

ON EIGEN-VALUES OF LAPLACIAN AND CURVATURE OF RIEMANNIAN MANIFOLD.

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1. Introduction. Let (M, g) be a compact connected orientable Riemannian manifold with fundamental tensor g , and Δ be the Laplace-Bertrami operator acting on differentiable functions of M , that is,

$$(1.1) \quad \Delta f = g^{ij} \nabla_i \nabla_j f,$$

where ∇_j denotes the covariant differentiation $\nabla_{\partial/\partial x_j}$ with respect to the Levi-Civita connection. Let $Sp(M, g) = \{0 = \lambda_0 > \lambda_1 = \dots = \lambda_n > \lambda_{n+1} = \dots\}$ be the set of eigen-values of Δ , where each eigen-value is written as many times as its multiplicity. Now there is an interesting problem: how the eigen-values of Δ determine the structure of (M, g) ? M. Berger has given a differential geometric approach to this problem. He used a formula of Minakshisundaram:

$$(1.2) \quad \sum_{k=0}^{\infty} e^{\lambda_k t} \underset{t \rightarrow 0}{\sim} (1/(4 \pi t)^{d/2}) \sum_{i=0}^{\infty} a_i t^i$$

where $d = \dim M$ and coefficients a_i 's are determined as follows. First take the normal coordinate system (U, y^i) about m . We put $g_{ij} = g(\partial/\partial y^i, \partial/\partial y^j)$. Then we define on U ,

$$(1.3) \quad c_0(y) = (\det(g_{ij}(y)))^{-1/4},$$

and for $i > 0$ define inductively

$$(1.4) \quad c_i(y) = c_0(y) \int_0^1 \frac{t^{i-1} \Delta c_{i-1}(ty)}{c_0(ty)} dt$$

for $y \in U$, where ty denotes the point with normal coordinates (ty^i) . Especially we have $c_i(0) = 1/i \Delta c_{i-1}(0)$. Now we define $u_i: M \rightarrow \mathbb{R}$ by $u_i(m) = c_i(0)$, and finally we have

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$$(1.5) \quad a_i = \int_M u_i \, dV,$$

where dV denotes the volume element of (M, g) . Berger ([1]) has calculated a_0 , a_1 , a_2 in terms of curvatures of M ,

$$(1.6) \quad a_0 = \text{volume } M,$$

$$(1.7) \quad a_1 = 1/6 \int_M \tau \, dV,$$

$$(1.8) \quad a_2 = 1/360 \int_M (5\tau^2 - 2|\rho|^2 + 2|R|^2) \, dV,$$

and has given some applications. One of which states that if (M, g) is of constant sectional curvature k , of dimension 2, 3, 4, and if $Sp(M, g) = Sp(M', g')$ holds good, then (M', g') must be of constant curvature k . See also [3].

In the present paper we shall calculate a_3 (Theorem 4.2) and give some applications. The main result is the following:

THEOREM. *Let (M, g) and (M', g') be compact connected orientable Einstein manifolds with dimension $M=6$. We assume that $\chi(M) = \chi(M')$ and $Sp(M, g) = Sp(M', g')$ hold where $\chi(M)$ denotes the Euler characteristic of M . Then (M, g) is locally symmetric if and only if (M', g') is locally symmetric.*

In §2 we shall give preliminaries for later use. In §3 and §4, we calculate a_3 in terms of curvatures of M and in final §5 we shall give applications.

2. Preliminaries. We shall give some formulas which are used in the subsequent sections. Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇ . Let $R(X, Y) Z = \nabla_{[X, Y]} Z - [\nabla_X, \nabla_Y] Z$ be a curvature tensor of ∇ . We put $R_{ijkh} = g(R(\partial/\partial y^i, \partial/\partial y^j)\partial/\partial y^k, \partial/\partial y^h)$, then $\rho_{ij} = R^a_{iaj}{}^*$ and ρ_a^a are Ricci tensor and scalar curvature respectively. They satisfy following fundamental formulas. Let; denote the covariant differentiation, and put $\nabla\tau = (\tau_{;i})$, $\nabla\rho = (\rho_{ij;k})$, $\nabla R = (R_{ijkl;i})$. $|\cdot|^2$ denotes the square of the length of a tensor“.”

(2.1) (Bianchi's identities)

$$(i) \quad R_{ijkh} + R_{ikjh} + R_{ihjk} = 0,$$

$$(ii) \quad R_{ijkh;i} + R_{ijhl;k} + R_{ijlk;h} = 0.$$

Especially we have

*> Throughout this paper we follow the Einstein's summation convention.

$$(2.2) \quad \begin{aligned} (i) \quad R^a_{ijk;a} &= \rho_{ik;j} - \rho_{ij;k}, \\ (ii) \quad \rho^a_{j;a} &= 1/2\tau_{;j}. \end{aligned}$$

(2.3) (Ricci's identity)

$$\nu_{h;kj} - \nu_{h;jk} = \nu_a R^a_{hkj}.$$

Now following (2.4)~(2.7) are all easily derived from (2.1).

$$(2.4) \quad \begin{aligned} (i) \quad R^{abcd;u} R_{adcb;u} &= 1/2 |\nabla R|^2, \\ (ii) \quad R^{abcd;u} R_{cbad;u} &= 1/2 |\nabla R|^2, \\ (iii) \quad R^{abcd;u} R_{abud;c} &= 1/2 |\nabla R|^2, \\ (iv) \quad R^{abcu;v} R_{abcv;u} &= 1/2 |\nabla R|^2, \\ (v) \quad R^{abcd;u} R_{ubcd;a} &= 1/2 |\nabla R|^2, \\ (vi) \quad R^{abcd;u} R_{ubad;c} &= 1/4 |\nabla R|^2, \end{aligned}$$

$$(2.5) \quad R^{abcd} R_{adcb;}^u u = 1/2 R^{abcd} R_{abcd;}^u u,$$

$$(2.6) \quad \begin{aligned} (i) \quad \rho^{uv} R_u^{ab} R_{vcba} &= 1/2 \rho^{uv} R_u^{ab} R_{vabc}, \\ (ii) \quad \rho^{uv} R_u^{ab} R_{vbca} &= -1/2 \rho^{uv} R_u^{ab} R_{vabc}, \end{aligned}$$

