $A = \lim_{n\to\infty} A^n$, and $A \prec C$ with $C < +\infty$, then the series A^n, $n \geqq 1$, A are all absolutely convergent and the limit can be construed as a numerical rather than a formal limit.

As in the outline of 2.2 we choose arbitrarily a Hölder index p, $1 < p < \infty$, and denote by q the conjugate index. R will denote a localised random variable, with $R \in L^q$; $r = max\{n : n \in \mathrm{loc}(R) \cup \{0\}\}$. We write, for $n \geqq 0$,

$$\begin{aligned} &E[R\exp(-gS(n;B))]\,C_n(g;B) \\ &\quad = E[R\exp(-gW(n;B))] \\ &\quad = E\left[R\exp\left(-g\sum_{k=0}^{n}\sum_{m=1}^{\infty} Z(k,m;B)\right)\right]. \end{aligned} \tag{1}$$

The right side of (1) is in the domain of the inductive expansion; denote it by $F_n(g, R; B)$. In this section, B, the truncation sequence, is fixed: $b_n = C_4(n+1)^{\beta_4}$, with C_4, β_4 as in B1. Note that for any $\theta > 1$ we then have, for all $n \geqq 0$, $k \geqq 0$,

$$b_{n+k}\theta^{-k} \leqq K(n+1)^{\beta_4}, \tag{2}$$

with $K = K(\theta) = \sup_{k \geqq 0, n \geqq 0} \left(1 + \frac{k}{n+1}\right)^{\beta_4} \theta^{-k} < \infty$.

We fix also the termination sequence J for the inductive expansion

$$j(n,m) = L\varrho^{n+m}. \tag{3}$$

Here $L > 0$ may be chosen arbitrarily, and $\varrho > 1$ will be specified later.

The main result of this section is

Lemma 7. *There exists a constant $\bar{g} > 0$, and positive functions $\Phi(g)$, $\Psi(g)$ defined on $[0, \bar{g})$, with $\Phi(g) \to 1$ as $g \to 0$, such that*

$$J\{F_n(g, R; B)\} \prec \|R\|_q \Phi(g)^r \Psi(g),$$

for $0 \leqq g < \bar{g}$, and all $n \geqq 0$.

Proof of Lemma 7. Define the connected sum

$$\mathrm{Co}(g, R; B) = \sum_s E[RP(s, g; B)]\,; \tag{4}$$

the sum in (4) is on J-admissible maps s such that the set

$$\mathrm{loc}(R) \cup \{-1, 0\} \cup \left(\bigcup_{(n,m):\, s(n,m) \neq 0} I(n, m)\right)$$

is connected (i.e. is a set of consecutive integers). If $R = 1$ we write simply $\mathrm{Co}(g; B)$;

$$\mathrm{Co}(g; B) = 1 + \sum_s E[P(s, g; B)].$$

The sum over s being over maps of weight $w \geqq 3$, satisfying the connectedness condition.

The following relations are immediate from the definition of the inductive expansion:

For any $n \geqq 0$, $J\{F_n(g,R;B)\} \subset J\{F_{n+1}(g,R;B)\}$, so $\lim_{n\to\infty} J\{F_n(g,R;B)\} = J\{F(g,R;B)\}$ (by definition) exists as a formal sum. If $R=1$ we write $J\{F(g;B)\} = J\{F(g,1;B)\}$.

$$J\{F(g,R;B)\} \subset \mathrm{Co}(g,R;B)(J\{F(g_1;B_1)\})^2, \tag{6}$$

and so by iteration

$$J\{F(g,R;B)\} \subset \mathrm{Co}(g,R;B) \prod_{k=1}^{\infty} [\mathrm{Co}(g_k;B_k)]^{2^k}; \tag{7}$$

in (7) the infinite product of the right is to be understood as the formal sum of products of finite total weight formed by multiplying the indicated factors.

We will use Hölder's inequality to estimate the terms in $\mathrm{Co}(g,R;B)$. Let $P(s,g;B)$ be a term of weight w. We factor

$$P(s,g;B) = \prod_{m \geqq 1} (P_m^0 P_m^1) \prod_{n \geqq 0} P_n^2, \tag{8}$$

with

$$P_m^i = \prod_{n:\left[\frac{n}{2m}\right] \equiv i (\mathrm{mod}\, 2)} \frac{[-gZ(n,m;B)]^{s(n,m)}}{s(n,m)!}, \tag{9}$$

$$P_n^2 = \prod_{m:s(n,m)=j(n,m)} \exp\{-gt(n,m)Z(n,m;B)\}, \tag{10}$$

and set $p(n) = 6(n+1)^2$, so $3\sum_{n=0}^{\infty} p(n)^{-1} < 1$. Hölder's inequality then gives

$$\begin{aligned} \|RP\|_1 &\leqq \|R\|_q \|P\|_p \\ &\leqq \|R\|_q \prod_{m \geqq 1} (\|P_m^0\|_{pp(m)} \|P_m^1\|_{pp(m)}) \prod_{n \geqq 0} \|P_n^2\|_{pp(n)}. \end{aligned} \tag{11}$$

We tackle first the final product in (11), and show that, for any $\varepsilon > 0$, and all $g \leqq 1$, there exists a constant $A = A(\varepsilon)$ such that

$$\prod_{n \geqq 0} \|P_n^2\|_{pp(n)} \leqq A(1+\varepsilon)^w. \tag{12}$$

Write $M(n) = \{m : s(n,m) = j(n,m)\}$, $c(n) =$ number of elements in $M(n)$, $w(n) = \sum_m s(n,m)$. Note that $w = \sum_n w(n)$, and $w(n) \geqq j(n, c(n))$.

$$\|P_n^2\|_{pp(n)} \leqq \prod_{m \in M(n)} \|\exp\{-gt(n,m)Z(n,m;B)\}\|_{pp(n)c(n)}. \tag{13}$$

Now if Z is a bounded random variable with $E[Z]=0$, $h(u) = E[\exp(uZ)]$ is increasing for $u \geqq 0$; $h'(u) = E[Z(\exp(uZ)-1)] = E\left[Z \sinh\left(u\frac{Z}{2}\right)\exp\left(-\frac{uZ}{2}\right)\right] \geqq 0$.

Hence

$$\begin{aligned}
&\|\exp\{-gt(n,m)Z(n,m;B)\}\|_{pp(n)c(n)}^{pp(n)c(n)}\\
&\quad=E[\exp\{-gpp(n)c(n)t(n,m)X(n,m;B)\}]\\
&\quad=E[E[\exp\{-gpp(n)c(n)t(n,m)X(n,m;B)\}\,|\,t(n,m)]]\\
&\quad\leqq E[\exp\{-gpp(n)c(n)X(n,m;B)\}] \qquad (14)
\end{aligned}$$

[recall $0\leqq t(n,m)\leqq 1$]. Replacing $X(n,m;B)$ by its expression as a martingale (2.2.7), and using successively the Cauchy-Schwarz and Jensen inequalities, we bound (14) by

$$\begin{aligned}
&(E[\exp\{-2gpp(n)c(n)E[Y(n;B)|F(n,m)]\})^{1/2}\\
&\qquad\cdot(E[\exp\{2gpp(n)c(n)E[Y(n;B)|F(n,m-1)]\}])^{1/2}\\
&\quad\leqq(E[\exp\{-2gpp(n)c(n)Y(n;B)\}])^{1/2}\\
&\qquad\cdot(E[\exp\{2gpp(n)c(n)Y(n;B)\}])^{1/2}. \qquad (15)
\end{aligned}$$

(13)–(15) combine to give

$$\begin{aligned}
\|P_n^2\|_{pp(n)}&\leqq(E[\exp\{-2gpp(n)c(n)Y(n;B)\}])^{(2pp(n))^{-1}}\\
&\qquad\cdot(E[\exp\{2gpp(n)c(n)Y(n;B)\}])^{(2pp(n))^{-1}}. \qquad (16)
\end{aligned}$$

We focus on the estimation of the second factor in (16), since it is here that the introduction of the truncation proves to be essential.

$$\begin{aligned}
&E[\exp\{2gpp(n)c(n)Y(n;B)\}]\\
&\quad=(E[\exp\{2gpp(n)c(n)2^{-n/2}Y(0;B_n)\}])^{2^n}\\
&\quad=(1+2g^2p^2p(n)^2c(n)^2 2^{-n}E[Y(0;B_n)^2\exp\{2gpp(n)c(n)2^{-n/2}t_2Y(0;B_n)\}])^{2^n}\\
&\quad\leqq\exp\{2g^2p^2p(n)^2c(n)^2E[X^2]\exp[2gpp(n)c(n)2^{-n/2}b_n]\}. \qquad (17)
\end{aligned}$$

If $p(n)c(n)b_n\leqq 2^{n/2}$, (17) is bounded, for $g\leqq 1$, by

$$\exp\{A_1p(n)^2c(n)^2\}, \qquad (18)$$

for some constant A_1. If $p(n)c(n)b_n\geqq 2^{n/2}$, we have, for $g\leqq 1$, the bound

$$\begin{aligned}
E[\exp\{2gpp(n)c(n)Y(n;B)\}]&\leqq\exp\{2pp(n)c(n)2^{n/2}b_n\}\\
&\leqq\exp\{A_2p(n)^2c(n)^2b_n^2\}, \qquad (19)
\end{aligned}$$

for some constant A_2, so that in any case, for some A_3,

$$E[\exp\{2gpp(n)c(n)Y(n;B)\}]\leqq\exp\{A_3p(n)^2c(n)^2b_n^2\}. \qquad (20)$$

For all but a finite number of pairs (n,c)

$$A_3p(n)^2b_n^2\leqq(\log(1+\varepsilon))L\varrho^{n+\varepsilon}. \qquad (21)$$

Denote by N_0 the number of exceptional pairs, and by A_4 the maximum of the left side of (21) taken over the exceptional pairs. Then we have, with $A_5=\exp\{N_0A_4\}$,

$$\prod_{n\geqq 0}E[\exp\{2gpp(n)c(n)Y(n;B)\}]\leqq A_5(1+\varepsilon)^w \qquad (22)$$

$\left[\text{recall } w=\sum_n w(n), \text{ and } w(n)\geqq j(n,c(n))=L\varrho^{n+c(n)}\right]$.

A similar estimation of the first factor in (16) gives

$$\prod_{n\geqq 0} E[\exp\{-2gpp(n)c(n)Y(n;B)\}]\leqq A_6(1+\varepsilon)^w, \tag{23}$$

the constant A_6, unlike A_5, being independent of the choice of truncation sequence B.

(16), (22), (23) combine to give (12).

We come now to the estimation of the factors $\|P^i_m\|_{pp(m)}$ in (11). Write

$$P^i_m=\prod_{r\geqq 0} P^i_{m,r} \qquad (i=0,1), \tag{24}$$

with $P^i_{m,r}$ the product of those factors in P^i_m for which $\left[\frac{n}{2m}\right]=2r+i$. Note that if n, n' are such that $\left|\left[\frac{n}{2m}\right]-\left[\frac{n'}{2m}\right]\right|\geqq 2$, then the localization intervals $I(n,m)$, $I(n',m)$ are disjoint; consequently the factors $P^i_{m,r}$ of P^i_m are mutually independent, and we have the factorisation

$$\|P^i_m\|_{pp(m)}=\prod_{r\geqq 0}\|P^i_{m,r}\|_{pp(m)} \qquad (i=0,1); \tag{25}$$

(25) is an essential improvement over Hölder's inequality. Finally we use Hölder's inequality once again to give

$$\|P^i_{m,r}\|_{pp(m)}\leqq \prod_{n:\left[\frac{n}{2m}\right]=2r+i}\frac{g^{s(n,m)}}{s(n,m)!}\|Z(n,m;B)\|^{s(n,m)}_{2mpp(m)s(n,m)}. \tag{26}$$

Fix the choice of ϱ in (3) so $1<\varrho<\tau_{10}^{-1}$, with τ_{10} the constant appearing in Lemma 2.5.6.

