On eigenvalue problems with Robin type boundary conditions having indefinite coefficients

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Abstract

A Robin type boundary condition with a sign-changing coefficient is treated. First the associated linear elliptic eigenvalue problem is studied, where the existence of a principal eigenvalue is discussed by use of a variational approach. Secondly the associated semilinear elliptic boundary value problem of logistic type is studied and the one parameter-dependent structure of positive solutions is investigated, where results obtained are due to the construction of suitable super and subsolutions by using the principal positive eigenfunctions of the linear eigenvalue problem.

Keywords: Eigenvalue problem; indefinite Robin boundary condition; Principal eigenvalue; Variational characterization; Lower estimate; Population dynamics

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1 Introduction

In this paper we consider the following elliptic eigenvalue problem.

$$\begin{cases} -\Delta u = \lambda(g(x) - cu)u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \boldsymbol{n}} = \lambda h(x)u & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 2$, with smooth boundary $\partial\Omega$; λ is a real parameter; $g \in C^\theta(\overline{\Omega})$ and $h \in C^{1+\theta}(\partial\Omega)$, $0 < \theta < 1$, are Hölder continuous functions in the closure $\overline{\Omega}$ and on the boundary $\partial\Omega$, respectively, which may be both *sign-changing*; c is a *nonnegative* constant; and $\boldsymbol{n}$ is the unit exterior normal to $\partial\Omega$.

From the viewpoint of population dynamics, our interest lies in positive solutions of (1.1) for *positive* parameter λ. Here a classical solution to $C^{2+\theta}(\overline{\Omega})$ of (1.1) is called *positive*

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if it is positive in Ω, and in fact a positive solution of (1.1) is positive in $\overline{\Omega}$ by virtue of the strong maximum principle and boundary point lemma (cf. [11]). More precisely, problem (1.1) denotes the steady state of the population density of some species, diffusing at rate $1/\lambda$ around the region Ω, and governed by the parabolic problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nabla \cdot (1/\lambda)\nabla u + (g(x) - cu)u & \text{in } (0,\infty) \times \Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \\ (1/\lambda)\nabla u \cdot \boldsymbol{n} = h(x)u & \text{on } (0,\infty) \times \partial\Omega. \end{cases}$$

Here g represents the birth or decay rate, c the crowding effect and the boundary condition a law of the population flux with the flux rate h on the border $\partial\Omega$. It should be emphasized that h is *indefinite*, which implies that the inflow of population to the region Ω occurs at $x \in \partial\Omega$ where $h(x) > 0$, while the outflow occurs at $x \in \partial\Omega$ where $h(x) < 0$. The usual Neumann and Robin boundary conditions are particular cases of ours, which correspond to the cases when $h \equiv 0$ and when $h \le 0$ not vanishing on the whole of $\partial\Omega$, respectively.

By Afrouzi and Brown [2], the existence of a principal eigenvalue is discussed for the following linear eigenvalue problem with a constant coefficient α being positive, zero or negative, in the case when g changes sign in Ω:

$$\begin{cases} -\Delta u = \lambda g(x) u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \boldsymbol{n}} + \alpha u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here a *principal eigenvalue* means an eigenvalue having an eigenfunction everywhere in $\overline{\Omega}$. It has been already proved by Brown and Lin [5] that there exists a positive principal eigenvalue of the Neumann problem ($\alpha = 0$) if and only if $\int_\Omega g\,dx < 0$ and $g \not\le 0$, and moreover that it is unique if exists. Meanwhile, by Saut and Scheurer [12] an *a priori* lower estimate is established for the unique positive principal eigenvalue of the Neumann problem in terms of the quantity $|\int_\Omega g\,dx|$.

In this paper we first consider the existence, uniqueness and lower estimate of positive principal eigenvalues for (1.1) with $c = 0$ in the indefinite case of coefficient h, including parameter λ in the boundary condition. Here our approach relies on variational techniques used in [2] and [12]. As an application of results obtained for (1.1) with $c = 0$, we secondly study the existence, uniqueness and limiting behavior of positive solutions of the semilinear problem (1.1) with $c > 0$ for positive parameter λ *via* the super and subsolution method. For other related works to ours we refer to [6, 7, 9].