$$(2.7) \quad \begin{aligned} (i) \quad R^{abcd} R_{ab}^{uv} R_{cudv} &= R^{abcd} R_{a}^{u}{}_{b}^{v} R_{cd}{}_{uv}, \\ &= 1/2 R^{abcd} R_{ab}^{uv} R_{cd}{}_{uv}, \\ (ii) \quad R^{abcd} R_{a}^{u}{}_{b}^{v} R_{cudv} &= 1/4 R^{abcd} R_{ab}^{uv} R_{cd}{}_{uv}, \\ (iii) \quad R^{abcd} R_{a}^{u}{}_{b}^{v} R_{cvdu} &= -R^{abcd} R_{a}^{u}{}_{b}^{v} R_{cu}{}_{dv} \\ &= -1/4 R^{abcd} R_{ab}^{uv} R_{cd}{}_{uv}, \\ (iv) \quad R^{abcd} R_{ac}^{uv} R_{bd}{}_{uv} &= 1/2 R^{abcd} R_{ab}^{uv} R_{cd}{}_{uv}, \\ (v) \quad R^{abcd} R_{a}^{u}{}_{c}^{v} R_{bd}{}_{uv} &= R^{abcd} R_{ac}^{uv} R_{bd}{}_{uv} \\ &= 1/2 R^{abcd} R_{ac}^{uv} R_{bd}{}_{uv} \\ &= 1/4 R^{abcd} R_{ab}^{uv} R_{cd}{}_{uv}, \\ (vi) \quad R^{abcd} R_{a}^{u}{}_{c}^{v} R_{bv}{}_{du} &= R^{abcd} R_{a}^{v}{}_{c}^{u} R_{bd}{}_{uv} \\ &= R^{abcd} R_{a}^{u}{}_{c}^{v} R_{bd}{}_{uv} - 1/4 R^{abcd} R_{ab}^{uv} R_{cd}{}_{uv}. \end{aligned}$$

Next, following (2.8)~(2.13) are easily derived from the definition of Laplacian.

$$(2.8) \quad \tau \Delta \tau = 1/2 \Delta \tau^2 - |\nabla \tau|^2,$$

$$(2.9) \quad \rho^{ab} \tau_{;ab} = (\rho^{ab} \tau)_{;ab} - 1/4 \Delta \tau^2 - 1/2 |\nabla \tau|^2,$$

$$(2.10) \quad \rho^{ab} \rho_{ab;c}^c = 1/2 \Delta |\rho|^2 - |\nabla \rho|^2,$$

$$(2.11) \quad R^{ab\cdots d}R_{abcd;u} = 1/2 \triangle |R|^2 - |\nabla R|^2,$$

$$(2.12) \quad \rho^{ab;c}\rho_{ac;b} = (\rho^{ab}\rho_a^c)_{;bc} - (\rho^{ab}\tau)_{;ab} + 1/4 \triangle \tau^2 + 1/4|\nabla \tau|^2 \\ - \rho^{ab}\rho_a^c\rho_{bc} + \rho^{ab}\rho^{cd}R_{acbd},$$

$$(2.13) \quad R^{ab\cdots d}\rho_{ac;bd} = (R^{ab\cdots d}\rho_{ac})_{;bd} + 2(\rho^{ab}\rho_a^c)_{;bc} - 3/2(\rho^{ab}\tau)_{;ab} \\ - 1/2 \triangle |\rho|^2 + 3/8 \triangle \tau^2 + 1/4|\nabla \tau|^2 - |\nabla \rho|^2 - \rho^{ab}\rho_a^c\rho_{bc} \\ + \rho^{ab}\rho^{cd}R_{acbd}.$$

Finally we shall give two integral formulas for compact orientable manifold M . We denote the volume element of M by dV .

$$(2.14) \quad \int_M R^{ab\cdots d}R_a^u{}_c^v R_{bvd}dV = \int_M \{1/4|\nabla \tau|^2 - |\nabla \rho|^2 + 1/4|\nabla R|^2 \\ - \rho^{ab}\rho_a^c\rho_{bc} + \rho^{ab}\rho^{cd}R_{acbd} + 1/2\rho^{uv}R_u^{ab}R_{vabc} \\ - 1/4R^{ab\cdots d}R_{ab}^{uv}R_{cd}{}_{uv}\} dV,$$

$$(2.15) \quad \int_M R^{ab\cdots d}R_a^u{}_c^v R_{bvd}dV = \int_M \{1/4|\nabla \tau|^2 - |\nabla \rho|^2 + 1/4|\nabla R|^2 \\ - \rho^{ab}\rho_a^c\rho_{bc} + \rho^{ab}\rho^{cd}R_{acbd} + 1/2\rho^{uv}R_u^{ab}R_{vabc} \\ - 1/2R^{ab\cdots d}R_{ab}^{uv}R_{cd}{}_{uv}\} dV.$$

Proof of (2.14) follows from Green's theorem by expanding $(R^{ab\cdots c}R^v{}_{abc})_{;uv}$ and using (2.4), (2.7). Then (2.15) follows from (2.14) and (2.7) (vi).

3. Calculation of a_3 —(1). 1° Let $m \in M$, and (U, y^i) be the normal coordinate system about m . Then $g_{ij} = g(\partial/\partial y^i, \partial/\partial y^j)$ on U can be represented in terms of curvature tensor and its successive covariant derivatives at m , in fact, we get

LEMMA 3.1. In U , g_{ij} may be expanded in the following way:

$$(3.1) \quad g_{ij} = \delta_{ij} - 1/3 R_{kijh}(m)y^ky^h - 1/3! R_{kijh; p}(m)y^ky^h y^p \\ + 1/5! (-6 R_{kijh; pq}(m) + 16/3 R_{kijh}(m)R_{pqru}(m))y^ky^h y^p y^q \\ + 1/6! (-8 R_{kijh; pqr}(m) + 16 R_{kijh}(m)R_{qjqu; r}(m) \\ + 16 R_{kjhu}(m)R_{piqu; r}(m))y^ky^h y^p y^q y^r + 1/7! (-10 R_{kijh; pqr}(m)$$

$$\begin{aligned}
& + 34 R_{kthu;pq}(m)R_{rjsu}(m) + 34 R_{kjhu;pq}(m)R_{risu}(m) \\
& + 55 R_{kthu;p}(m)R_{qjru;s}(m) - 16 R_{kthu}(m)R_{pjqu}(m)R_{rusv}(m)) \\
& \times y^k y^h y^p y^q y^r y^s + o(|\cdot|^6).
\end{aligned}$$