Then

$$\begin{aligned}2mpp(m)s(n,m)&\leqq 2mpp(m)j(n,m)\\ &\leqq 2mpp(m)L\varrho^{n+m}\\ &\leqq D\tau_{10}^{-(n+m)},\end{aligned} \tag{27}$$

for some constant D. We choose $C_{10}=D$ in Lemma 6 so that the Hölder indices which appear in (26) lie in the domain of validity of 2.5.28.

Write $l=\sum_{n,m} ms(n,m)$, $\tilde{w}(m)=\sum_n s(n,m)$, $C_{14}=C_{13}(1+\varepsilon)(2p)^{1/2}$. Then the bound given by (11), (12), (25), (26), 2.5.28 is

$$\|RP\|_1\leqq A\|R\|_q\tau_{11}^l\prod_m[mp(m)]^{1/2\tilde{w}(m)}\prod_{n,m}(C_{14}g)^{s(n,m)}\frac{[s(n,m)]^{1/2s(n,m)}}{s(n,m)!}. \tag{28}$$

Choose $\eta>0$ so $\tau_{16}=(1+\eta)\tau_{11}<1$. For $m\geqq m_0(\eta)$

$$1/2\log(mp(m))\leqq m\log(1+\eta).$$

Write $C_{15}=[m_0p(m_0)]^{1/2}$, $C_{16}=C_{14}C_{15}$, so that

$$\prod_m[mp(m)]^{1/2\tilde{w}(m)}\leqq C_{15}^w(1+\eta)^l,$$

and we obtain the final form of our bound for the individual terms of the connected sum

$$\|RP\|_1 \leqq A\|R\|_q \tau_{16}^l \prod_{n,m} (C_{16}g)^{s(n,m)} \frac{[s(n,m)]^{1/2\,s(n,m)}}{s(n,m)!}. \tag{29}$$

We will show that, for sufficiently small g, the sum over s of the right side of (29) is finite. We will make no further use of the condition that s be J-admissible, and, regarding the remaining condition on the maps s in (4) (the connectedness condition), we will use only the fact that it implies

$$s(n,m)=0 \quad \text{for} \quad n>r+2l. \tag{30}$$

Write $\tau_{17}=\tau_{16}^{1/2}<1$. We will first fix l and estimate

$$K(l)=\sum_s \prod_{n,m} (C_{16}g)^{s(n,m)} \frac{[s(n,m)]^{1/2\,s(n,m)}}{s(n,m)!} \tau_{17}^{m\,s(n,m)}, \tag{31}$$

the sum being taken over all maps $s:[0,r+2l]\times\mathbb{Z}^+\to\mathbb{Z}^+$, and then show

$$\sum_l A\|R\|_q \tau_{17}^l K(l)<\infty, \tag{32}$$

for sufficiently small g.

Let $Y(n)$, $n\geqq 0$, be independent Gaussian random variables of mean 0, variance 1. For some $c>0$, and all $p\geqq 1$,

$$\|Y(n)\|_p \geqq cp^{1/2}, \tag{33}$$

so

$$\begin{aligned} K(l) &\leqq \sum_s \prod_{n,m} (C_{16}gc^{-1})^{s(n,m)} \frac{E[(|Y(n)|\tau_{17}^m)^{s(n,m)}]}{s(n,m)!} \\ &\leqq \sum_s E\left[\prod_{n,m} \frac{[C_{16}gc^{-1}|Y(n)|\tau_{17}^m]^{s(n,m)}}{s(n,m)!}\right] \\ &= E\left[\exp\left\{\sum_{n,m} C_{16}gc^{-1}|Y(n)|\tau_{17}^m\right\}\right] \\ &= E\left[\exp\left\{\sum_{n=0}^{r+2l} C_{16}gc^{-1}\tau_{17}(1-\tau_{17})^{-1}|Y(n)|\right\}\right] \\ &= [H(C_{17}g)]^{r+2l+1}, \end{aligned} \tag{34}$$

with $H(x)=E[\exp\{x|Y|\}]$, Y Gaussian 0, 1, $C_{17}=C_{16}c^{-1}\tau_{17}(1-\tau_{17})^{-1}$. Since $H(x)\to 1$ as $x\to 0$, $\tau_{17}[H(C_{17}g)]^2<1$ for g in some interval $[0,\bar{g})$, $0<\bar{g}\leqq 1$, and for g in this interval

$$\mathrm{Co}(g,R;B) \prec \|R\|_q [H(C_{17}g)]^{r+1} A(1-\tau_{17}[H(C_{17}g)]^2)^{-1}. \tag{35}$$

For $R=1$, (29) gives

$$\mathrm{Co}(g;B) \prec 1+A(g), \tag{36}$$

with $A(g)=\sum_{w\geqq 3} A_w g^w$ a power series with positive coefficients, convergent for $g<\bar{g}$ [by (35), with $R=1$]. We claim that we can arrange to have

$$\mathrm{Co}(g_k; B_k) \prec 1+A(g_k), \tag{37}$$

for any $k\geqq 0$. Indeed the estimates used to derive (36) were uniform in B over the set B_k, $k\geqq 0$, as remarked in 2.5, with the sole exception being the estimation of the second factor in (16). At that point in the argument we have now to replace $b_n g$ by $b_{n+k} g_k = (b_{n+k} 2^{-k/2}) g$; here (2), with $\theta = 2^{1/2}$, gives the necessary bound uniform in k, and (37) follows.

At this point we must interpolate the simple

Lemma 8. *If* $f(z)=\sum_{n=0}^{\infty} f_n z^n$ *is a power series with coefficients* $f_n \geqq 0$, $f_0=1$, $f_1=f_2=0$, *convergent for* $|z|<R$, *then the series*

$$J(z)=\prod_{k=0}^{\infty} [f(z2^{-k/2})]^{2k}$$

also converges for $|z|<R$.

The proof of Lemma 7 is completed by combining (7), (35), (37) and Lemma 8. □

Proof of Lemma 8. For some r, $0<r\leqq R$, $\log[f(z)]$ is holomorphic in $|z|<r$, and in that circle satisfies, for some $C>0$, the bound

$$|\log[f(z)]| \leqq C|z|^3 .$$

The series $\sum_{k=0}^{\infty} 2^k \log[f(z2^{-k/2})]$ is therefore for $|z|<r$ majorised by

$$\sum_{k=0}^{\infty} C 2^{-k/2} |z|^3 < \infty ,$$

and it follows that $J(z)$ is holomorphic in $|z|<r$. In this circle $J(z)$ satisfies the functional equation

$$J(z)=f(z)\,[J(z2^{-1/2})]^2 . \tag{38}$$

But the right side of (38) is holomorphic in $|z|<\min\{R, 2^{1/2}r\}$, so (38) can be used to extend J holomorphically to this possibly larger circle. For sufficiently large N, $R=\min\{R, 2^{N/2} r\}$, so by iteration of the extension argument we conclude J is holomorphic in $|z|<R$. □

2.8. Existence of the Limit

In this section B and J are fixed as in 2.7; p is a fixed Holder index, $1<p<\infty$, with conjugate index q, and $R\in L^q(\Omega, G(m), \mu)$ as in 2.2. We are to prove 2.2.5, 2.2.6. In view of the corollary to Lemma 2, it suffices to prove the statements 2.2.5′, 2.2.6′ obtained by replacing $C_n(g)$ by $C_n(g; B)$ in 2.2.5, 2.2.6.

Let t_1 be an independent random variable with uniform distribution on $[0, 1]$, as in 2.4, and define for $k \geqq 0$, $s > m$

$$L(k) = \sum_{j=0}^{k-1} [X(j) - X(j; B)] + t_1[X(k) - X(k; B)], \tag{1}$$

$$H_1(k) = \exp\{-gL(k)\}, \tag{2}$$

$$H_2(k) = X(k) - X(k; B), \tag{3}$$

$$H(k) = H_1(k) H_2(k), \tag{4}$$

$$H(k, m) = E[H(k) | G(m)], \tag{5}$$

$$H(k, s) = E[H(k) | G(s)] - E[H(k) | G(s-1)]. \tag{6}$$

Note that $L(k) \geqq 0$, so $H_1(k) \leqq 1$, and that

$$H(k) = \sum_{s \geqq m} H(k, s). \tag{7}$$

We have the identity

$$\begin{aligned} &E[R \exp\{-gS(n)\}] C_n(g; B)^{-1} \\ &\quad = E[R \exp\{-gS(n); B)\}] C_n(g; B)^{-1} \\ &\qquad - g \sum_{k=0}^{n} E[RH(k) \exp\{-gS(n; B)\}] C_n(g; B)^{-1} \\ &\quad = F_n(g, R; B) - g \sum_{k=0}^{n} F_n(g, RH(k); B) \\ &\quad = F_n(g, R; B) - g \sum_{k=0}^{n} \sum_{s=m}^{\infty} F_n(g, RH(k, s); B) \\ &\quad = J\{F_n(g, R; B)\} - g \sum_{k=0}^{n} \sum_{s=m}^{\infty} J\{F_n(g, RH(k, s); B)\}. \end{aligned} \tag{8}$$

We choose l, $1 < l < q$, and define r by $l^{-1} = q^{-1} + r^{-1}$ so

$$\|RH(k, s)\|_l \leqq \|R\|_q \|H(k, s)\|_r. \tag{9}$$

We may then use Lemma 7, with the index q of the statement of that lemma replaced by l, to obtain, for sufficiently small g, the following majorant for (8)

$$\Psi(g) \|R\|_q \left\{ \Phi(g)^m + g \sum_{k=0}^{\infty} \sum_{s \geqq m} \Phi(g)^2 \|H(k, s)\|_r \right\}. \tag{10}$$

We will prove that (10) is finite, for sufficiently small g, and this establishes 2.2.6′ [with $K(m, n, k)$ the sum of the appropriate subset of terms of the majorant]. If $R = 1$ we may write the majorant in the form $1 + gA(g)$, with $A(g)$ bounded as $g \to 0$, and hence obtain 2.2.5′.

By applying Lemma 3(b) as in the derivation of 2.5.13, but taking this time not $A = 1$ but $A = 2r$, we obtain

$$\|Y(k) - Y(k; B)\|_{2r} \leqq K_1 \varrho^k, \tag{11}$$

for some constants K_1, ϱ, $\varrho<1$. 2.5.6 gives, for some K_2,

$$E[X(k)-X(k;B)]\leqq K_2\varrho^k, \tag{12}$$

so

$$\|X(k)-X(k;B)\|_{2r}\leqq K_3\varrho^k, \tag{13}$$

with $K_3=K_1+K_2$. Lemma 6, with $C_{10}=2r$, gives, for $s>k$

$$\|X(k)-E[X(k)|G(s-1)]\|_{2r}\leqq K_4\tau^{s-k}, \tag{14}$$

$$\|X(k;B)-E[X(k;B)|G(s-1)]\|_{2r}\leqq K_4\tau^{s-k} \tag{15}$$

for some K_4, and $\tau<1$ [since $G(s-1)\supset F(k,s-k-1)$].