The next section is devoted to the statement of our main results.

2 Main results

To comprehend positive principal eigenvalues of (1.1) with $c = 0$, we first in this section consider principal eigenvalues of the following linear eigenvalue problem for an eigenvalue

$\mu(\lambda)$.

$$\begin{cases} -\Delta\phi = \lambda g(x)\phi + \mu(\lambda)\phi & \text{in } \Omega, \\ \dfrac{\partial\phi}{\partial \boldsymbol{n}} = \lambda h(x)\phi & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Now the following theorem contains the existence of principal eigenvalues of (2.1) and their properties.

Theorem 2.1. *Assume that either $g(x) \not\leq 0$ in Ω, or $h(x) \not\leq 0$ on $\partial\Omega$. Then the following three assertions hold true:*

(1) *For any $\lambda \in \mathbb{R}$ there exists a unique principal eigenvalue $\mu_1(\lambda)$ of (2.1), characterized variationally as follows.*

$$\mu_1(\lambda) = \inf \left\{ \int_\Omega |\nabla u|^2\, dx - \lambda \int_\Omega g u^2\, dx - \lambda \int_{\partial\Omega} h u^2\, d\sigma : u \in W^{1,2}(\Omega),\ \int_\Omega u^2\, dx = 1 \right\}. \tag{2.2}$$

Here $d\sigma$ denotes the surface element of $\partial\Omega$.

(2) *Mapping $\lambda \mapsto \mu_1(\lambda)$ is concave and satisfies $\mu_1(\lambda) \to -\infty$ as $\lambda \to \infty$.*

(3) *The unique principal eigenvalue $\mu_1(\lambda)$ has a local maximum (i.e. global maximum) if $\int_\Omega g\, dx + \int_{\partial\Omega} h\, d\sigma \leq 0$. Moreover, this is unique and the sign of the unique global maximum point is equal to that of $-(\int_\Omega g\, dx + \int_{\partial\Omega} h\, d\sigma)$, if it exists.*

Remark 2.1. Having in mind $\mu_1(0) = 0$, it follows from assertions (2) and (3) of the theorem that if $\int_\Omega g\, dx + \int_{\partial\Omega} h\, d\sigma < 0$, then there exists a unique positive principal eigenvalue $\lambda_1(g, h)$ of (1.1) with $c = 0$, meaning $\mu_1(\lambda_1(g, h)) = 0$, while no positive principal eigenvalue otherwise.

Now, let c_1 and c_2 be constants, respectively, due to the Poincaré inequality (cf. [8, p. 164])

$$\|w\|_{L^2(\Omega)} \leq c_1 \|\nabla w\|_{L^2(\Omega)}, \quad \forall w \in \left\{ w \in W^{1,2}(\Omega) : \int_\Omega w\, dx = 0 \right\}, \tag{2.3}$$

and the continuous imbedding (cf. [1])

$$\|u\|_{L^2(\partial\Omega)} \leq c_2 \|u\|_{W^{1,2}(\Omega)}, \quad \forall u \in W^{1,2}(\Omega). \tag{2.4}$$

The following theorem gives the variational characterization of the unique positive principal eigenvalue $\lambda_1(g, h)$ of (1.1) with $c = 0$, as well as its *a priori* lower bound in terms of the quantity $|\int_\Omega g\, dx + \int_{\partial\Omega} h\, d\sigma|$.

Theorem 2.2. *Under the assumption of Theorem 2.1, problem (1.1) with $c = 0$ has a unique positive principal eigenvalue $\lambda_1(g,h)$ if and only if $\int_\Omega g\,dx + \int_{\partial\Omega} h\,d\sigma < 0$, and it is characterized by the formula*

$$\lambda_1(g,h) = \inf\left\{ \frac{\int_\Omega |\nabla u|^2\,dx}{\int_\Omega gu^2\,dx + \int_{\partial\Omega} hu^2\,d\sigma} : u \in W^{1,2}(\Omega),\ \int_\Omega gu^2\,dx + \int_{\partial\Omega} hu^2\,d\sigma > 0 \right\}, \tag{2.5}$$

and is estimated below by some constant $C(c_1,c_2) > 0$, depending only on the constants c_1 and c_2, as follows.

$$\lambda_1(g,h) \geq C(c_1,c_2)\left(\|g^+\|_{C(\overline{\Omega})} + \|h^+\|_{C(\partial\Omega)} + \frac{\|g\|^2_{L^2(\Omega)} + \|h\|^2_{L^2(\partial\Omega)}}{|\int_\Omega g\,dx + \int_{\partial\Omega} h\,d\sigma|} \right)^{-1} =: \underline{\lambda}(g,h), \tag{2.6}$$

where $f^+ = \max(f,0)$ for a continuous function f.