PROOF. Let $\gamma(t) = (a^i t)$ be a geodesic with the initial point m and the initial direction $a^i \partial/\partial y^i$ such that $\gamma(l) = y$. Then $J_i(t) = t/l \partial/\partial y^i(\gamma(t))$ is a Jacobi field along γ which satisfies $J_i(0) = 0$, $J'_i(0) = 1/l \partial/\partial y^i(m)$. Put $f(t) = g(J_i(t), J_j(t))$, then $g_{ij}(y) = f(l)$ and we get (3.1) from the Taylor expansion of $f(l)$ and using the Jacobi equation $\nabla_{\dot{\gamma}(t)} \nabla_{\dot{\gamma}(t)} J_i(t) + R(\dot{\gamma}(t), J_i(t))\dot{\gamma}(t) = 0$.

COROLLARY 3.2.

$$\begin{aligned}
(3.2) \quad g^{ij} &= \delta_{ij} + 1/3 R_{kthi}(m)y^k y^h + 1/3 ! R_{kthj;i}(m)y^k y^h y^l \\
& + 1/5 ! (6 R_{kthj;pq}(m) + 8 R_{kthu}(m)R_{pjqu}(m))y^k y^h y^p y^q + o(|\cdot|^4).
\end{aligned}$$

COROLLARY 3.3.

$$\begin{aligned}
(3.3) \quad \Gamma_{ab}^c &= -1/3(R_{kabc}(m) + R_{kbac}(m))y^k - 1/3 ! \{(R_{kabc;h}(m) \\
& + R_{kbac;h}(m)) + 1/2(R_{akch;b}(m) + R_{bkch;a}(m) - R_{akhb;c}(m))\}y^k y^h \\
& - 1/5 ! [6(R_{kabc;hl}(m) + R_{kbac;hl}(m)) + 3(-R_{kahb;cl}(m) \\
& - R_{kahb;lc}(m) + R_{kbhc;al}(m) + R_{kbhc;la}(m) + R_{kahc;bl}(m) \\
& + R_{kahc;lb}(m)) - 8(R_{kahu}(m)R_{bcu}(m) + R_{kbhu}(m)R_{actu}(m)) \\
& + 32/3 R_{hclu}(m)(R_{kabu}(m) + R_{kbau}(m))]y^k y^h y^l + o(|\cdot|^3).
\end{aligned}$$

Proof of Corollary 3.2 is direct, and (3.3) comes from $\Gamma_{ab}^c = 1/2 g^{cd}(\partial_a g_{bc} + \partial_b g_{ca} - \partial_c g_{ab})$, (3.1), (3.2), and Bianchi's identity (2.1). Next we shall find the formula for $c_0 = (\det g_{ij})^{-1/4}$. First from $\det g_{ij} = \sum_{(i_1 \dots i_n) \in S_n} \epsilon_{i_1 i_2 \dots i_n} \prod_{a=1}^d g_{a i_a}$ and (3.1), we have

LEMMA 3.4. In normal coordinate neighborhood,

$$(3.4) \quad \det g_{ij} = 1 + A + B + C + D + E + o(|\cdot|^6)$$

where

$$\begin{aligned}
A &= -1/3 \rho_{kh}(m)y^k y^h, \\
B &= -1/3 ! \rho_{kh; p}(m)y^k y^h y^p, \\
C &= 1/4! (-6/5 \rho_{kh; pq}(m) + 4/3 \rho_{kh}(m)\rho_{pq}(m) - 4/15 R_{kuhv}(m)R_{puqv}(m))y^k y^h y^p y^q,
\end{aligned}$$

$$\begin{aligned}
D &= 1/5 ! \ 4/3(-\rho_{kh;pq,r}(m) + 5\rho_{kh}(m)\rho_{pq;r}(m) - R_{kuhv}(m)R_{puqv;r}(m))y^k y^h y^p y^q y^r, \\
E &= 1/6 ! \ \{-10/7 \rho_{kh;pq,rs}(m) + 12\rho_{kh}(m)\rho_{pq;rs}(m) - 16/7 R_{kuhv}(m)R_{puqv;rs}(m) \\
&\quad + 10\rho_{kh;p}(m)\rho_{qr;s}(m) - 15/7 R_{kuhv;p}(m)R_{qurv;s}(m) - 40/9 \rho_{kh}(m)\rho_{pq}(m)\rho_{rs}(m) \\
&\quad + 8/3 \rho_{kh}(m)R_{puqv}(m)R_{rusv}(m) \\
&\quad - 32/63 R_{kuhv}(m)R_{pvqw}(m)R_{rwsv}(m)\}y^k y^h y^p y^q y^r y^s.
\end{aligned}$$

Thus from the formula $f(x) = (1+x)^{-1/4} = 1 - 1/4 x + 5/32 x^2 - 15/128 x^3 + \dots$, we have

LEMMA 3.5.

$$\begin{aligned}
(3.5) \quad c_0 &= 1 + 1/12 \rho_{kh}(m)y^k y^h + 1/24 \rho_{kh;p}(m)y^k y^h y^p + 1/4 ! \ (3/10 \rho_{kh;pq}(m) \\
&\quad + 1/12 \rho_{kh}(m)\rho_{pq}(m) + 1/15 R_{kuhv}(m)R_{puqv}(m))y^k y^h y^p y^q \\
&\quad + 1/5 ! \ (1/3 \rho_{kh;pqr}(m) + 5/12 \rho_{kh}(m)\rho_{pq;r}(m) + 1/3 R_{kuhv}(m)R_{puqv;r}(m)) \\
&\quad \times y^k y^h y^p y^q y^r + 1/6 ! \ \{5/14 \rho_{kh;pq,rs}(m) + 3/4 \rho_{kh}(m)\rho_{pq;rs}(m) \\
&\quad + 4/7 R_{kuhv}(m)R_{puqv;rs}(m) + 5/8 \rho_{kh;p}(m)\rho_{qr;s}(m) + 15/28 R_{kuhv;p}(m) \\
&\quad \times R_{qurv;s}(m) + 5/72 \rho_{kh}(m)\rho_{pq}(m)\rho_{rs}(m) + 1/6 \rho_{kh}(m)R_{puqv}(m)R_{rusv}(m) \\
&\quad + 8/63 R_{kuhv}(m)R_{pvqw}(m)R_{rwsv}(m)\}y^k y^h y^p y^q y^r y^s + o(|\cdot|^6).
\end{aligned}$$