For $s\leqq\max(k,m)$ we estimate

$$\|H(k,s)\|_r\leqq\|H(k)\|_r\leqq\|H_2(k)\|_r\leqq K_3\varrho^k. \tag{16}$$

For $s>\max(k,m)$

$$\begin{aligned}\|H(k,s)\|_r&\leqq\|H(k)-E[H(k)|G(s-1)]\|_r\\&\leqq 2\|H_1(k)H_2(k)-E[H_1(k)|G(s-1)]\,E[H_2(k)|G(s-1)]\|_r\\&\quad\text{(cf. the first remark in the proof of Lemma 1)}\\&\leqq 2\|H_1(k)\,[H_2(k)-E[H_2(k)|G(s-1)]]\|_r\\&\quad+2\|(H_1(k)-E[H_1(k)|G(s-1)])\,E[H_2(k)|G(s-1)]\|_r\\&\leqq 2\|H_2(k)\|_r^{1/2}\,\|H_2(k)-E[H_2(k)|G(s-1)]\|_r^{1/2}\\&\quad+2\|H_1(k)-E[H_1(k)|G(s-1)]\|_{2r}\|H_2(k)\|_{2r}.\end{aligned} \tag{17}$$

Now

$$\begin{aligned}&\|H_1(k)-E[H_1(k)|G(s-1)]\|_{2r}\\&\quad\leqq 2\|\exp\{-gL(k)\}-\exp\{-gE[L(k)|G(s-1)]\}\|_{2r}\\&\quad\leqq 2g\|L(k)-E[L(k)|G(s-1)]\|_{2r}\\&\qquad(\text{since } |e^{-x}-e^{-y}|\leqq|x-y|, \text{ for } x,y\geqq 0)\\&\quad\leqq 4gK_4(k+1)\tau^{s-k},\end{aligned} \tag{18}$$

by (1), (14), (15).

(13)–(15), (17), (18) give the following bound for the sum in (10)

$$\sum_{k=0}^{\infty}\{\Phi(g)^{\max(k,m)}(k+1)K_3\varrho^k$$
$$+\Phi(g)^k[4K_3^{1/2}K_4^{1/2}\varrho^{k/2}+8gK_3K_4(k+1)\varrho^k]\sum_{s>k}(\tau^{1/2}\Phi(g))^{s-k}\}$$

which is finite, provided g is sufficiently small that $\Phi(g)<\min(\varrho^{-1/2},\tau^{-1/2})$.

The proof of Theorem 1 is complete.

3. Construction of the Measure for the Polymer Problem

33.1. *Mixtures of Tied-Down Gaussian Processes*

Let $(X, \mathfrak{X})$ be a measure space, $C: X \times X \to \mathbb{R}$ a measurable covariance on X, $X_0 = \{\xi : C(\xi, \xi) = 0\}$. Denote by $\mathfrak{H}$ the reproducing kernel Hilbert space defined by C; $\mathfrak{H}$ is the Hausdorff completion of the pre-Hilbert space whose elements are functions on X of the form

$$f(\cdot) = \sum_{i=1}^{N} c_i C(\xi_i, \cdot),$$

with $c_i \in \mathbb{R}$, $\xi_i \in X$, $1 \leqq i \leqq N$, and whose inner product is the polarisation of

$$\langle f, f \rangle = \sum_{i,j=1}^{N} c_i c_j C(\xi_i, \xi_j)$$

[15]. For $\xi \in X$, denote by $e(\xi) \in \mathfrak{H}$ the vector corresponding to the element $C(\xi, \cdot)$ of the pre Hilbert space. The map

$$e : \xi \to e(\xi)$$

of $(X, \mathfrak{X})$ into $(\mathfrak{H}, B)$ is measurable; here B is the Borel σ-algebra of the weak topology of $\mathfrak{H}$. The map

$$i : f \to \langle e(\cdot), f \rangle$$

is an injection of $\mathfrak{H}$ into the vector space of measurable functions on X, zero on X_0.

We will suppose $\mathfrak{H}$ separable. Let $\{e_n, n \geqq 1\}$ be an orthonormal basis for $\mathfrak{H}$, and $\{y_n, n \geqq 1\}$ independent Gaussian random variables of mean 0, variance 1, realised on a probability measure space (Ω, M, μ). For $e \in \mathfrak{H}$ define

$$\varphi(e) = \sum_n \langle e_n, e \rangle y_n$$

[convergence in $L^2(\Omega, M, \mu)$]; then

$$\varphi : \mathfrak{H} \to L^2(\Omega, M, \mu)$$

is measurable [relative to the Borel σ-algebras of the weak topologies of $\mathfrak{H}$, $L^2(\Omega, M, \mu)$]. Hence also the composed map $x = \varphi \circ e$

$$x : X \to L^2(\Omega, M, \mu)$$

is measurable.

Now let λ be a positive σ-finite measure on $(X, \mathfrak{X})$. If $g \in L^2(X \times \Omega, \lambda \times \mu)$, we may choose a measurable function $g(\xi, \omega)$ representing g such that

$$\hat{g} : \xi \to g(\xi, \cdot)$$

is a measurable map of X into $L^2(\Omega, M, \mu)$, and

$$\|g\|_2^2 = \int \|\hat{g}(\xi)\|^2 d\lambda(\xi).$$

Then

$$L(g) = \int \langle x(\xi), \hat{g}(\xi) \rangle c(\xi) d\lambda(\xi) \qquad (1)$$

defines a continuous linear functional $L(\cdot)$ on $L^2(X\times\Omega,\lambda\times\mu)$, for any measurable function $c(\cdot)$ such that

$$\int C(\xi,\xi)c(\xi)^2\,d\lambda(\xi)<\infty.$$

Since λ is σ-finite we may choose such a $c(\cdot)$ with $c(\xi)\neq 0$ for all ξ. By the Riesz representation theorem there exists a measurable function $k(\xi,\omega)$ such that

$$L(g)=\int k(\xi,\omega)g(\xi,\omega)\,d\lambda(\xi)\,d\mu(\omega). \tag{2}$$

We may suppose also that

$$\hat{k}:\xi\to g(\xi,\cdot)$$

is measurable map of X into $L^2(\Omega,M,\mu)$, and

$$\|k\|_2^2=\int\|\hat{k}(\xi)\|_2^2\,d\lambda(\xi).$$

Then (2) can also be written in the form

$$L(g)=\int\langle\hat{k}(\xi),\,\hat{g}(\xi)\rangle\,d\lambda(\xi), \tag{3}$$

and comparison of (3) with (1) gives

$$\int\langle c(\xi)x(\xi)-\hat{k}(\xi),\hat{g}(\xi)\rangle\,d\lambda(\xi)=0, \tag{4}$$

for all $g\in L^2(X\times\Omega,\,\lambda\times\mu)$. If $f\in L^2(\Omega,\mu)$ we may consider in (4) g of the form $h(\xi)f(\omega)$, and infer

$$\langle c(\xi)x(\xi)-\hat{k}(\xi),f\rangle=0\ \text{a.e.}\,(\lambda). \tag{5}$$

Since $\mathfrak{H}$ is separable, $L^2(\Omega,\mu)$ is separable. Therefore by letting f in (5) run through an orthonormal basis for $L^2(\Omega,\mu)$, we obtain

$$x(\xi)=c(\xi)^{-1}\hat{k}(\xi)\qquad\text{a.e.}\,(\lambda). \tag{6}$$

Define

$$x(\xi,\omega)=c(\xi)^{-1}\,k(\xi,\omega). \tag{7}$$

Then $x(\xi,\cdot)$ is a stochastic process realised on (Ω,M,μ), is measurable in ξ, and for $\xi\notin N$, the null set implicit in (6), $x(\xi,\cdot)$ is Gaussian with mean zero and covariance the restriction of C to $(X-N)\times(X-N)$.

For v an integer $\geqq 1$, denote by $x^v(\cdot)$ the Gaussian process indexed by X with values in $\mathbb{R}^v$ given by

$$x^v(\cdot)=(x_1(\cdot),x_2(\cdot),\dots,x_v(\cdot)),$$

with $x_j(\cdot)$, $1\leqq j\leqq v$, independent copies of $x(\cdot)$, the Gaussian process with covariance C. Denote by δ^v the Dirac distribution on $\mathbb{R}^v$, and let λ be a positive σ-finite measure on X. The purpose of this section is to show, under suitable conditions on λ, that the formal expression

$$J(\lambda)=(2\pi)^{v/2}\int\delta^v(x^v(\xi))\,d\lambda(\xi) \tag{8}$$

has a natural interpretation as a positive random variable.

Let $\{\delta_\varepsilon(x), \varepsilon>0\}$ be an approximation to $\delta^\nu(x)$ i.e. a set of functions on $\mathbb{R}^\nu$ such that

(i) $\delta_\varepsilon(x)$ is positive, bounded and continuous, for each $\varepsilon>0$,

(ii) $\int \delta_\varepsilon(x)dx=1$, for each $\varepsilon>0$

(with dx Lebesgue measure on $\mathbb{R}^\nu$),

(iii) $\lim_{\varepsilon\to 0+} \int f(x)\delta_\varepsilon(x)dx=f(0)$,

for every bounded and continuous function f on $\mathbb{R}^\nu$.

Let $x(\xi,\omega)$ be the function given by (7). Then

$$x^\nu(\xi,\omega_1,\ldots,\omega_\nu)=(x(\xi,\omega_1),\ldots,x(\xi,\omega_\nu)),$$

is a realisation on $(\Omega,M,\mu)^\nu$ of the restriction of $x(\cdot)$ to $X-N$, and

$$J_\varepsilon(\lambda)=(2\pi)^{\nu/2}\int \delta_\varepsilon(x(\xi))d\lambda(\xi) \tag{9}$$

is well-defined (though possibly equal to $+\infty$). Suppose now that λ satisfies the condition

$$\int \frac{d\lambda(\xi_1)d\lambda(\xi_2)}{[C(\xi_1,\xi_1)C(\xi_2,\xi_2)-C(\xi_1,\xi_2)^2]^{\nu/2}}<\infty, \tag{10}$$

so that a fortiori

$$\int \frac{d\lambda(\xi)}{[C(\xi,\xi)]^{\nu/2}}<\infty. \tag{11}$$

(11) implies $J_\varepsilon(\lambda)\in L^1$, so $J_\varepsilon(\lambda)$ is almost surely finite. (10) implies that $\lim_{\varepsilon\to 0+} J_\varepsilon(\lambda)$ exists in L^2;for

$$\begin{aligned}
&E[(J_{\varepsilon_1}(\lambda)-J_{\varepsilon_2}(\lambda))^2]\\
&=(2\pi)^\nu E[\{\int[\delta_{\varepsilon_1}(x(\xi))-\delta_{\varepsilon_2}(x(\xi))]d\lambda(\xi)\}^2]\\
&=(2\pi)^\nu\int d\lambda(\xi_1)d\lambda(\xi_2)E[(\delta_{\varepsilon_1}(x(\xi_1))-\delta_{\varepsilon_2}(x(\xi_1)))(\delta_{\varepsilon_1}(x(\xi_2))-\delta_{\varepsilon_2}(x(\xi_2)))]\\
&=\int\frac{d\lambda(\xi_1)d\lambda(\xi_2)}{[C(\xi_1,\xi_1)C(\xi_2,\xi_2)-C(\xi_1,\xi_2)^2]^{\nu/2}}\int_{\mathbb{R}^\nu\times\mathbb{R}^\nu}(\delta_{\varepsilon_1}(x_1)-\delta_{\varepsilon_2}(x_1)(\delta_{\varepsilon_1}(x_2)\\
&-\delta_{\varepsilon_2}(x_2))\exp[-\tfrac{1}{2}Q]dx_1dx_2
\end{aligned}$$

[With $Q=Q(x_1,x_2;\xi_1,\xi_2)$ a positive quadratic form]$\to 0$ as ε_1, $\varepsilon_2\to 0+$. Thus if (10) holds we may define $J(\lambda)=\lim_{\varepsilon\to 0+} J_\varepsilon(\lambda)$.