Remark 2.2. We see in (2.6) that $\underline{\lambda}(g_j,h_j) \to 0$ as $\int_\Omega g_j\,dx + \int_{\partial\Omega} h_j\,d\sigma \nearrow 0$ if $\|g_j\|^2_{L^2(\Omega)} + \|h_j\|^2_{L^2(\partial\Omega)}$ is bounded below by a positive constant, as an example.

Finally we study the semilinear problem (1.1) with $c > 0$. The following theorem contains the existence, uniqueness and nonexistence of positive solutions.

Theorem 2.3. *Assume that either $g(x) \not\leq 0$ in Ω, or $h(x) \not\leq 0$ on $\partial\Omega$. Then there exists a unique positive solution u_λ of (1.1) with $c > 0$ for each $\lambda > \lambda_1(g,h)$, meanwhile no positive solution for any $0 < \lambda \leq \lambda_1(g,h)$. Here it is understood that $\lambda_1(g,h) = 0$ when $\int_\Omega g\,dx + \int_{\partial\Omega} h\,d\sigma \geq 0$. Moreover we have the a priori upper bound*

$$\|u_\lambda\|_{L^3(\Omega)} \leq c^{-1}\left(-\frac{\mu_1(\lambda)}{\lambda}\right)|\Omega|^{1/3}, \quad \forall\lambda > \lambda_1(g,h), \tag{2.7}$$

and the limiting behavior

$$\lim_{\lambda\downarrow\lambda_1(g,h)} u_\lambda = \frac{\max\{\int_\Omega g\,dx + \int_{\partial\Omega} h\,d\sigma,\ 0\}}{c|\Omega|} \quad \text{in } C^2(\overline{\Omega}). \tag{2.8}$$

Remark 2.3. The corresponding results to those of Theorem 2.3 can be established if the constant coefficient c is replaced by a Hölder continuous function $c \in C^\theta(\overline{\Omega})$ satisfying $c(x) > 0$ in $\overline{\Omega}$.

Sections 3, 4 and 5 are devoted to the proofs of Theorems 2.1, 2.2 and 2.3, respectively.

3 Proof of Theorem 2.1

The proof of Theorem 2.1 is due to a variational argument by Smoller [13, Chap. 11]. We consider the minimizer of the functional

$$S_\lambda = \left\{ \int_\Omega |\nabla u|^2\,dx - \lambda\int_\Omega gu^2\,dx - \lambda\int_{\partial\Omega} hu^2\,d\sigma : u \in W^{1,2}(\Omega),\ \int_\Omega u^2\,dx = 1 \right\}.$$

We first prove the following lemma.

Lemma 3.1. *S_λ is bounded below.*

Proof. The following result is crucial ([2, Lemma 1]).

Proposition 3.1. *For any $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that*

$$\int_{\partial\Omega} u^2 \, d\sigma \leq \varepsilon \int_\Omega |\nabla u|^2 \, dx + C(\varepsilon) \int_\Omega u^2 \, dx, \quad \forall u \in W^{1,2}(\Omega).$$

From Proposition 3.1 it follows that if $\int_\Omega u^2 \, dx = 1$, then we have

$$\left| \lambda \int_{\partial\Omega} hu^2 \, d\sigma \right| \leq |\lambda|\varepsilon \|h\|_{C(\partial\Omega)} \int_\Omega |\nabla u|^2 \, dx + |\lambda| C(\varepsilon) \|h\|_{C(\partial\Omega)}.$$