COROLLARY 3.6.

$$\begin{aligned}
(3.6) \quad c_0^{-1} &= 1 - 1/12 \rho_{kh}(m)y^k y^h - 1/24 \rho_{kh;p}(m)y^k y^h y^p - 1/4 ! \ \{3/10 \rho_{kh;pq}(m) \\
&\quad - 1/12 \rho_{kh}(m)\rho_{pq}(m) + 1/15 R_{kuhv}(m)R_{puqv}(m)\}y^k y^h y^p y^q + o(|\cdot|^4).
\end{aligned}$$

2°. Next we put $\Delta c_0 = x + x_i y^i + x_{ij} y^i y^j + x_{ijk} y^i y^j y^k + x_{ijkl} y^i y^j y^k y^l + o(|\cdot|^4)$ and represent $u_3(m)$ in terms of x, x_i, x_{ij}, x_{ijk} and x_{ijkl} . Firstly we have

$$\begin{aligned}
(3.7) \quad c_1 &= c_0 \int_0^1 \frac{\Delta c_0(ty)}{c_0(ty)} dt \\
&= x + 1/2 x_i y^i + 1/3(x_{ij} + 1/6 x \rho_{ij}(m))y^i y^j + 1/4(x_{ijk} \\
&\quad + 1/12 x_i \rho_{jk}(m) + 1/8 x \rho_{ij;k}(m))y^i y^j y^k + \{1/5(x_{ijkl} \\
&\quad + 1/18 x_{ij} \rho_{kh}(m) + 1/16 x_i \rho_{jk;h}(m)) + 1/5 ! x \{6/5 \rho_{ij;kh}(m) \\
&\quad + 2/9 \rho_{ij}(m)\rho_{kh}(m) + 4/15 R_{iu;fv}(m)R_{kuhv}(m)\}y^i y^j y^k y^l + o(|\cdot|^4)\}.
\end{aligned}$$

Secondly we shall calculate $c_2 = c_0 \int_0^1 \frac{t \Delta c_1(ty)}{c_0(ty)} dt$. Since $\Delta c_1 = g^{ab} \partial_b \partial_a c_1$

$-g^{ab}\Gamma_{ab}^c\partial_c c_1$, by (2.2), (2.3), (3.2), (3.3) and (3.7) we get

$$\begin{aligned}
 (3.8) \quad c_2 &= 1/6(2x_{aa} + 1/3x\tau(m)) + 1/12\{2(x_{aa} + x_{aia} + x_{ia}) \\
 &\quad + 1/6x_i\tau(m) - x_a\rho_{ai}(m) + 1/2x\tau_{;i}(m)\}y^i + [1/10(x_{aa} + x_{aia}) \\
 &\quad + x_{aia} + x_{ia} + x_{ia} + x_{ia}] + 1/20\{1/9x_{ij}\tau(m) + 7/18x_{aa}\rho_{ij}(m) \\
 &\quad - 8/9(x_{ai} + x_{ia})\rho_{aj}(m) + 10/9x_{uv}R_{iu,jv}(m)\} + 1/20\{1/4x_i\tau_{;j}(m) \\
 &\quad + x_a(1/3\rho_{ij;a}(m) - \rho_{ia;j}(m))\} + 1/5!x\{3/5(3\tau_{;ij}(m) + \rho_{ij;aa}(m)) \\
 &\quad + 1/2\tau(m)\rho_{ij}(m) - 26/45\rho_{iu}(m)\rho_{ju}(u) + 8/45\rho_{uv}(m)R_{iu,jv}(m) \\
 &\quad + 2/5R_{iabc}(m)R_{jab}(m)\}\}y^iy^j + o(|\cdot|^2).
 \end{aligned}$$

LEMMA 3.7.

$$\begin{aligned}
 (3.9) \quad u_3(m) &= 2/15(x_{aabb} + x_{abab} + x_{abba}) + 1/30(1/2x_{aa}\tau(m) \\
 &\quad - 2/3x_{ab}\rho_{ab}(m)) + 1/360x_a\tau_{;a}(m) + 1/5!x(8/5\Delta\tau \\
 &\quad + 1/3\tau^2 - 4/15|\rho|^2 + 4/15|R|^2).
 \end{aligned}$$

PROOF. This follows from $u_3(m) = c_i(0) = 1/3\Delta c_2(0)$ and (3.8).

4. Calculation of a_3 — (2). 1°. By (3.7), to get the formula for a_3 it suffices to represent x, x_i, x_{ij}, x_{ijkh} in terms of curvatures of M . Since $\Delta c_0 = g^{ab}\partial_b\partial_a c_0 - g^{ab}\Gamma_{ab}^c\partial_c c_0$, we first calculate $g^{ab}\partial_b\partial_a c_0$. By (3.5), (2.3), (3.2) etc., we have