For any $\xi_0\in X$,

$$C_{\xi_0}(\xi_1,\xi_2)=C(\xi_1,\xi_2)-\frac{C(\xi_1,\xi_0)C(\xi_2,\xi_0)}{C(\xi_0,\xi_0)}$$

is a measurable covariance on X [if $\xi_0\in X_0$, the definition is to be read as $C_{\xi_0}(\xi_1,\xi_2)=C(\xi_1,\xi_2)$]. We will refer to the corresponding Gaussian process indexed by X as the process obtained by tying down at ξ_0 the process with covariance C; it may be realised in terms of the process with covariance C

$$x(\xi;\xi_0)=x(\xi)-\frac{C(\xi,\xi_0)x(\xi_0)}{C(\xi_0,\xi_0)}, \tag{12}$$

with the second term construed as 0 in case $\xi_0 \in X_0$. If λ is a probability measure on $(X, \mathfrak{X})$, we construct $x(\xi, \omega)$ as above, and define a process $x(\xi; \lambda)$ on $(X \times \Omega, \lambda \times \mu)$

$$x(\xi; \lambda)(\xi_0, \omega) = x(\xi, \omega) - \frac{C(\xi, \xi_0)\, x(\xi_0, \omega)}{C(\xi_0, \xi_0)} \tag{13}$$

(the second term being zero if $\xi_0 \in X_0$). $x(\cdot\,; \lambda)$ can be considered a mixture of the processes $x(\cdot\,; \xi_0)$. Similarly we define on $(X \times \Omega^\nu, \lambda \times \mu^\nu)$ the vector valued process $x^\nu(\cdot\,; \lambda)$

$$x^\nu(\xi, \lambda)(\xi_0, \omega^\nu) = x^\nu(\xi, \omega^\nu) - \frac{C(\xi, \xi_0)\, x^\nu(\xi_0, \omega^\nu)}{C(\xi_0, \xi_0)}. \tag{14}$$

Denote by $\tilde{\lambda}$ the measure on X given by

$$d\tilde{\lambda}(\xi) = C(\xi, \xi)^{\nu/2}\, d\lambda(\xi). \tag{15}$$

Then for any integer $N \geqq 1$, and bounded function F on $\mathbb{R}^{\nu N}$, we have, assuming $\tilde{\lambda}$ satisfies (10), for any $\xi_1, \ldots, \xi_N \in X$

$$E[F(x^\nu(\xi_1; \lambda), \ldots, x^\nu(\xi_N; \lambda))] = E[J(\tilde{\lambda}) F^\nu(x^\nu(\xi_1), \ldots, x^\nu(\xi_N))] \tag{16}$$

(by a short calculation). Thus (10) (with λ replaced by $\tilde{\lambda}$) implies absolute continuity of the measure $\sigma(\lambda)$ induced by $x(\xi; \lambda)$ on the space $F(M, \mathbb{R}^\nu)$ of function $f: M \to \mathbb{R}^\nu$ with respect to the measure σ induced on $F(M, \mathbb{R}^\nu)$ by $x(\xi)$;

$$J(\tilde{\lambda}) = \frac{d\sigma(\lambda)}{d\sigma}. \tag{17}$$

It is natural to take (17) as the definition of $J(\tilde{\lambda})$ whenever $\sigma(\lambda) \ll \sigma$; however, we do not know explicit necessary and sufficient conditions on λ for this to be the case.

We give two examples of these general considerations

Example 1. Take $X = [0, \infty)$, $\mathfrak{X}$ = Borel σ-algebra of X, $C(\sigma, \tau) = \min(\sigma, \tau)$ for $\sigma, \tau \in X$. Then the Gaussian process $x(\cdot)$ with covariance C is the standard Brownian motion on $\mathbb{R}^1$ starting at 0, and $x^\nu(\cdot)$ is the standard Brownian motion on $\mathbb{R}^\nu$ starting at $\mathbf{0}$. Let $\lambda(t)$ for $t \geqq 0$ be given by $d\lambda(t) = 1_{[0,t]} dt$. Then for $\nu = 1$ (10) holds, so $T(t) = J(\lambda(t))$ is defined; (up to a normalisation factor) $T(t)$ is P. Lévy's local time at 0 [16]. For $\nu > 1$ (10) fails; indeed $x^\nu(c) \neq 0$ a.s. for c in any interval $[\varepsilon, \infty)$, $\varepsilon > 0$, so that we will not have $\sigma(\lambda) \ll \sigma$ for any probability measure λ other than the unit mass at 0.

Example 2. Take $X = [0, \infty) \times [0, \infty)$, $\mathfrak{X}$ = Borel σ-algebra of X, $C(\sigma_1, \sigma_2; \tau_1, \tau_2) = \min(\sigma_1, \tau_1) + \min(\sigma_2, \tau_2)$. The Gaussian process $x(\cdot)$ with covariance C may be realised as $x(\sigma_1, \sigma_2) = x_1(\sigma_1) - x_2(\sigma_2)$, with $x_1(\cdot)$, $x_2(\cdot)$ independent standard Brownian motions as in Exercise 1. For $t_1, t_2 \geqq 0$ define $\lambda(t_1, t_2)$ by $d\lambda(t_1, t_2) = 1_{[0,t_1] \times [0,t_2]} dt_1\, dt_2$. For $\nu \leqq 3$ (10) holds, so $T_\nu(t_1, t_2) = J(\lambda(t_1, t_2))$ is defined, and gives a measure of the time spent at points of intersection $\{\sigma_1, \sigma_2 : x_1(\sigma_1) = x_2(\sigma_2), \sigma_1 \leqq t_1, \sigma_2 \leqq t_2\}$. For $\nu > 3$ (10) fails; in fact Brownian motion in $\mathbb{R}^4$ has no double points ([17]; note that by writing $x(\sigma) = x_1(|\sigma|)\, (\sigma < 0)$, $x(\sigma) = x_2(\sigma)\, (\sigma \geqq 0)$, we may regard the intersection points of $x_1(\cdot)$, $x_2(\cdot)$ as double points of the Brownian motion $x(\cdot)$). For $\nu=1$ the construction of $T_1(t_1,t_2)$ can be subsumed to that of Lévy

$$T_1(t_1,t_2)=(2\pi)^{-1/2}\int_{\mathbb{R}} T^1(t_1,x)T^2(t_2,x)dx\,, \tag{18}$$

with $T^j(t,x)$ $(j=1,2)$ the local time at x for $x_j(\cdot)$

$$T(t,x)=(2\pi)^{1/2}\int_0^t \delta(x(\sigma)-x)d\sigma\,. \tag{19}$$

For $\nu=2$ Brownian paths have points of multiplicity m for any integer $m\geqq 2$ [18] (and indeed of multiplicity m equal to the power of the continuum [19]!), so that one may expect to be able to define random variables $T(t_1,t_2,\ldots,t_m)$, generalising the above construction for $m=2$; this has been done by Wolpert [20], who discusses also a relation to $P(\varphi)_2$ Euclidean field theories [21]. For $\nu\geqq 3$ Brownian paths have no points of multiplicity $m>2$ [22]. (See also [23, 24] for further information on multiple points of Brownian motion, and related questions.)

3.2. *Application of the Limit Theorem*

In this section we show how the limit theorem of Sect. 2 may be used to construct the probability measure for the polymer problem.

We indicate elements of $\mathbb{R}^3$, and functions taking their values in $\mathbb{R}^3$ by boldface letters; thus $\mathbf{z}=(z_1,z_2,z_3)$, $\mathbf{h}(\cdot)=(h_1(\cdot),h_2(\cdot),h_3(\cdot))$. Denote by (Ω,M) the space of continuous maps $\boldsymbol{\omega}:[0,1]\to\mathbb{R}^3$, $\boldsymbol{\omega}(0)=\mathbf{0}$, with the Borel σ-algebra of the topology of uniform convergence, and by $\mathbf{x}(s)(\cdot)$, $0\leqq s\leqq 1$, the evaluation maps $\mathbf{x}(s)(\boldsymbol{\omega})=\boldsymbol{\omega}(s)$. Then there exists a probability measure μ on (Ω,M) such that $\mathbf{x}(s)(\cdot)$ is standard Brownian motion on $\mathbb{R}^3$ starting at $\mathbf{0}$ [16]. In the notation of 3.1, $\mathbf{x}(s)$ is a realisation of the vector valued Gaussian process defined by the covariance $C(s,t)=\min(s,t)$ on $[0,1]$, and $\nu=3$. The corresponding reproducing kernel Hilbert space $\mathfrak{H}$ is given by

$$\mathfrak{H}=\{f: f \text{ is absolutely continuous, } f(0)=0,\ f'\in L^2[0,1]\}$$

and $\|f\|_{\mathfrak{H}}=\|f'\|_2$. Denote by $\boldsymbol{\varphi}$ the continuous linear map $\mathfrak{H}\to L^2(\Omega,M;\mathbb{R}^3)$ characterized by $\boldsymbol{\varphi}(e(s))=\mathbf{x}(s)$ [with $e(s)$ as in 3.1].

Denote by (Ω_0,M_0) [resp. (Ω_1,M_1)] the space of continuous maps $\boldsymbol{\omega}:[0,1/2]\to\mathbb{R}^3$, $\boldsymbol{\omega}(0)=\mathbf{0}$ (resp. $\boldsymbol{\omega}:[1/2,1]\to\mathbb{R}^3$, $\boldsymbol{\omega}(1/2)=\mathbf{0}$), with the Borel σ-algebra of the topology of uniform convergence, and define $p_0:\Omega\to\Omega_0$ (resp. $p_1:\Omega\to\Omega_1$) by $p_0(\boldsymbol{\omega})(s)=\boldsymbol{\omega}(s)$, $s\in[0,1/2]$ (*resp.* $p_1(\boldsymbol{\omega})(s)=\boldsymbol{\omega}(s)-\boldsymbol{\omega}(1/2)$, $s\in[1/2,1]$). Then $(\Omega,M)=(\Omega_0,M_0)\times(\Omega_1,M_1)$, with p_0,p_1 the corresponding projections. Since Brownian motion has independent increments, the measure induced by μ on $\Omega_0\times\Omega_1$ is a product measure $\mu_0\times\mu_1$. Define also scaling maps j_0,j_1; $j_0:\Omega_0\to\Omega$ (resp. $j_1:\ \Omega_1\to\Omega$) is given by $j_0(\boldsymbol{\omega})(s)=2^{1/2}\boldsymbol{\omega}\left(\frac{s}{2}\right)$ $\left[\text{resp. } j_1(\boldsymbol{\omega})(s)=2^{1/2}\boldsymbol{\omega}\left(\frac{1+s}{2}\right)\right]$. Then j_0, j_1 are isomorphisms of (Ω_0,M_0,μ_0), (Ω_1,M_1,μ_1) onto (Ω,M,μ) (scale covariance of Brownian motion [25]). Define $\mathbf{x}_0(s)=2^{1/2}\left[\mathbf{x}(1/2)-\mathbf{x}\left(\frac{1-s}{2}\right)\right]$, $\mathbf{x}_1(s)$

$=2^{1/2}\left[\mathbf{x}(1/2)-\mathbf{x}\left(\frac{1+s}{2}\right)\right]$, $0 \leqq s \leqq 1$; then $\mathbf{x}_0(\cdot)$, $\mathbf{x}_1(\cdot)$ are independent standard Brownian motions in $\mathbb{R}^3$ starting at $\mathbf{0}$, and we may construct the random variable

$$X = T_3(1,1) = (2\pi)^{3/2} \int_0^1 \int_0^1 \delta^3(\mathbf{x}_0(s) - \mathbf{x}_1(t))\, ds\, dt, \tag{1}$$

as in 3.1, Example 2. We define also the σ-algebra $G = \sigma\{\mathbf{x}(1)\}$; this completes the identification of the data in the statement of the limit theorem.