Hence we have

$$\begin{aligned} &\int_\Omega |\nabla u|^2 \, dx - \lambda \int_\Omega gu^2 \, dx - \lambda \int_{\partial\Omega} hu^2 \, d\sigma \\ &\geq (1 - |\lambda|\varepsilon \|h\|_{C(\partial\Omega)}) \int_\Omega |\nabla u|^2 \, dx - |\lambda| \|g\|_{C(\overline{\Omega})} - |\lambda| C(\varepsilon) \|h\|_{C(\partial\Omega)}. \end{aligned}$$

Lemma 3.1 is verified if ε has been already taken so small that $|\lambda|\varepsilon\|h\|_{C(\partial\Omega)} < 1$. □

In the same manner as in [13, Theorem 11.10], the infimum of S_λ is attained by some nonnegative function $\phi_1(\lambda)$, which is of class $C^{2+\theta}(\overline{\Omega})$ by elliptic regularity (cf. [8]). By the maximum principle and boundary point lemma, we have $\phi_1(\lambda) > 0$ in $\overline{\Omega}$, so that formula (2.2) is proved in the same manner as in [13, Theorem 11.4]. For the uniqueness result we use a contradiction argument. We assume to the contrary that μ ($\neq \mu_1(\lambda)$) is a principal eigenvalue of (2.1) whose eigenfunction ϕ is positive in $\overline{\Omega}$, then it follows by integration by parts that $\int_\Omega \phi\phi_1(\lambda) \, dx = 0$, a contradiction. Hence the assertion (1) has been verified.

For a fixed $u \in W^{1,2}(\Omega)$ the mapping

$$\lambda \longmapsto \int_\Omega |\nabla u|^2 \, dx - \lambda \int_\Omega gu^2 \, dx - \lambda \int_{\partial\Omega} hu^2 \, d\sigma$$

is affine and so concave, so that the infimum is also concave. It is possible to choose a nontrivial function $\hat{u} \in C^1(\overline{\Omega})$ satisfying $\int_\Omega \hat{u}^2 \, dx = 1$, such that $\hat{u}$ is strictly positive in $\{x \in \partial\Omega : h(x) > 0\}$ whose support is contained in a very thin tubular neighborhood of $\partial\Omega$ if $h \not\leq 0$ on $\partial\Omega$, or that its support is compact in $\{x \in \Omega : g(x) > 0\}$ if $g \not\leq 0$ in Ω. Then we have

$$\mu_1(\lambda) \leq \int_\Omega |\nabla \hat{u}|^2 \, dx - \lambda \int_\Omega g\hat{u}^2 \, dx - \lambda \int_{\partial\Omega} h\hat{u}^2 \, d\sigma \longrightarrow -\infty \quad \text{as } \lambda \to \infty.$$

The assertion (2) has been verified.

Finally we verify the assertion (3). We note

$$\begin{cases} -\Delta\phi_1(\lambda) = \lambda g\phi_1(\lambda) + \mu_1(\lambda)\phi_1(\lambda) & \text{in } \Omega, \\ \dfrac{\partial\phi_1(\lambda)}{\partial \boldsymbol{n}} = \lambda h\phi_1(\lambda) & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Differentiate (3.1) with respect to λ and we have

$$\begin{cases} -\Delta\phi_1'(\lambda) = g\phi_1(\lambda) + \lambda g\phi_1'(\lambda) + \mu_1'(\lambda)\phi_1(\lambda) + \mu_1(\lambda)\phi_1'(\lambda) & \text{in } \Omega, \\ \dfrac{\partial\phi_1'(\lambda)}{\partial\boldsymbol{n}} = h\phi_1(\lambda) + \lambda h\phi_1'(\lambda) & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

By integration by parts, we see

$$\int_\Omega \{-\Delta\phi_1(\lambda)\phi_1'(\lambda) + \phi_1(\lambda)\Delta\phi_1'(\lambda)\}\,dx = \int_{\partial\Omega}\left\{-\frac{\partial\phi_1(\lambda)}{\partial\boldsymbol{n}}\phi_1'(\lambda) + \phi_1(\lambda)\frac{\partial\phi_1'(\lambda)}{\partial\boldsymbol{n}}\right\}d\sigma.$$

Substitute (3.1) and (3.2) into the equality and we derive

$$\mu_1'(\lambda) = -\frac{\int_\Omega g\phi_1(\lambda)^2\,dx + \int_{\partial\Omega} h\phi_1(\lambda)^2\,d\sigma}{\int_\Omega \phi_1(\lambda)^2\,dx}. \tag{3.3}$$