$$\begin{aligned}
 (4.1) \quad g^{ab}\partial_b\partial_a c_0 &= 1/6\tau(m) + 1/6\tau_{;i}(m)y^i + 1/4!(9/5\tau_{;ij}(m) \\
 &\quad + 3/5\rho_{ij;uu}(m) + 1/3\tau(m)\rho_{ij}(m) + 28/15\rho_{iu}(m)\rho_{ju}(m) \\
 &\quad + 2/5\rho_{uv}(m)R_{iu,jv}(m) + 2/5R_{iuvw}(m)R_{juvw}(m))y^iy^j \\
 &\quad + (\text{term of order 3}) + 1/6![5/7\{2(\tau_{;ijkh}(m) + \rho_{iuj;jkhu}(m) \\
 &\quad + \rho_{iu;jkhu}(m) + \rho_{iu;jkhu}(m)) + (\rho_{ij;uukh}(m) + \rho_{ij;uukh}(m) + \rho_{ij;uukh}(m) \\
 &\quad + \rho_{ij;kuuh}(m) + \rho_{ij;kuuh}(m) + \rho_{ij;kuuh}(m))\} + 3/2\{\tau(m)\rho_{ij;kh}(m) \\
 &\quad + 3\rho_{ij}(m)\tau_{;kh}(m) + \rho_{ij}(m)\rho_{kh;uu}(m) + 2\rho_{iu}(m)(\rho_{jk;uh}(m) + \rho_{jk;uh}(m)) \\
 &\quad + 4\rho_{iu}(m)\rho_{uj;kh}(m)\} + 2/7\{25\rho_{uv}(m)R_{iu,jv;kh}(m) \\
 &\quad + 4R_{iu,jv}(m)R_{kuhv;ww}(m) + 8R_{iuvw}(m)(R_{juvw;kh}(m) + R_{juvw;kh}(m))
 \end{aligned}$$

$$\begin{aligned}
& + R_{jukw;hv}(m) + R_{jukw;vh}(m)) + R_{iu;jv}(m)(41 \rho_{uv;kh}(m) + 21 \rho_{kh;uv}(m) \\
& + 26\rho_{ku;vh}(m) + 42 \rho_{ku;hv}(m)) + 5(\tau_i(m)\rho_{jk;h}(m) + \rho_{ij;u}(m)\rho_{ku;h}(m) \\
& + \rho_{iu;j}(m)\rho_{ku;h}(m)) + 5/4 \rho_{ij;u}(m)\rho_{kh;u}(m) \\
& + 15/7 \{23/3 R_{iu;jv;k}(m)\rho_{uv;h}(m) + 22/3 R_{iu;jv;k}(m)\rho_{hu;v}(m) \\
& + 1/2 R_{iu;jv;w}(m)R_{ku;hv;w}(m) + R_{iu;vw;j}(m)(R_{ku;vw;h}(m) + R_{kw;vu;h}(m) \\
& + 2R_{ku;hv;v}(m))\} + \{5/12 \tau(m)\rho_{ij}(m)\rho_{kh}(m) \\
& + 14/3 \rho_{ij}(m)\rho_{ku}(m)\rho_{hu}(m)\} + \{\rho_{ij}(m)\rho_{uv}(m)R_{ku;hv}(m) \\
& + 20/3 \rho_{iu}(m)\rho_{ju}(m)R_{ku;hv}(m)\} + \{1/3 \tau(m)R_{ku;hv}(m)R_{iu;jv}(m) \\
& + 104/21 \rho_{iu}(m)R_{uv;jw}(m)R_{kv;hw}(m) \\
& + 184/21 \rho_{uv}(m)R_{iu;jw}(m)R_{kv;hw}(m) + 2/3 \rho_{ij}(m)R_{ku;vw}(m)(R_{hv;vw}(m) \\
& + R_{hv;vw}(m))\} + 8/21 R_{iu;jv}(m)\{R_{kv;hw}(m)R_{uv;vx}(m) \\
& + 11 R_{kx;uw}(m)(R_{hx;vw}(m) - R_{hv;vx}(m))\}]y^i y^j y^k y^h + o(|\cdot|^4).
\end{aligned}$$

Secondly, by (3.2), (3.3), (2.3) etc., we get

$$\begin{aligned}
(4.2) \quad g^{ab}\Gamma_{ab}^c \partial_c c_0 & = 1/9 \rho_{iu}(m)\rho_{ju}(m)y^i y^j + (\text{term of order 3}) \\
& + 1/6 ! [\{36 \rho_{iu}(m)\rho_{ju;kh}(m) - 3 \rho_{iu}(m)\rho_{jk;uh}(m) \\
& + 9 \rho_{iu}(m)\rho_{jk;hu}(m)\} + \{30 \rho_{iu;j}(m)\rho_{ku;h}(m) + 10 \rho_{ij;u}(m)\rho_{ku;h}(m) \\
& - 5/2 \rho_{ij;u}(m)\rho_{kh;u}(m)\} + \{20/3 \rho_{iu}(m)\rho_{ju}(m)\rho_{kh}(m) \\
& + 46/3 \rho_{iu}(m)\rho_{ju}(m)R_{ku;hv}(m) + 16 \rho_{iu}(m)R_{uv;jw}(m)R_{kv;hw}(m)\}y^i y^j y^k y^h \\
& + o(|\cdot|^4)].
\end{aligned}$$

LEMMA 4.1. x, x_i, x_{ij}, x_{ijkh} are given as follows:

$$(4.3) \quad x = 1/6 \tau(m),$$

$$(4.4) \quad x_i = 1/6 \tau_{;i}(m),$$

$$\begin{aligned}
(4.5) \quad x_{ij} & = 1/5 ! \{9\tau_{;ij}(m) + 3 \rho_{ij;uu}(m) + 5/3 \tau(m)\rho_{ij}(m) - 4\rho_{iu}(m)\rho_{ju}(m) \\
& + 2 \rho_{uv}(m)R_{iu;jv}(m) + 2R_{iu;vw}(m)R_{ju;vw}(m)\},
\end{aligned}$$

$$(4.6) \quad x_{ijkh} = 1/6 ! (\textcircled{1} + \textcircled{2} + \textcircled{3} + \textcircled{4}),$$