For $s \in [0,1]$ denote by $s = 0 \cdot s_1 s_2 \ldots s_n \ldots$ the dyadic expansion of s. Define sets $\Delta(v) \subset [0,1] \times [0,1]$, $v \in T$:

$$\Delta(0) = ([0,1/2] \times [1/2,1]) \cup ([1/2,1] \times [0,1/2]),$$

and for $n > 0$

$$\Delta(n; i_1 \ldots i_n) = \{(s,t) : n = \max[m : s_m = t_m], s_j = t_j = i_j, 1 \leqq j \leqq n\}.$$

By appropriate change of variables in the time integrations we obtain the identifications

$$X(v) = 2\pi^{3/2} \int_{\Delta(v)} \delta^3(\mathbf{x}(s) - \mathbf{x}(t))\, ds\, dt, \tag{2}$$

for $v \in T$, and hence

$$S(n) = 2\pi^{3/2} \int_{R(n)} \delta^3(\mathbf{x}(s) - \mathbf{x}(t))\, ds\, dt, \tag{3}$$

with $R(n) = \{(s,t) : \max[m : s_m = t_m] \leqq n\}$. As $n \to \infty$, $R(n) \uparrow [0,1] \times [0,1]$, so that formally the measures $v_n(g)$ defined by 2.1.4 should converge to the polymer measure with coupling constant $2\pi^{3/2} g$.

We begin now the verification of conditions *C1–C6*. Define intervals $I(v) \subset [0,1]$, $v \in T : I(0) = [0,1]$, and for $n > 0$, $I(n; i_1 \ldots i_n) = \{s : s_1 = i_1, \ldots, s_n = i_n\}$, and unit vectors $g(v) \in \mathfrak{H} : g(v)' = 2^{n/2} 1_{I(v)}$. Note that $g(0) = e(1)$, so $G = G(0) = \sigma\{\mathbf{x}(1)\}$ $= \sigma\{\boldsymbol{\varphi}(g(0))\}$, and we find $G(v) = \sigma\{\boldsymbol{\varphi}(g(v))\}$, for all $v \in T$. Since $g(0) = 2^{-1/2}[g(1;0)$ $+ g(1,1)]$, $G(0) \subset G(1)$ (*C1*).

Define $f = 2^{-1/2}[g(1;0) - g(1;1)]$, so that $\langle f, g(0) \rangle = 0$ and $\operatorname{span}\{f, g(0)\}$ $= \operatorname{span}\{g(1;0), g(1;1)\}$; this implies $G(1) = G(0) \vee F$, with $F = \sigma\{\boldsymbol{\varphi}(f)\}$ independent of $G(0)$ (*C2*). For $v \in T$ define $f(v) \in \mathfrak{H} : f(0) = f$, and for $n > 0$, $f(n; i_1 \ldots i_n)$ $= 2^{-1/2}[g(n+1; i_1 \ldots i_n 0) - g(n+1; i_1 \ldots i_n 1)]$. Then $\{g(0), f(v), v \in T\}$ is an orthonormal basis for $\mathfrak{H}$, which is mapped onto the Haar basis of $L^2[0,1]$ by the isomorphism $f \to f'$ of $\mathfrak{H}$ onto $L^2[0,1]$, and $F(v) = \sigma\{\boldsymbol{\varphi}(f(v))\}$, $v \in T$. Write $\mathbf{y}(v)$ $= \boldsymbol{\varphi}(f(v))$, $v \in T$. For $0 \leqq s \leqq 1$ we may expand $e(s) \in \mathfrak{H}$ in the basis $\{g(0), f(v), v \in T\}$

$$e(s) = s g(0) + \sum_{v \in T} f(v)(s) f(v), \tag{4}$$

and apply the map $\boldsymbol{\varphi}$ to both sides to give

$$\mathbf{x}(s) = s\mathbf{x}(1) + \sum_{v \in T} f(v)(s) \mathbf{y}(v), \tag{5}$$

(5) is Ciesielski's representation of Brownian motion [16].

C3 is evident from the definition of X. The remaining conditions *C4–C6* will be verified in the next section, so Theorem 1 is applicable. It gives for $g \leqq \bar{g}$ a martingale f_m relative to the increasing sequence of σ-algebras $G(m)$, and hence a consistent sequence of measures $\nu^m(g)$

$$\nu^m(g)[B] = E[1_B f_m], \tag{6}$$

$B \in G(m)$. Since $G(m) = \sigma\{\mathbf{x}(1), \mathbf{y}(v), n(v) \leqq m\}$, Kolmogorov's extension theorem may be used to construct a measure $\nu(g)$ on $\mathbb{R}^\infty$, the product of countably many copies of $\mathbb{R}$ indexed by $(\{1\} \cup T) \times \{1, 2, 3\}$, so that the coordinate variables may be written $x_j(1)$, $y_j(v)$, $v \in T$, $j \in \{1, 2, 3\}$. (5) may then be used to transfer this measure to $\{\Omega, M)$, as in Ciesielski's construction of Brownian motion, i.e., we will prove that if $\mathbf{x}(1)$, $\mathbf{y}(v)$, $v \in T$, are distributed according to $\nu(g)$ then (5) defines a stochastic process which may be realised on (Ω, M). In 2.8 we proved the existence of $\lim_{n \to \infty} E_{\nu_n(g)}[R] = E_{\nu(g)}[R]$ for $R \in L^q(\Omega, G(m), \mu)$; the proof gave a bound for $E_{\nu(g)}[R]$ which may be written

$$|E_{\nu(g)}[R]| \leqq C^m \|R\|_q, \tag{7}$$

for some $C > 0$. For $v \in T$, $j \in \{1, 2, 3\}$, $p \geqq 1$, take $R = |y_j(v)|^{mp}$, $m = n(v) + 1$, in (7) to give

$$\|y_j(v)\|'_p \leqq D p^{1/2} (n+1)^{1/2}, \tag{8}$$

for some $D > 0$, with $\|\cdot\|'_p$ the norm on $L^p(\Omega, G(m), \nu(g))$. According to Varadhan [26], an estimate of the form

$$E[|x(s) - x(t)|^\beta] \leqq M |s - t|^{1+\alpha}, \tag{9}$$

for some $\alpha > 0$, $\beta > 0$, implies a.s. Hölder continuity of the sample paths of the process $x(\cdot)$ for any exponent $\varrho < \min(\alpha\beta^{-1}, 1)$. The functions $f(v)(s)$ in (5) enjoy the following properties:

$$\operatorname{supp} f(v) = I(v); \; 0 \leqq f(v)(s) \leqq 2^{-n/2-1}; \; |f(v)(s) - f(v)(t)| \leqq 2^{n/2} |s - t|. \tag{10}$$

Using (8), (10) we obtain, by checking Varadhan's condition, a.s. Hölder continuity of the sample paths of the process defined by (5) and the measure $\nu(g)$, for any exponent $\varrho < 1/2$, just as in the case of the standard Brownian motion.

3.3. *Three Estimates*

C4. We will show that the random variable X defined by 3.2.1 satisfies

$$C_1 p^{3/2} \leqq \|X\|_p \leqq C_2 p^{3/2} \tag{1}$$

for $p \geqq 1$, and some constants C_1, C_2. The upper bound verifies condition *C4* of 2.1; the lower bound is of interest because it implies divergence of the Taylor expansion of $E[\exp(-gX)]$ about $g = 0$.
For $\mathbf{z} \in \mathbb{R}^3$ write $\mathbf{z}^2 = z_1^2 + z_2^2 + z_3^2$, and denote by $d\mathbf{z}$ Lebesgue measure. We may suppose p an integer. By definition

$$X = \lim_{\varepsilon \to 0+} (2\pi)^{3/2} \int_0^1 \int_0^1 \delta_\varepsilon(\mathbf{x}_0(s) - \mathbf{x}_1(t)) \, ds \, dt, \tag{2}$$

so

$$\begin{aligned}\|X\|_p^p &= E[X^p] \\ &= (p!)^2 \int ds_1 \dots ds_p \int dt_1 \dots dt_p\, f(s,t) \\ &\quad 0 \leqq s_1 \leqq \dots \leqq s_p \leqq 1, 0 \leqq t_1 \leqq \dots \leqq t_p \leqq 1,\end{aligned} \tag{3}$$

with

$$f(s,t) = \frac{1}{p!} \sum_{\pi \in S_p} f_\pi(s,t) \tag{4}$$

(S_p denoting the symmetric group on p letters, and

$$\begin{aligned}f_\pi(s,t) &= \lim_{\varepsilon \to 0+} E\left[\prod_{i=1}^{p} \{(2\pi)^{3/2} \delta_\varepsilon(\mathbf{x}_0(s_i) - \mathbf{x}_i(t_{\pi(i)}))\}\right] \\ &= \lim_{\varepsilon \to 0+} \int d\boldsymbol{\xi}_1 \dots d\boldsymbol{\xi}_p\, d\boldsymbol{\eta}_1 \dots d\boldsymbol{\eta}_p \\ &\quad \cdot \prod_{i=1}^{p} \left\{(2\pi)^{-3/2} \delta_\varepsilon(\boldsymbol{\xi}_i - \boldsymbol{\eta}_{\pi(i)}) \exp\left[-\frac{(\boldsymbol{\xi}_i - \boldsymbol{\xi}_{i-1})^2}{2(s_i - s_{i-1})} - \frac{(\boldsymbol{\eta}_i - \boldsymbol{\eta}_{i-1})^2}{2(t_i - t_{i-1})}\right]\right. \\ &\quad \left. \cdot [(s_i - s_{i-1})(t_i - t_{i-1})]^{-3/2}\right\},\end{aligned}$$

with $\boldsymbol{\xi}_0 = \boldsymbol{\eta}_0 = \mathbf{0}$, $s_0 = t_0 = 0$,

$$\begin{aligned}&= \int d\boldsymbol{\eta}_1 \dots d\boldsymbol{\eta}_p \\ &\quad \cdot \prod_{i=1}^{p} \left\{(2\pi)^{-3/2} \exp\left[-\frac{(\boldsymbol{\eta}_{\pi(i)} - \boldsymbol{\eta}_{\pi(i-1)})^2}{2(s_i - s_{i-1})} - \frac{(\boldsymbol{\eta}_i - \boldsymbol{\eta}_{i-1})^2}{2(t_i - t_{i-1})}\right]\right. \\ &\quad \left. \cdot [(s_i - s_{i-1})(t_i - t_{i-1})]^{-3/2}\right\},\end{aligned} \tag{5}$$

since Brownian motion has independent increments.

Change integration variables in (3) : $(s,t) \to (\alpha, \beta)$, with

$$\alpha_i = s_i - s_{i-1}, \quad \beta_i = t_i - t_{i-1}, \quad 1 \leqq i \leqq p,$$

and interchange the (α, β) and $\boldsymbol{\eta}$ integrations to give

$$\|X\|_p^p = (p!)^2 (2\pi)^{-\frac{3p}{2}} \frac{1}{p!} \sum_{\pi \in S_p} \int \mathrm{d}\boldsymbol{\eta}_1 \dots d\boldsymbol{\eta}_p\, g_\pi(\boldsymbol{\eta})$$

with $g_\pi(\boldsymbol{\eta}) = \int d\alpha_1 \dots d\alpha_p \int d\beta_1 \dots d\beta_p$

$$\alpha_i \geqq 0, \quad \sum_i \alpha_i \leqq 1, \quad \beta_i \geqq 0, \quad \sum_i \beta_i \leqq 1$$

$$\cdot \prod_{i=1}^{p} \left\{\exp\left[-\frac{(\boldsymbol{\eta}_{\pi(i)} - \boldsymbol{\eta}_{\pi(i-1)})^2}{2\alpha_i} - \frac{(\boldsymbol{\eta}_i - \boldsymbol{\eta}_{i-1})^2}{2\beta_i}\right] (\alpha_i \beta_i)^{-3/2}\right\}. \tag{7}$$

To obtain an upper bound for $\|X\|_p^p$ we replace the integration region in (7) by $\{(\alpha, \beta) : \alpha_i \geqq 0, \beta_i \geqq 0, \sum_i (\alpha_i + \beta_i) \leqq 2\}$; denote by $h_\pi(\boldsymbol{\eta})$ the resulting integral, so that $g_\pi(\boldsymbol{\eta}) \leqq h_\pi(\boldsymbol{\eta})$.