By definition we note that $\phi_1(0)$ is a positive constant. Hence assertion (3.3) implies

$$\mu_1'(0) = -\frac{\int_\Omega g\,dx + \int_{\partial\Omega} h\,d\sigma}{|\Omega|}. \tag{3.4}$$

From (3.3) we find that λ_0 is a critical point of $\mu_1(\lambda)$, that is, $\mu_1'(\lambda_0) = 0$ if and only if $\int_\Omega g\phi_1(\lambda_0)^2\,dx + \int_{\partial\Omega} h\phi_1(\lambda_0)^2\,d\sigma = 0$. In fact, combining the assertion (2) and (3.4) ensures the existence of a critical point if $\int_\Omega g\,dx + \int_{\partial\Omega} h\,d\sigma \le 0$. Moreover the uniqueness can be obtained in the following manner. Let λ_0 be a critical point of $\mu_1(\lambda)$, that is, $\int_\Omega g\phi_1(\lambda_0)^2\,dx + \int_{\partial\Omega} h\phi_1(\lambda_0)^2\,d\sigma = 0$. Here the positive eigenfunction $\phi_1(\lambda_0)$ is normalized as $\int_\Omega \phi_1(\lambda_0)^2\,dx = 1$. Then, thanks to the concavity of $\mu_1(\lambda)$, it suffices to prove that $\mu_1(\lambda) < \mu_1(\lambda_0)$ for $\lambda \neq \lambda_0$. Note that $\mu_1(\lambda_0) = \int_\Omega |\nabla\phi_1(\lambda_0)|^2\,dx$. By definition we see, for any $\lambda \neq \lambda_0$,

$$\begin{aligned} \mu_1(\lambda) &\le \int_\Omega |\nabla\phi_1(\lambda_0)|^2\,dx - \lambda\int_\Omega g\phi_1(\lambda_0)^2\,dx - \lambda\int_{\partial\Omega} h\phi_1(\lambda_0)^2\,d\sigma \\ &= \int_\Omega |\nabla\phi_1(\lambda_0)|^2\,dx = \mu_1(\lambda_0). \end{aligned}$$

If we assume for a contradiction that there exists a $\lambda_1 \neq \lambda_0$ such that $\mu_1(\lambda_1) = \mu_1(\lambda_0)$, then $\phi_1(\lambda_0)$ attains the infimum of S_{λ_1}, and it follows that

$$\begin{cases} -\Delta\phi_1(\lambda_0) = \lambda_1 g\phi_1(\lambda_0) + \mu_1(\lambda_1)\phi_1(\lambda_0) & \text{in } \Omega, \\ \dfrac{\partial\phi_1(\lambda_0)}{\partial\boldsymbol{n}} = \lambda_1 h\phi_1(\lambda_0) & \text{on } \partial\Omega. \end{cases}$$

Meanwhile, we know

$$\begin{cases} -\Delta\phi_1(\lambda_0) = \lambda_0 g\phi_1(\lambda_0) + \mu_1(\lambda_0)\phi_1(\lambda_0) & \text{in } \Omega, \\ \dfrac{\partial\phi_1(\lambda_0)}{\partial\boldsymbol{n}} = \lambda_0 h\phi_1(\lambda_0) & \text{on } \partial\Omega. \end{cases}$$

From the assumption that $\mu_1(\lambda_1) = \mu_1(\lambda_0)$, it follows that

$$(\lambda_1 - \lambda_0) g \phi_1(\lambda_0) = 0 \quad \text{in } \Omega, \quad \text{and} \quad (\lambda_1 - \lambda_0) h \phi_1(\lambda_0) = 0 \quad \text{on } \partial\Omega.$$

Since $\phi_1(\lambda_0) > 0$ in $\overline{\Omega}$, it follows that $g \equiv 0$ in Ω and $h \equiv 0$ on $\partial\Omega$, a contradiction. The uniqueness has been verified. Finally it is clear from (3.4) that the sign of the unique global maximum point coincides with that of $-(\int_\Omega g\,dx + \int_{\partial\Omega} h\,d\sigma)$. The assertion (3) has been verified.