where

$$\begin{aligned}
\textcircled{1} &= 5/7 \{ 2(\tau_{;ijkh}(m) + \rho_{iu;jukh}(m) + \rho_{iu;jkuh}(m) + \rho_{iu;jkhu}(m)) + (\rho_{ij;uuhh}(m) \\
&\quad + \rho_{ij;ukuh}(m) + \rho_{ij;ukhu}(m) + \rho_{ij;kuuh}(m) + \rho_{ij;kuhu}(m) + \rho_{ij;khuu}(m)) \}, \\
\textcircled{2} &= 3/2(\tau(m)\rho_{ij;kh}(m) + 3\rho_{ij}(m)\tau_{;kh}(m) + \rho_{ij}(m)\rho_{kh;uu}(m)) \\
&\quad + 6\rho_{iu}(m)(\rho_{jk;uh}(m) - \rho_{jk;hu}(m) - 5\rho_{ju;kh}(m)) + 2/7 \{ 25\rho_{uv}(m)R_{iu;jv;kh}(m) \\
&\quad + 4R_{iu;jv}(m)R_{ku;hv;ww}(m) + 8R_{iuvw}(m)(R_{ju;vw;kh}(m) \\
&\quad + R_{jwvu;kh}(m) + R_{ju;kv;hv}(m) + R_{ju;kv;vh}(m)) + R_{iu;jv}(m)(41\rho_{uv;kh}(m) \\
&\quad + 21\rho_{kh;uv}(m) + 26\rho_{ku;vh}(m) + 42\rho_{ku;hv}(m)) \}, \\
\textcircled{3} &= 5\{\tau_{;i}(m)\rho_{jk;h}(m) - \rho_{ij;u}(m)\rho_{ku;h}(m) - 5\rho_{iu;j}(m)\rho_{ku;h}(m)\} \\
&\quad + 15/4\rho_{ij;u}(m)\rho_{kh;u}(m) + 15/7\{23/3R_{iu;jv;k}(m)\rho_{uv;h}(m) \\
&\quad + 22/3R_{iu;jv;k}(m)\rho_{hu;v}(m) + 1/2R_{iu;jv;w}(m)R_{ku;hv;w}(m) \\
&\quad + R_{iuvw;f}(m)(R_{ku;vw;h}(m) + R_{kw;vv;h}(m) + 2R_{ku;hw;v}(m)) \}, \\
\textcircled{4} &= \{5/12\tau(m)\rho_{ij}(m)\rho_{kh}(m) - 2\rho_{iu}(m)\rho_{ju}(m)\rho_{kh}(m) \\
&\quad + \rho_{ij}(m)\rho_{uv}(m)R_{ku;hv}(m) - 26/3\rho_{iu}(m)\rho_{ju}(m)R_{ku;hv}(m) \\
&\quad + 1/3\tau(m)R_{iu;jv}(m)R_{ku;hv}(m)\} + \{-232/21\rho_{iu}(m)R_{uv;jw}(m)R_{kv;hw}(m) \\
&\quad + 184/21\rho_{uv}(m)R_{iu;jw}(m)R_{kv;hw}(m) + 2/3\rho_{ij}(m)R_{ku;vw}(m)(R_{hv;vw}(m) \\
&\quad + R_{hw;vw}(m))\} + 8/21R_{iu;jv}(m)\{R_{kw;hx}(m)R_{uw;vx}(m) + 11R_{kx;ux}(m) \\
&\quad \times (R_{hx;vw}(m) - R_{hw;vx}(m))\}.
\end{aligned}$$

2.^o Next we shall calculate $x_{aabb} + x_{abab} + x_{abba}$ from (4.6). With full use of (2.1)~(2.7) we have

$$x_{aabb} + x_{abab} + x_{abba} = 1/6\{\textcircled{1}' + \textcircled{2}' + \textcircled{3}' + \textcircled{4}'\},$$

where \textcircled{i}' comes from the corresponding \textcircled{i} in Lemma 4.1, $i = 1, 2, 3, 4$.

$$\begin{aligned}
\textcircled{1}' &= \operatorname{div} X_1(m), \text{ for some vector field } X_1, \\
\textcircled{2}' &= 9\tau(m)\Delta\tau(m) - 87/7\rho_{ab}(m)\tau_{;ab}(m) + 51/7\rho_{ab}(m)\rho_{ab;cc}(m) \\
&\quad - 174/7\rho_{ab}(m)\rho_{bc}(m)\rho_{ac}(m) + 174/7\rho_{ab}(m)\rho_{cd}(m)R_{ac;bd}(m) \\
&\quad + 36/7R_{abcd}(m)R_{abcd;uu}(m) + 48R_{abcd}(m)\rho_{ac;bd}(m) \\
&\quad + 48/7\rho_{uv}(m)R_{uabc}(m)R_{vabc}(m) - 192/7R_{abcd}(m)R_{aucv}(m)R_{bucd}(m)
\end{aligned}$$

$$\begin{aligned}
& + 96/7 R_{abcd}(m)R_{aucv}(m)R_{bvd़u}(m), \\
\textcircled{3}' & = 5 |\nabla \tau|^2(m) + 195/7 |\nabla \rho|^2(m) - 330/7 \rho_{ab;c}(m)\rho_{ac;b}(m) + 135/14 |\nabla R|^2(m), \\
\textcircled{4}' & = 5/12 \tau^3(m) + 1/6 \tau(m) |\rho|^2(m) + 3/2 \tau(m) |R|^2(m) \\
& - 82/21 \rho_{ab}(m)\rho_{ac}(m)\rho_{bc}(m) + 34/7 \rho_{uv}(m)R_{uabc}(m)R_{vabc}(m) \\
& - 80/21 R_{abcd}(m)R_{aucv}(m) (R_{bvd़u}(m) + R_{bvd़u}(m)).
\end{aligned}$$

Thus we have

$$\begin{aligned}
(4.7) \quad & x_{aabb} + x_{abab} + x_{abba} = 1/6 ! [\{9 \tau(m) \triangle \tau(m) - 87/7 \rho_{ab}(m)\tau_{;ab}(m) \\
& + 51/7 \rho_{ab}(m)\rho_{ab;cc}(m) + 36/7 R_{abcd}(m)R_{abcd;uu}(m) \\
& + 48R_{abcd}(m)\rho_{ac;ba}(m)\} + \{5 |\nabla \tau|^2(m) + 195/7 |\nabla \rho|^2(m) + 135/14 |\nabla R|^2(m) \\
& - 330/7 \rho_{ab;c}(m)\rho_{ac;}(m)\}_b(m) + \{5/12 \tau^3(m) + 1/6 \tau(m) |\rho|^2(m) \\
& + 3/2 \tau(m) |R|^2(m) - 604/21 \rho_{ab}(m)\rho_{bc}(m)\rho_{ac}(m) \\
& + 174/7 \rho_{ab}(m)\rho_{cd}(m)R_{acbd}(m) + 82/7 \rho_{uv}(m)R_{uabc}(m)R_{vabc}(m) \\
& - 656/21 R_{abcd}(m)R_{aucv}(m)R_{bvd़u}(m) + 208/21 R_{abcd}(m)R_{aucv}(m)R_{bvd़u}(m)\}] \\
& + \text{div } X_1(m).
\end{aligned}$$

for some vector field X_1 on M .