For any $\lambda > 0$ we may make the change of integration variables

$$\alpha'_1 = \lambda\alpha_i, \quad \beta'_i = \lambda\beta_i, \quad \boldsymbol{\eta}'_i = \lambda^{1/2}\boldsymbol{\eta}_i,$$

and write $\int d\boldsymbol{\eta}_1 \dots d\boldsymbol{\eta}_p h_\pi(\boldsymbol{\eta})$ in the alternative form

$$\int d\boldsymbol{\eta}'_1 \dots d\boldsymbol{\eta}'_p h^\lambda_\pi(\boldsymbol{\eta}'),$$

with

$$h^\lambda_\pi(\boldsymbol{\eta}') = \lambda^{-p/2} \int d\alpha'_1 \dots d\alpha'_p d\beta'_1 \dots d\beta'_p$$

$$a'_i \geqq 0, \beta'_i \geqq 0, \sum_i (\alpha'_1 + \beta'_i) \leqq 2\lambda$$

$$\cdot \prod_{i=1}^{p} \left\{ \exp\left[-\frac{(\boldsymbol{\eta}'_{\pi(i)} - \boldsymbol{\eta}'_{\pi(i-1)})^2}{2\alpha'_i} - \frac{(\boldsymbol{\eta}'_i - \boldsymbol{\eta}'_{i-1})^2}{2\beta'_i} \right] (\alpha'_i \beta'_i)^{-3/2} \right\}. \tag{8}$$

After the change of variables the primes on the new integration variables may be dropped. Now let $f(\lambda)$ be some positive function on $[0, \infty)$ with $\int_0^\infty f(\lambda) d\lambda = 1$. Then we may average over the alternative forms to write our upper bound as

$$\|X\|_p^p \leqq (p!)^2 (2\pi)^{\frac{-3p}{2}} \frac{1}{p!} \sum_{\pi \in S_p} \int d\boldsymbol{\eta}_1 \dots d\boldsymbol{\eta}_p k_\pi(\boldsymbol{\eta}), \tag{9}$$

with

$$k_\pi(\boldsymbol{\eta}) = \int_0^\infty \frac{f(\lambda)}{\lambda^{p/2}} d\lambda \int_{\alpha_i \geqq 0, \beta_i \geqq 0, \sum_i (\alpha_i + \beta_i) \leqq 2\lambda}$$

$$\cdot \prod_{i=1}^{p} \left\{ \exp\left[-\frac{(\boldsymbol{\eta}_{\pi(i)} - \boldsymbol{\eta}_{\pi(i-1)})^2}{2\alpha_i} - \frac{(\boldsymbol{\eta}_i - \boldsymbol{\eta}_{i-1})^2}{2\beta_i} \right] (\alpha_i \beta_i)^{-3/2} \right\}. \tag{10}$$

Choose $f(\lambda) = [\Gamma(1 + p/2)]^{-1} \lambda^{p/2} e^{-\lambda}$, and interchange the λ and (α, β) integrations in (10). Since

$$\int_{1/2 \sum_i (\alpha_i + \beta_i)}^\infty e^{-\lambda} d\lambda = \exp\left\{ -1/2 \sum_i (\alpha_i + \beta_i) \right\},$$

$k_\pi(\boldsymbol{\eta})$ then assumes a product form

$$k_\pi(\boldsymbol{\eta}) = [\Gamma(1 + p/2)]^{-1} \prod_{i=1}^{p} \{ H(|\boldsymbol{\eta}_{\pi(i)} - \boldsymbol{\eta}_{\pi(i-1)}|) H(|\boldsymbol{\eta}_i - \boldsymbol{\eta}_{i-1}|) \}, \tag{11}$$

with

$$H(R) = \int_0^\infty d\alpha \alpha^{-3/2} \exp\left[-\frac{1}{2} \left(\frac{R^2}{\alpha} + \alpha \right) \right]. \tag{12}$$

The Cauchy-Schwarz inequality now gives, for each $\pi \in S_p$,

$$\begin{aligned}\int d\boldsymbol{\eta}_1 \ldots d\boldsymbol{\eta}_p k_\pi(\boldsymbol{\eta}) \leqq & \left[\Gamma\left(1+\frac{p}{2}\right)\right]^{-1} \\ & \cdot\left(\int d\boldsymbol{\eta}_1 \ldots d\boldsymbol{\eta}_p \prod_{i=1}^{p}\left[H(|\boldsymbol{\eta}_{\pi(i)}-\boldsymbol{\eta}_{\pi(i-1)}|)\right]^2\right)^{1/2} \\ & \cdot\left(\int d\boldsymbol{\eta}_1 \ldots d\boldsymbol{\eta}_p \prod_{i=1}^{p}\left[H(|\boldsymbol{\eta}_i-\boldsymbol{\eta}_{i-1}|)\right]^2\right)^{1/2} \\ = & \left[\Gamma\left(1+\frac{p}{2}\right)\right]^{-1} K^p, \end{aligned} \tag{13}$$

with $K=\int d\boldsymbol{\eta}[H|\boldsymbol{\eta}|)]^2=(2\pi)^2<\infty$.

(9), (13) give the upper bound in (1).

To obtain the lower bound for $\|X\|_p^p$ we replace the integration region (7) by $\left\{(\alpha, \beta): \alpha_i \geqq 0,\ \beta_i \geqq 0,\ \sum_i(\alpha_i+\beta_i) \leqq 1\right\}$, and make the same transformations as in the derivation of the upper bound to give

$$\|X\|_p^p \geqq 2^{-p/2}(p!)^2(2\pi)^{\frac{-3p}{2}} \frac{1}{p!} \sum_{\pi \in S_p} \int d\boldsymbol{\eta}_1 \ldots d\boldsymbol{\eta}_p k_\pi(\boldsymbol{\eta}). \tag{14}$$

For each $\pi \in S_p$

$$\begin{aligned}\int d\boldsymbol{\eta}_1 \ldots d\boldsymbol{\eta}_p k_\pi(\boldsymbol{\eta}) \geqq & \left[\Gamma\left(1+\frac{p}{2}\right)\right]^{-1} \\ & \cdot \int_{R_1} \prod_{i=1}^{p}\left\{H(|\boldsymbol{\eta}_{\pi(i)}-\boldsymbol{\eta}_{\pi(i-1)}|) H(|\boldsymbol{\eta}_i-\boldsymbol{\eta}_{i-1}|)\right\} d\boldsymbol{\eta}_1 \ldots d\boldsymbol{\eta}_p,\end{aligned}$$

with

$$\begin{aligned} & R_1=\{(\boldsymbol{\eta}_i): |\boldsymbol{\eta}_i| \leqq 1, 1 \leqq i \leqq p\} \\ & \geqq\left[\Gamma\left(1+\frac{p}{2}\right)\right]^{-1} K_1^p, \end{aligned} \tag{15}$$

with $K_1=H(2)^2 \frac{4\pi}{3}$ [since H is a decreasing function of R, and $|\boldsymbol{\eta}_i-\boldsymbol{\eta}_{i-1}|$, $|\boldsymbol{\eta}_{\pi(i)}-\boldsymbol{\eta}_{\pi(i-1)}| \leqq 2$ for $(\boldsymbol{\eta}_i) \in R_1$].

(14), (15) give the lower bound in (1).

In verifying *C5*, *C6* we will make use of a formula which we state in the general framework of 3.1. With the notations of 3.1, denote by $\varphi^\nu: \mathfrak{H} \rightarrow L^2(\Omega^\nu, M^\nu, \mu^\nu; \mathbb{R}^\nu)$ the map given by $\varphi^\nu(e)=(\varphi(e, \omega_1), \ldots, \varphi(e, \omega_\nu))$, and, for $\mathfrak{L} \subset \mathfrak{H}$ a closed subspace of $\mathfrak{H}$, define the σ-algebra $B(\mathfrak{L}) \subset M^\nu$ by

$$B(\mathfrak{L})=\sigma\{\varphi_j^\nu(e), 1 \leqq j \leqq \nu, e \in \mathfrak{L}\}.$$

Denote by P the orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{L}$. Let $p \geqq 2$ be an even integer, and λ a measure on X such that $J(\lambda)$ exists and is in L^p. Then $\|J(\lambda)-E[J(\lambda) \mid B(\mathfrak{L})]\|_p^p$

$$\int d\lambda(\xi_1) \ldots d\lambda(\xi_p)\left(\sum_{A \subset\{1, \ldots, p\}}(-1)^{|A|}[\det C(P, A, \xi)]^{-\nu/2}\right), \tag{16}$$

with $|A|$ the number of elements in A, and $C(P,A,\xi)$ the $\nu\times\nu$ matrix given by

$$\left.\begin{aligned}[C(P,A,\xi)]_{ij}&=\langle e(\xi_i),e(\xi_j)\rangle \quad \text{if} \quad i=j \quad \text{or} \quad i,j\in A\\ &=\langle e(\xi_i),Pe(\xi_j)\rangle \quad \text{if} \quad i\neq j \quad \text{and} \quad i \quad \text{or} \quad j\notin A\end{aligned}\right\}. \tag{17}$$

(16) results from a straightforward calculation of Gaussian integrals. Note that if, for some index i_0 and all $i\neq i_0$, $\langle e(\xi_i),(1-P)e(\xi_{i_0})\rangle=0$, the integrand in (16) is zero at ξ; analytically this results from

$$C(P,A,\xi)=C(P,A\cup\{i_0\},\xi)$$

for every A with $i_0\notin A$; probabilistically it is clear since the random variables $\{\varphi_j^\nu(e(\xi_{i_0})),1\leqq j\leqq\nu\}$, $\{\varphi_j^\nu(e(\xi_i)),\ 1\leqq j\leqq\nu,\ i\neq i_0\}$ are then independent conditionally on $B(\mathfrak{L})$.