The proof of Theorem 2.1 is now complete.

4 Proof of Theorem 2.2

In this section we prove Theorem 2.2. First we verify that the infimum (2.5) is well-defined by a positive constant. The following lemma is crucial.

Lemma 4.1. *Let $\int_\Omega g\,dx + \int_{\partial\Omega} h\,d\sigma < 0$. Then there exists a constant $c_0 > 0$ such that, $\int_\Omega |\nabla u|^2\,dx \geq c_0$ if $u \in W^{1,2}(\Omega)$ satisfies that $\int_\Omega u^2\,dx + \int_{\partial\Omega} u^2\,d\sigma = 1$ and $\int_\Omega g u^2\,dx + \int_{\partial\Omega} h u^2\,d\sigma > 0$.*

Proof. For a contradiction we assume that a sequence $\{u_n\} \subset W^{1,2}(\Omega)$ can be chosen such that

$$\int_\Omega u_n^2\,dx + \int_{\partial\Omega} u_n^2\,d\sigma = 1,$$
$$\int_\Omega g u_n^2\,dx + \int_{\partial\Omega} h u_n^2\,d\sigma > 0,$$
$$\int_\Omega |\nabla u_n|^2\,dx \leq \frac{1}{n}.$$

It follows that u_n is bounded in $W^{1,2}(\Omega)$. Since the imbedding $W^{1,2}(\Omega) \subset L^2(\Omega)$ is compact, we can take a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that $u_n \to \hat{u}$ in $L^2(\Omega)$ for some $\hat{u} \in L^2(\Omega)$. Moreover we see

$$\begin{aligned}\|u_n - u_m\|^2_{W^{1,2}(\Omega)} &= \int_\Omega |u_n - u_m|^2\,dx + \int_\Omega |\nabla(u_n - u_m)|^2\,dx \\ &\leq \int_\Omega |u_n - u_m|^2\,dx + 2\left(\frac{1}{n} + \frac{1}{m}\right) \longrightarrow 0, \quad n, m \to \infty.\end{aligned}$$

By the completeness of $W^{1,2}(\Omega)$ it follows that $u_n \to \hat{u}$ in $W^{1,2}(\Omega)$, and hence that

$$\int_\Omega |\nabla \hat{u}|^2\,dx = \lim_{n\to\infty} \int_\Omega |\nabla u_n|^2\,dx = 0,$$
$$\int_\Omega \hat{u}^2\,dx + \int_{\partial\Omega} \hat{u}^2\,d\sigma = \lim_{n\to\infty}\left(\int_\Omega u_n^2\,dx + \int_{\partial\Omega} u_n^2\,d\sigma\right) = 1.$$

This implies that $\hat{u}$ is a nonzero constant, so that

$$\int_\Omega g\hat{u}^2\,dx + \int_{\partial\Omega} h\hat{u}^2\,d\sigma = \hat{u}^2\left(\int_\Omega g\,dx + \int_{\partial\Omega} h\,d\sigma\right) < 0.$$

However, since we see

$$\int_\Omega g u_n^2\,dx + \int_{\partial\Omega} h u_n^2\,d\sigma \longrightarrow \int_\Omega g\hat{u}^2\,dx + \int_{\partial\Omega} h\hat{u}^2\,d\sigma, \quad n \to \infty,$$

we conclude that n large enough implies $\int_\Omega g u_n^2\,dx + \int_{\partial\Omega} h u_n^2\,d\sigma < 0$, a contradiction. The proof of Lemma 4.1 is complete. □

Lemma 4.1 shows that the infimum (2.5) is positive. Indeed, let $u \in W^{1,2}(\Omega)$ be such that $\int_\Omega g u^2\,dx + \int_{\partial\Omega} h u^2\,d\sigma > 0$. Then, for some $\delta > 0$ the function $v = \delta u$ verifies that $\int_\Omega v^2\,dx + \int_{\partial\Omega} v^2\,d\sigma = 1$, and that

$$\frac{\int_\Omega |\nabla u|^2\,dx}{\int_\Omega g u^2\,dx + \int_{\partial\Omega} h u^2\,d\sigma} = \frac{\int_\Omega |\nabla v|^2\,dx}{\int_\Omega g v^2\,dx + \int_{\partial\Omega} h v^2\,d\sigma} \geq \frac{c_0}{\|g^+\|_{C(\overline{\Omega})} + \|h^+\|_{C(\partial\Omega)}} > 0. \tag{4.1}$$