Now by Lemma 3.7, (4.7) and formulas (2.8)~(2.13) we have

$$\begin{aligned}
(4.8) \quad u_3(m) & = 1/6 ! [-54/35 |\nabla \tau|^2(m) - 114/35 |\nabla \rho|^2(m) + 3/5 |\nabla R|^2(m) \\
& + 5/9 \tau^3(m) - 2/3 \tau(m) |\rho|^2(m) + 2/3 \tau(m) |R|^2(m) \\
& - 1076/315 \rho_{ab}(m)\rho_{cb}(m)\rho_{ac}(m) + 332/105 \rho_{ab}(m)\rho_{cd}(m)R_{acbd}(m) \\
& + 136/105 \rho_{uv}(m)R_{uabc}(m)R_{vabc}(m) \\
& - 1312/315 R_{abcd}(m)R_{aucv}(m)R_{bvd़u}(m) \\
& + 416/315 R_{abcd}(m)R_{aucv}(m)R_{bvd़u}(m)] + \text{div } X.
\end{aligned}$$

for some vector field X . Finally we get

THEOREM 4.2. $a_3 = \int_M u_3 dV$ is represented as follows :

$$(4.9) \quad a_3 = 1/6 ! \int_M \{ -142/63 |\nabla \tau|^2 - 26/63 |\nabla \rho|^2 - 1/9 |\nabla R|^2 + 5/9 \tau^3 \\ - 2/3 \tau |\rho|^2 + 2/3 \tau |R|^2 - 4/7 \rho^{ab} \rho_b^c \rho_{ac} + 20/63 \rho^{ab} \rho^{cd} R_{acbd} \\ - 8/63 \rho^{uv} R_u^{abc} R_{vabc} + 8/21 R^{abcd} R_{ab}^{uv} R_{cduv} \} dV.$$

PROOF. This formula follows from (4.8), (2.14), (2.15) and Green's theorem.

Remark to Theorem 4.2. If $\dim M = 2$, we have

$$a_2 = 1/60 \int_M \tau^2 dV, \quad a_3 = 2/7 ! \int_M \{ -9 |\nabla \tau|^2 + 4 \tau^3 \} dV.$$

For case of $\dim M = 3$, see Remark to Theorem 5.3.

5. Applications. First we shall prove a result which is essentially due to Berger ([1]).

THEOREM 5.1. *Let M, M' be compact connected orientable Einstein manifolds such that $Sp(M, g) = Sp(M', g')$ holds. Then M is of constant sectional curvature k if and only if M' is of constant sectional curvature k .*

PROOF. In Einsteinian case formulas (1.6), (1.7), (1.8) and (4.9) take the form

$$(5.1) \quad a_0 = \text{vol } M,$$

$$(5.2) \quad a_1 = 1/6 \tau \text{ vol } M,$$

where τ is the constant scalar curvature.

$$(5.3) \quad a_2 = 1/360(5 - 2/d) \tau^2 \text{ vol } M + 1/180 \int_M |R|^2 dV,$$

$$(5.4) \quad a_3 = 1/6 ! (5/9 - 2/3 \cdot 1/d - 16/63 \cdot 1/d^2) \tau^3 \text{ vol } M \\ + 1/6 ! (2/3 - 8/63 \cdot 1/d) \tau \int_M |R|^2 dV \\ + 1/5 ! \cdot 4/63 \int_M R^{abcd} R_{ab}^{uv} R_{cduv} dV - 1/6 ! 1/9 \int_M |\nabla R|^2 dV.$$

Thus from the assumption of the theorem, we have

$$\dim M = \dim M' = d, \quad \text{vol } M = \text{vol } M', \quad \tau = \tau', \quad \int_M |R|^2 dV = \int_{M'} |R'|^2 dV'.$$

Now suppose that M is of constant curvature k . Then $|R|^2 = 2/(d-1) |\rho|^2 = 2/\{d(d-1)\} \tau^2$ holds. ([1] proposition 2.2) Thus we have

$$\begin{aligned} \int_{M'} |R'|^2 dV' &= \int_M |R|^2 dV = 2/\{d(d-1)\} \int_M \tau^2 dV \\ &= 2/\{d(d-1)\} \int_{M'} \tau'^2 dV'. \end{aligned}$$

On the other hand $|R'|^2 \geq 2/\{d(d-1)\} \tau'^2$ holds and from the above equality we get $|R'|^2 = 2/\{d(d-1)\} \tau'^2 = 2/(d-1) |\rho'|^2$, and M' must be of constant curvature. ([1]. proposition 2.2) This constant curvature is k because $\tau = \tau'$ holds.

COROLLARY 5.2. *Under the same assumption as Theorem 5.1, M is a sphere (respectively, a real projective space) of constant curvature 1 if and only if M' is so.*

PROOF. If M is a sphere (respectively, a real projective space) of constant sectional curvature 1, then under the assumption of Theorem 5.1, M' is of constant sectional curvature 1 and $\text{vol } M = \text{vol } S^d$ (respectively, $\text{vol } RP^d$). Thus M' must be a sphere (respectively, RP^d).

REMARK. With respect to the corollary, M. Obata ([2]) has already obtained a better result which states that, if compact connected orientable Einsteinian M of $\dim M = d$ has the first non-zero eigen-value $\lambda_1 = -d$ and M is of constant scalar curvature 1, then M is isometric with a unit sphere. Thus the assumption on eigen-values of Δ is considerably weaker than that of Corollary 4.2, however he assumed that $\dim M = d$ and scalar curvature of M is 1. Thus it seems natural to consider the Einsteinian case in the following situation.

THEOREM 5.3. *Let M, M' be compact connected orientable Einsteinian manifolds. We assume that $Sp(M, g) = Sp(M', g')$ and $\int_M R^{abcp} R_{ab}{}^{uv} R_{cd}{}_{uv} dV = \int_{M'} R'^{abcd} R'_{ab}{}^{uv} R'_{cd}{}_{uv} dV'$ hold good. Then M is locally symmetric if and only if M' is locally symmetric.*

PROOF. From the assumption we have easily $\int_M |\nabla R|^2 dV = \int_{M'} |\nabla' R'|^2 dV'$.