Some linear algebra (Lemma 1 below) gives the inequality

$$\det C(P,A,\xi)\geqq 2^{-p}\det C(\xi). \tag{18}$$

Denote by $\chi(\xi)$ the characteristic function of the set

$$\{\xi:\text{for each index } i,\ 1\leqq i\leqq p,\ \text{there is an index } j\neq i,\ 1\leqq j\leqq p,\ \text{such that } \langle e(\xi_i),(1-P)e(\xi_j)\rangle\neq 0\}, \tag{19}$$

then from (16) and the following remark we obtain the bound

$$\begin{aligned}&\|J(\lambda)-E[J(\lambda)|B(\mathfrak{L})]\|_p^p\\ &\quad\leqq 2^{p(1+\nu/2)}\int d\lambda(\xi_1)\dots d\lambda(\xi_p)\chi(\xi)[\det C(\xi)]^{-\nu/2}.\end{aligned} \tag{20}$$

Note that $[\det C(\xi)]^{-\nu/2}$ is the integrand in the integral formula for $\|J(\lambda)\|_p^p$

$$\|J(\lambda)\|_p^p=\int d\lambda(\xi_1)\dots d\lambda(\xi_p)[\det C(\xi)]^{-\nu/2}. \tag{21}$$

Lemma 1. *Let $\mathfrak{H}$ be a real Hilbert space, $P:\mathfrak{H}\to\mathfrak{H}$ an orthogonal projection. Let $p\geqq 1$ be an integer, and $e_1,\dots,e_p$, p vectors in $\mathfrak{H}$. Define, for any $A\subset\{1,\dots,p\}$, a $p\times p$ matrix $C(P,A)$*

$$\begin{aligned}[C(P,A)]_{ij}&=\langle e_i,e_j\rangle \quad \textit{if} \quad i=j \quad \textit{or} \quad i,j\in A\\ &=\langle e_i,Pe_j\rangle \quad \textit{if} \quad i\neq j \quad \textit{and} \quad i \quad \textit{or} \quad j\notin A,\end{aligned}$$

write $C=C(P,\{1,\dots,p\})$. Then

$$\det C\leqq 2^{|A^c|}\det C(P,A).$$

Proof. For any set $\{v_1,\dots,v_k\}$ of vectors of $\mathfrak{H}$, we denote by $\Lambda\{v_1,\dots,v_k\}$ their exterior product (in the order indicated by their indices; thus $\Lambda\{v_2,v_1\}=\Lambda\{v_1,v_2\}=v_1\Lambda v_2$), and by $G(v_1,\dots,v_k)=\|\Lambda\{v_1,\dots,v_k\}\|^2$ their Gram determinant.

For any l, $1\leqq l\leqq k$, we have

$$G(v_1,\dots,v_k)\leqq G(v_1,\dots,v_l)G(v_{l+1},\dots,v_k). \tag{$*$}$$

$(*)$ is a variant of Hadamard's inequality.

We have

$$\begin{aligned}\det C &= \|A\{u_1, \ldots, u_p\}\|^2 \\ &= \Big\|\sum_{S\subset A^c} A\{u_i, i\in A; Pu_i, i\in S; (1-P)u_i, i\notin S\}\Big\|^2 \\ &\leqq \Big(\sum_{S\subset A^c} \|A\{u_i, i\in A; Pu_i, i\in S; (1-P)u_i, i\notin S\}\|\Big)^2 \\ &\leqq 2^{|A^c|} \sum_{S\subset A^c} G(u_i, i\in A; Pu_i, i\in S; (1-P)u_i, i\notin S)\end{aligned}$$

(by the Cauchy-Schwarz inequality)

$$\leqq 2^{|A^c|} \sum_{S\subset A^c} G(u_i, i\in A; Pu_i, i\in S) \prod_{i\notin S} \|(1-P)u_i\|^2$$

(by the variant of Hadamard's inequality given above)

$$= 2^{|A^c|} \det C(P, A).$$

[At the last step we use the observation that $C(P, A)$ is the sum of the Gram matrix of the vectors u_i, $i\in A$; Pu_i, $i\notin A$, and a diagonal matrix whose diagonal entries are 0, $i\in A$, $\|(1-P)u_i\|^2$, $i\notin A$.] □

C5. To verify condition *C5* of 2.1 it is sufficient to obtain an estimate of the form 2.1.2 for $p\geqq 2$ an even integer, and $n\geqq 1$.

From 3.2.5, and $\operatorname{supp} f(v)=I(v)$ $(v\in T)$, we have, for all $t\in[0,1]$

$$\mathbf{x}(t)\in\sigma\{\mathbf{x}(1), \mathbf{y}(0), \mathbf{y}(1, t_1)\, \mathbf{y}(2, t_1 t_2), \ldots, \mathbf{y}(m, t_1 t_2 \ldots t_m), \ldots\}.$$

Now $G(n)=\sigma\{\mathbf{x}(1),\ \mathbf{y}(v),\ n(v)<n\}$, so two increments $\mathbf{x}(t)-\mathbf{x}(s)$, $\mathbf{x}(w)-\mathbf{x}(u)$, of Brownian motion, with $s, u\in[0, 1/2]$ and $t, w\in[1/2, 1]$, are independent conditional on $G(n)$ unless $s_1 s_2 \ldots s_n = u_1 u_2 \ldots u_n$ or $t_1 t_2 \ldots t_n = w_1 w_2 \ldots w_n$, i.e. unless two rooks placed on squares $(s_1 s_2 \ldots s_n,\ t_1 t_2 \ldots t_n)$, $(u_1 u_2 \ldots u_n,\ w_1 w_2 \ldots w_n)$ of a $N\times N$ chess board, $N=2^n$, can capture each other.

$G(n)=B(\mathfrak{L})$, with $\mathfrak{L}=\operatorname{span}\{g(v),\ v\in T(n)\}$, and $X=J(\lambda)$, with $d\lambda(s,t) = 2^{-1/2} 1_{A(0)}(s,t)\,ds\,dt$, so we may use (20) to estimate $\|X-E[X\,|\,G(n)]\|_p$. We use the remark following (20), and refer to the calculation of $\|X\|_p^p$ to write the resulting bound in the form

$$\|X-E[X\,|\,G(n)]\|_p^p \leqq 2^{\frac{5p}{2}} (p!)^2 \int\limits_{0\leqq s_1\leqq \ldots s_p\leqq 1} ds_1 \ldots ds_p \int\limits_{0\leqq t_1\leqq \ldots \leqq t_p\leqq 1} dt_1 \ldots dt_p f^n(s,t), \tag{22}$$

with

$$f^n(s,t) = \frac{1}{p!} \sum_{\pi\in S_p} \mathrm{f}_\pi(\mathrm{s}, \mathrm{t})\, \chi_\pi^n(s,t), \tag{23}$$

and $\chi_\pi^n(s,t)$ the characteristic function of the set

$\{(s,t)$: for each i, $1\leqq i\leqq p$, there is an index $j\neq i$, $1\leqq j\leqq p$ such that either $[2^{n-1}s_i]=[2^{n-1}s_j]$ or $[2^{n-1}t_{\pi(i)}]=[2^{n-1}t_{\pi(j)}]\}$.

(Here $[x]$ denotes the greatest integer $\leqq x$.)

Proceeding as in the derivation of $C4$ we write the right side of (22) as

$$2^{5p/2}(p!)^2(2\pi)^{-3p/2}\frac{1}{p!}\sum_{\pi\in S_p}\int d\boldsymbol{\eta}_1\dots d\boldsymbol{\eta}_p g_\pi^n(\boldsymbol{\eta}), \tag{24}$$

with

$$g_\pi^n(\boldsymbol{\eta})=\int_{\alpha_i\geqq 0,\Sigma\alpha\leqq 1} d\alpha_1\dots d\alpha_p \int_{\beta_i\geqq 0,\Sigma\beta_i\leqq 1} d\beta_1\dots d\beta_p \bar{\chi}_\pi^n(\alpha,\beta) \cdot\prod_{i=1}^p\left\{\exp\left[-\frac{(\boldsymbol{\eta}_{\pi(i)}-\boldsymbol{\eta}_{\pi(i-1)})^2}{2\alpha_i}-\frac{(\boldsymbol{\eta}_i-\boldsymbol{\eta}_{i-1})^2}{2\beta_i}\right](\alpha_i\beta_i)^{-3/2}\right\}, \tag{25}$$

and $\bar{\chi}_\pi^n$ the transform of χ_π^n under the change of variables $(s,t)\to(\alpha,\beta)$. Choose q with $1<q<4/3$, and apply Hölder's inequality to give

$$g_\pi^n(\boldsymbol{\eta})\leqq \|\bar{\chi}_\pi^n\|_{q'}I_\pi(\boldsymbol{\eta})^{1/q}, \tag{26}$$

with

$$I_\pi(\boldsymbol{\eta})=\int_{\alpha_i\geqq 0,\Sigma\alpha_i\leqq 1} d\alpha_1\dots d\alpha_p \int_{\beta_i\geqq 0,\Sigma\beta_i\leqq 1} d\beta_1\dots d\beta_p \cdot\prod_{i=1}^p\left\{\exp\left[-\frac{q(\boldsymbol{\eta}_{\pi(i)}-\boldsymbol{\eta}_{\pi(i-1)})^2}{2\alpha_i}-\frac{q(\boldsymbol{\eta}_i-\boldsymbol{\eta}_{i-1})^2}{2\beta_i}\right](\alpha_i\beta_i)^{-3q/2}\right\}. \tag{27}$$

Continuing to follow the pattern of the derivation of $C4$, we extend the integration region in (27) to $\left\{(\alpha,\beta)\colon \alpha_i\geqq 0,\ \beta_i\geqq 0,\ \sum_i(\alpha_i+\beta_i)\leqq 2\right\}$, and then exploit the transformation property of the integrand under scale transformations to give

$$\|X-E[X|G(n)]\|_p^p\leqq 2^{5p/2}(p!)^2(2\pi)^{-3p/2}\frac{1}{p!}\sum_{\pi\in S_p}\|\bar{\chi}_\pi^n\|_{q'}\int d\boldsymbol{\eta}_1\dots d\boldsymbol{\eta}_p h_\pi(\boldsymbol{\eta}), \tag{28}$$

with

$$h_\pi(\boldsymbol{\eta})=\int_0^\infty f(\lambda)\,\lambda^{-\left[\frac{2}{q}-\frac{3}{2}\right]p}[J_\pi(\boldsymbol{\eta},\lambda)]^{1/q}d\lambda. \tag{29}$$

Here $f(\lambda)$ is any probability density on $[0,\infty)$, and $J_\pi(\boldsymbol{\eta},\lambda)$ is the integral obtained by changing the integration region in (27) to $\left\{(\alpha,\beta)\colon \alpha_i\geqq 0,\ \beta_i\geqq 0,\ \sum_i(\alpha_i+\beta_i)\leqq 2\lambda\right\}$. We choose $f(\lambda)=\left[\Gamma\left(\left\{\frac{2}{q}-\frac{3}{2}\right\}p+1\right)\right]^{-1}\lambda^{[2/q-3/2]p}\exp(-\lambda)$, and apply Hölder's inequality to (29) to give

$$h_\pi(\boldsymbol{\eta})\leqq\left[\Gamma\left(\left\{\frac{2}{q}-\frac{3}{2}\right\}p+1\right)\right]^{-1}\left(\int_0^\infty\exp\left[\frac{-q'\lambda}{2}\right]d\lambda\right)^{1/q'}\left(\int_0^\infty\exp\left[\frac{-q\lambda}{2}\right]J_\pi(\boldsymbol{\eta},\lambda)d\lambda\right)^{1/q}$$
$$=\left[\Gamma\left(\left\{\frac{2}{q}-\frac{3}{2}\right\}p+1\right)\right]^{-1}\left(\frac{2}{q'}\right)^{1/q'}(L_\pi(\boldsymbol{\eta}))^{1/q}\quad\text{(say)}. \tag{30}$$

Interchange of the λ and (α,β) integrations in the integral representation of $L_\pi(\boldsymbol{\eta})$ now gives

$$L_\pi(\boldsymbol{\eta})=\frac{2}{q}\prod_{i=1}^p\{K(|\boldsymbol{\eta}_{\pi(i)}-\boldsymbol{\eta}_{\pi(i-1)}|)K(|\boldsymbol{\eta}_i-\boldsymbol{\eta}_{i-1}|)\}, \tag{31}$$

with

$$K(R)=\int_0^\infty d\alpha\,\alpha^{-3q/2}\exp\left[\frac{-q}{2}\left(\frac{R^2}{\alpha}+\frac{\alpha}{2}\right)\right]. \tag{32}$$

By the Cauchy-Schwarz inequality

$$\int d\mathbf{\eta}_1\ldots d\mathbf{\eta}_p(L_\pi(\mathbf{\eta}))^{1/q}\leqq\left(\frac{2}{q}\right)^{1/q}L^p, \tag{33}$$

with $L=\int d\mathbf{\eta}[K(|\mathbf{\eta}|)]^{2/q}<\infty$, since $q<\frac{4}{3}$.