Let λ_* be a positive constant given by the infimum (2.5). To establish formula (2.5) it suffices to show that $\mu_1(\lambda_*) = 0$ in view of Remark 2.1. Let $u \in W^{1,2}(\Omega)$. If $\int_\Omega g u^2\,dx + \int_{\partial\Omega} h u^2\,d\sigma > 0$, then by the definition of λ_* we find

$$\int_\Omega |\nabla u|^2\,dx - \lambda_* \int_\Omega g u^2\,dx - \lambda_* \int_{\partial\Omega} h u^2\,d\sigma \geq 0. \tag{4.2}$$

From the fact $\lambda_* > 0$, assertion (4.2) also holds true even if $\int_\Omega g u^2\,dx + \int_{\partial\Omega} h u^2\,d\sigma \leq 0$. By definition it follows that $\mu_1(\lambda_*) \geq 0$.

On the other hand, it is derived from (2.5) that there exists a sequence $\{u_n\} \subset W^{1,2}(\Omega)$ such that

$$\int_\Omega g u_n^2\,dx + \int_{\partial\Omega} h u_n^2\,d\sigma > 0,$$

$$\int_\Omega u_n^2\,dx = 1, \tag{4.3}$$

$$\left(1 + \frac{1}{n}\right)\lambda_* \geq \frac{\int_\Omega |\nabla u_n|^2\,dx}{\int_\Omega g u_n^2\,dx + \int_{\partial\Omega} h u_n^2\,d\sigma}. \tag{4.4}$$

The choice (4.3) is in fact possible by arguing in the same way as in (4.1). Thanks to Proposition 3.1, for any $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that

$$\begin{aligned}\int_{\partial\Omega} h u_n^2\,d\sigma &\leq \|h^+\|_{C(\partial\Omega)} \int_{\partial\Omega} u_n^2\,d\sigma \\ &\leq \varepsilon \|h^+\|_{C(\partial\Omega)} \int_\Omega |\nabla u_n|^2\,dx + C(\varepsilon)\|h^+\|_{C(\partial\Omega)} \int_\Omega u_n^2\,dx.\end{aligned} \tag{4.5}$$

By (4.3) and (4.5), assertion (4.4) implies

$$(1 - 2\lambda_* \varepsilon \|h^+\|_{C(\partial\Omega)}) \int_\Omega |\nabla u_n|^2 \, dx \le 2\lambda_* (\|g^+\|_{C(\overline{\Omega})} + C(\varepsilon)\|h^+\|_{C(\partial\Omega)}).$$

Here it has been used that $1 + (1/n) \le 2$. Taking $\varepsilon > 0$ so small that

$$1 - 2\lambda_* \varepsilon \|h^+\|_{C(\partial\Omega)} > \frac{1}{2},$$

we have

$$\int_\Omega |\nabla u_n|^2 \, dx \le 4\lambda_* \left(\|g^+\|_{C(\overline{\Omega})} + C(\varepsilon)\|h^+\|_{C(\partial\Omega)} \right) < \infty.$$

Combining this assertion and (4.3) shows that u_n is bounded in $W^{1,2}(\Omega)$, and thus that also in $L^2(\partial\Omega)$ by virtue of the continuous imbedding $W^{1,2}(\Omega) \subset L^2(\partial\Omega)$. By the boundedness, assertion (4.4) implies

$$\begin{aligned}
&\int_\Omega |\nabla u_n|^2 \, dx - \lambda_* \int_\Omega g u_n^2 \, dx - \lambda_* \int_{\partial\Omega} h u_n^2 \, d\sigma \\
&\le \frac{\lambda_*}{n} \left(\int_\Omega g u_n^2 \, dx + \int_{\partial\Omega} h u_n^2 \, d\sigma \right) \\
&\le \frac{\lambda_*}{n} \left(\|g^+\|_{C(\overline{\Omega})} + \|h^+\|_{C(\partial\Omega)} \int_{\partial\Omega} u_n^2 \, d\sigma \right) \longrightarrow 0, \quad n \to \infty.
\end{aligned}$$

This means $\mu_1(\lambda_*) \le 0$ and hence we conclude $\mu_1(\lambda_*) = 0$.