REMARK. For three dimensional case we get the following:

Let M, M' be compact orientable Riemannian manifolds of constant scalar curvature with $\dim M = 3$. If $Sp(M, g) = Sp(M', g')$ and $\int_M \rho^{ab} \rho_a^c \rho_{bc} dV = \int_{M'} \rho'^{ab} \rho'_a^c \rho'_{bc} dV'$ hold, then (M, g) is locally symmetric if and only if (M', g') is locally symmetric.

PROOF. In case of $\dim M = 3$, following equalities hold good.

$$(5.5) \quad R_{ijkl} = (g_{ik} \rho_{jl} + g_{jl} \rho_{ik}) - (g_{il} \rho_{jk} + g_{jk} \rho_{il}) - \tau/2(g_{ik} g_{jl} - g_{il} g_{jk}),$$

$$(5.6) \quad R^{ab}{}^{cd} R_{ab}{}^{uv} R_{cd}{}_{uv} = -8 \rho^{ab} \rho_a^c \rho_{bc} + 12 \tau |\rho|^2 - 3\tau^3,$$

$$(5.7) \quad \rho^{uv} R_u{}^{ab}{}^c R_{vabc} = -2\rho^{ab} \rho_a^c \rho_{bc} + 4 \tau |\rho|^2 - \tau^3,$$

$$(5.8) \quad \rho^{ab} \rho^{cd} R_{abcd} = -2 \rho^{ab} \rho_a^c \rho_{bc} + 5/2 \tau |\rho|^2 - 1/2 \tau^3,$$

$$(5.9) \quad |R|^2 = 4 |\rho|^2 - \tau^2,$$

$$(5.10) \quad |\nabla R|^2 = 4 |\nabla \rho|^2 - |\nabla \tau|^2.$$

Thus if M is a 3-dimensional compact orientable Riemannian manifold, we have

$$(5.11) \quad a_2 = 1/120 \int_M \{\tau^2 + 2 |\rho|^2\} dV,$$

$$(5.12) \quad a_3 = 1/6 ! \int_M \{-15/7 |\nabla \tau|^2 - 6/7 |\nabla \rho|^2 - 9/7 \tau^3 + 48/7 \tau |\rho|^2 - 4 \rho^{ab} \rho_a^c \rho_{bc}\} dV.$$

Let M, M' be of constant scalar curvature and have the same spectrum. If $\int_M \rho^{ab} \rho_a^c \rho_{bc} dV = \int_{M'} \rho'^{ab} \rho'_a^c \rho'_{bc} dV'$ hold, we have from (1.6), (1.7), (5.11) and (5.12)

$$(5.13) \quad 1/4 \int_M |\nabla R|^2 dV = \int_M |\nabla \rho|^2 dV = \int_{M'} |\nabla' \rho'|^2 dV' \\ = 1/4 \int_{M'} |\nabla' R'|^2 dV'.$$

In case of six dimension we have

THEOREM 5.4. Let M, M' be compact connected orientable Einstein manifolds with $\dim M = 6$. We denote the Euler characteristic of M by $\chi(M)$. Now we assume that $\chi(M) = \chi(M')$ and $Sp(M, g) = Sp(M', g')$ hold good. Then M is locally symmetric if and only if M' is locally symmetric.

PROOF. First we shall give the following lemma :

LEMMA 5.5. Let M be a 6-dimensinal compact orientable Riemannian manifold. Then $\chi(M)$ is given as follows :

$$(5.14) \quad \begin{aligned} \chi(M) = & 1/\{384 \cdot \pi^3\} \int_M \{\tau^3 - 12\tau |R|^2 + 3\tau |R|^2 + 16 \rho^{ab}\rho_a^c\rho_{bc} \\ & + 24 \rho^{ab}\rho^{cd}R_{acba} - 24 \rho^{uv}R_u^{ab}R_{vabc} - 8 R^{abcd}R_a^u{}_c^vR_{bvdu} \\ & + 2R^{abcd}R_{ab}^{uv}R_{cduv}\} dV. \end{aligned}$$

PROOF of Lemma 5.5. This is an explicit representation by contractin of Chern's integrand of well-known Gauss-Bonnet formula

$$(5.15) \quad \chi(M) = \frac{1}{2^9 \pi^3 3!} \int_M \epsilon_{i_1 \dots i_6} \epsilon_{j_1 \dots j_6} R_{i_1 i_2 j_1 j_2} R_{i_3 i_4 j_3 j_4} R_{i_5 i_6 j_5 j_6} dV.$$

referred to orthonormal frames.

When M ($\dim M = 6$) is Einsteinian, (5.15) takes the form

$$(5.16) \quad \begin{aligned} \chi(M) = & 1/\{384 \pi^3\} \int_M \{1/9 \tau^3 - \tau |R|^2 - 8R^{abcd}R_a^u{}_c^vR_{bvdu} \\ & + 2R^{abcp}R_{ab}^{uv}R_{cduv}\} dV \\ = & (-1)/\{384 \pi^3\} \int_M \{2 |\nabla R|^2 - 6 R^{abcd}R_{ab}^{uv}R_{cduv} - 1/9 \tau^3 \\ & + 5/3 \tau |R|^2\} dV \end{aligned}$$

by (2.15). Thus from the assumption of the theorem we have

$$\begin{aligned} & \int_M \{7 |\nabla R|^2 - 24 R^{abcd}R_{ab}^{uv}R_{cduv}\} dV \\ &= \int_{M'} \{7 |\nabla' R'|^2 - 24 R'^{abcd}R'_{ab}^{uv}R'_{cduv}\} dV' \\ & \int_M \{|\nabla R|^2 - 3 R^{abcd}R_{ab}^{uv}R_{cduv}\} dV \\ &= \int_{M'} \{|\nabla' R'|^2 - 3 R'^{abcd}R'_{ab}^{uv}R'_{cduv}\} dV' \end{aligned}$$

and consequently $\int_M |\nabla R|^2 dV = \int_{M'} |\nabla' R'|^2 dV'$.

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Especially I learned that the assumption of orientability of M in the present paper can be removed. In fact, we may consider the canonical measure of (M, g) instead of the volume element of (M, g) .

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