To estimate $\|\bar{\chi}_\pi^n\|_{q'}$ in (28) we rely upon the lemma of the rooks:

Lemma 2. *Let p rooks be placed at random on a $N\times N$ chessboard (without restriction as to the number of rooks that may be placed on a single square). Then the probability that each rook is in a position to capture some other rook is* $\leqq 6^p p^{2p} N^{-p/2}$.

Proof. Denote by $\omega=(\omega_1,\ldots,\omega_p)$ a typical rook configuration, and write $X_j(\omega)=\omega_j$ for the position of the j^{th} rook, $1\leqq j\leqq p$. The X_j are assumed independent, each being uniformly distributed over the N^2 squares of the board. Denote by R the set of configurations having the property in question. For $\omega\in R$ define inductively sets $S^j(\omega)\subset\{1,2,\ldots,p\}$: $S^1(\omega)=\{1\}$, and for $1\leqq j\leqq p-1$

$S^{j+1}(\omega)=S^j(\omega)\cup\{j+1\}$ if for no $k\in S^j(\omega)$ can a rook placed on $X_{j+1}(\omega)$ capture a rook on $X_k(\omega)$

$=S^j(\omega)$ otherwise.

Write $S(\omega)=S^p(\omega)$, and define sets $S_1(\omega)$, $S_2(\omega)\subset S(\omega)^c$:

$S_1(\omega)=\{k\in S(\omega)^c$: a rook placed on $X_k(\omega)$ can capture exactly one of the rooks with index $j\in S(\omega)\}$,

$S_2(\omega)=\{k\in S(\omega)^c$: a rook placed on $X_k(\omega)$ can capture exactly two of the rooks with index $j\in S(\omega)\}$.

Note that for $k\in S(\omega)^c$, rook k can capture a rook $j\in S(\omega)$ with $j<k$ [by the inductive definition if $S(\omega)$], and can capture at most two rooks $j\in S(\omega)$ (one by a move on a horizontal, one by a move on a vertical file) [otherwise we should have two rooks $j_1, j_2\in S(\omega)$ which could be captured by k by moves of the same type (both horizontal or both vertical); but then j_1, j_2 could capture each other, which contradicts the definition of $S(\omega)$]. Thus $S(\omega)^c=S_1(\omega)\cup S_2(\omega)$. A rook $j\in S(\omega)$ cannot capture another rook $\in S(\omega)$ so, since $\omega\in R$, it must be able to capture a rook $\in S(\omega)^c$. Hence $|S(\omega)|\leqq|S_1(\omega)|+2|S_2(\omega)|$.

For a given partition (S,S_1,S_2) of $\{1,2,\ldots,p\}$ we estimate

$$\begin{aligned}\Pr\{\omega\in R, S_1(\omega)=S_1, S_2(\omega)=S_2\}&\leqq\left(\frac{2p}{N}\right)^{|S_1|}\left(\frac{p^2}{N^2}\right)^{|S_2|}\\&\leqq 2^p p^{2p} N^{-p/2}\end{aligned}$$

(if the set is non-empty, we must have $|S| \leqq |S_1| + 2|S_2|$; this implies $p = |S| + |S_1| + |S_2| \leqq 2|S_1| + 3|S_2| \leqq 2[|S_1| + 2|S_2|]$). Thus

$$\Pr\{\omega \in R\} \leqq \sum_{(S, S_1, S_2)} \Pr\{\omega \in R, S_1(\omega) = S_1, S_2(\omega) = S_2\}$$
$$\leqq 3^p \cdot 2^p p^{2p} N^{-p/2}. \quad \square$$

Lemma 2 implies $\|\chi_\pi^n\|_{q'} \leqq [6^p p^{2p} N^{-p/2}]^{1/q'}$, with $N = 2^{n-1}$. Combining this bound with (28), (30), (33) we obtain *C5* with $\beta_2 = \frac{11}{2} - \frac{4}{q}$, $\tau_2 = 2^{-1/2q'}$ (so e.g. choosing $q = 5/4$, $\beta_2 = 5.3$, $\tau_2 = 2^{-0.1}$).

C6. We show first that *C6* may be replaced by a condition which is more easily verified:

Lemma 3. *In Theorem 1 condition C6 may be replaced by the following condition C6′*

C6′. For some constants $K > 0$, *and* $\tau < 1$, *and all* $n \geqq 0$

$$\|X(n) - E[X(n) | F^+(0)]\|_2 \leqq K\tau^n. \tag{34}$$

Proof. From *C4* and Lemma 5, Sect. 2 we have for some $K_1 > 0$, and all $n \geqq 0$, $p \geqq 1$

$$\|X(n) - E[X(n) | F^+(0)]\|_p = \|Y(n) - E[Y(n) | F^+(0)]\|_p$$
$$\leqq \|Y(n)\|_p \leqq K_1 p^{\beta_1 + 1/2}. \tag{35}$$

Given $n \geqq 0$ write $n = n_1 + n_2$, with $n_1 = \left[\frac{n}{2}\right]$. Note that

$$F^+(0) \supset \bigvee_{v \in T(n_1)} F^+(v),$$

so that for all $p \geqq 1$

$$\|X(n) - E[X(n) | F^+(0)]\|_p \leqq 2 \left\| X(n) - E\left[X(n) \,\middle|\, \bigvee_{v \in T(n_1)} F^+(v)\right]\right\|_p. \tag{36}$$

Now

$$X(n) - E\left[X(n) \,\middle|\, \bigvee_{v \in T(n_1)} F^+(v)\right]$$
$$= 2^{\frac{-n_1}{2}} \sum_{v \in T(n_1)} \zeta(v)^* [X(n_2) - E[X(n_2) | F^+(0)]]$$
$$\simeq S_{N_1}\{X(n_2) - E[X(n_2) | F^+(0)]\},$$

with $N_1 = 2^{n_1}$. (34), (35) imply that the hypotheses of Lemma 3(b), Sect. 2 are satisfied by the sequence of random variables $X(n_2) - E[X(n_2) | F^+(0)]$ (with $N = N_1$, $C = K_1$, $\beta = \beta_1 + 1/2$, $D = K$, $\delta = \log(\tau^{-1})(\log 2)^{-1}$), so we conclude from that lemma that for some $\alpha > 0$, $K_2 > 0$, and all $p \leqq N_1^\alpha$ (we take $A = 1$)

$$\left\| X(n) - E\left[X(n) \,\middle|\, \bigvee_{v \in T(n_1)} F^+(v)\right]\right\|_p \leqq K_2 N_1^{-\alpha}. \tag{37}$$

Since $N_1 \geqq 2^{1/2(n-1)}$, (36), (37) *give*

$$\|X(n) - E[X(n)|F^+(0)]\|_p \leqq 2K_2 2^{-1/2\alpha(n-1)}, \tag{38}$$

for $p \leqq 2^{1/2\alpha(n-1)}$. But for $p > 2^{1/2\alpha(n-1)}$, (35) gives

$$\|X(n) - E[X(n)|F^+(0)]\|_p \leqq K_1 p^{\beta_1 + 1/2} \leqq K_1 p^{\beta_1 + 3/2} 2^{-1/2\alpha(n-1)}. \tag{39}$$

(38), (39) combine to give *C6* (with $\beta_3 = \beta_1 + \frac{3}{2}$, $\tau_3 = 2^{-1/2\alpha}$). □

To verify condition *C6'* note that $F^+(0) = B(\mathfrak{L})$, with $\mathfrak{L}$ = closed span $\{f(v), v \in T\}$, and $X(n) = 2^{-1/2} J(\lambda)$, with $d\lambda(s,t) = 1_{\Delta(n)}(s,t)\,ds\,dt$. Here $\Delta(n) = \bigcup_{v \in T(n)} \Delta(v)$. Hence, by (16),

$$\begin{aligned} &\|X(n) - E[X(n)|F^+(0)]\|_2^2 \\ &= 2^{-1} \int_{\Delta(n)} ds_1\,dt_1 \int_{\Delta(n)} ds_2\,dt_2 \sum_{A \subset \{1,2\}} (-1)^{|A|} [\det C(P, A, s, t)]^{-3/2}. \end{aligned} \tag{40}$$

Since $\mathfrak{L}^\perp = \operatorname{span}\{e(1)\}$, the orthogonal projection P with range $\mathfrak{L}$ is given by

$$P = 1 - \langle e(1), \cdot\,\rangle e(1). \tag{41}$$

(40) simplifies to

$$2^{-1} \int_{\Delta(n)} ds_1\,dt_1 \int_{\Delta(n)} ds_2\,dt_2 \{[\det C]^{-3/2} - [\det C(P, \{1,2\})]^{-3/2}\} \tag{42}$$

(since $\|X(n) - E[X(n)|F^+(0)]\|_2^2 = \|X(n)\|_2^2 - \|E[X(n)|F^+(0)]\|_2^2$). For $(s,t) \in \Delta(v_1) \times \Delta(v_2)$ with $v_1 \neq v_2$, we have

$$\det C \geqq \det C(P, \{1,2\}),$$

since C, $C(P, \{1,2\})$ have the same diagonal elements, and C is diagonal, so that the integrand in (42) is negative. We may therefore bound (42) by

$$2^{-1} \sum_{v \in T(n)} \int_{\Delta(v) \times \Delta(v)} ds_1\,dt_1\,ds_2\,dt_2 \{[\det C]^{-3/2} - [\det C(P, \{1,2\})]^{-3/2}\}. \tag{43}$$

The terms in (43) are all equal. This equality may be displayed explicitly by making, for each $v \in T(n)$, the change of variables $s_i' = Ns_i - [Ns_i]$, $t_i' = Nt_i - [Nt_i]$, with $N = 2^n$. After dropping the primes on the new variables we obtain

$$2 \int_0^{1/2} ds_1 \int_0^{1/2} dt_1 \int_0^{1/2} ds_2 \int_0^{1/2} dt_2 \{d^{-3/2} - d_N^{-3/2}\}, \tag{44}$$

with

$$\left.\begin{aligned} d &= |I_1|\,|I_2| - |I_1 \cap I_2|^2 \\ d_N &= d + 2N^{-1} |I_1 \cap I_2|\,|I_1|\,|I_2| - N^{-2} |I_1|^2 |I_2|^2 \end{aligned}\right\}. \tag{45}$$

In (45) $I_1 = [t_1, s_1]$, $I_2 = [t_2, s_2]$. (44) is bounded by the integral obtained by replacing d_N by

$$d_N' = d + 2N^{-1} |I_1 \cap I_2|\,|I_1|\,|I_2|. \tag{46}$$

Choose α with $0<\alpha<1/2$. Then for some constant B and all x,y with $x \geqq y>0$

$$y^{-3/2}-x^{-3/2} \leqq B(x-y)^{\alpha} y^{-3/2-\alpha}. \tag{47}$$

(44) is thus bounded by $KN^{-\alpha}$, with

$$K=2^{1+\alpha} B \int_0^{1/2} ds_1 \int_0^{1/2} dt_1 \int_0^{1/2} ds_2 \int_0^{1/2} dt_2 \frac{|I_1 \cap I_2|^{\alpha} |I_1|^{\alpha} I_2|^{\alpha}}{d^{3/2+\alpha}} < \infty, \tag{48}$$

and $C6'$ is verified.

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