Next we prove (2.6). Let $X = W^{2,2}(\Omega)$, and let us introduce the operator $P : X \to W = \{u \in X : \int_\Omega u \, dx = 0\}$, defined as

$$Pu = u - \frac{1}{|\Omega|} \int_\Omega u \, dx,$$

which produce the unique decomposition $X = \mathbb{R} \oplus W$. Let $Y = L^2(\Omega)$, and let us introduce the operator $Q : Y \to \{v \in Y : \int_\Omega v \, dx = 0\}$, defined as

$$Q[v] = v - \frac{\int_\Omega v \, dx}{\int_\Omega g \, dx + \int_{\partial\Omega} h \, d\sigma} \left(g + \frac{1}{|\Omega|} \int_{\partial\Omega} h \, d\sigma \right), \quad v \in Y.$$

It is easy to see that operator Q also produce the unique decomposition $Y = Q[Y] \oplus (1 - Q)[Y]$.

Now let us take a solution $u \in C^{2+\theta}(\overline{\Omega})$ of (1.1) with $c = 0$ for $\lambda > 0$. Then Green's formula shows

$$Q[-\Delta u] = -\Delta w + \frac{\lambda \int_{\partial\Omega} h(\alpha + w) \, d\sigma}{\int_\Omega g \, dx + \int_{\partial\Omega} h \, d\sigma} \left(g + \frac{1}{|\Omega|} \int_{\partial\Omega} h \, d\sigma \right),$$

where $u = \alpha + w$ for some $\alpha \in \mathbb{R}$ and $w \in W$ uniquely determined. Since

$$Q[\lambda g u] = \lambda g(\alpha + w) - \frac{\lambda \int_\Omega g(\alpha + w) \, dx}{\int_\Omega g \, dx + \int_{\partial\Omega} h \, d\sigma} \left(g + \frac{1}{|\Omega|} \int_{\partial\Omega} h \, d\sigma \right),$$

it follows that

$$\begin{aligned}
&-\Delta w + \frac{\lambda \int_{\partial\Omega} h(\alpha + w)\, d\sigma}{\int_\Omega g\, dx + \int_{\partial\Omega} h\, d\sigma} \left(g + \frac{1}{|\Omega|} \int_{\partial\Omega} h\, d\sigma \right) \\
&= \lambda g(\alpha + w) - \frac{\lambda \int_\Omega g(\alpha + w)\, dx}{\int_\Omega g\, dx + \int_{\partial\Omega} h\, d\sigma} \left(g + \frac{1}{|\Omega|} \int_{\partial\Omega} h\, d\sigma \right).
\end{aligned} \tag{4.6}$$

Multiplying (4.6) by w and integrating by parts, we have

$$\begin{aligned}
\int_\Omega |\nabla w|^2\, dx = &\int_{\partial\Omega} \frac{\partial w}{\partial \boldsymbol{n}} w\, d\sigma + \lambda \int_\Omega g(\alpha + w) w\, dx \\
&- \frac{\lambda \int_\Omega gw\, dx}{\int_\Omega g\, dx + \int_{\partial\Omega} h\, d\sigma} \left(\int_\Omega g(\alpha + w)\, dx + \int_{\partial\Omega} h(\alpha + w)\, d\sigma \right),
\end{aligned}$$

from the fact that $\int_\Omega w\, dx = 0$. It follows that

$$\begin{aligned}
\int_\Omega |\nabla w|^2\, dx = \lambda &\int_{\partial\Omega} h(\alpha + w) w\, d\sigma + \lambda \int_\Omega g w^2\, dx \\
&- \frac{\lambda \int_\Omega gw\, dx}{\int_\Omega g\, dx + \int_{\partial\Omega} h\, d\sigma} \left(\int_\Omega gw\, dx + \int_{\partial\Omega} hw\, d\sigma \right).
\end{aligned} \tag{4.7}$$

We note that by Green's formula, i.e. $\int_\Omega \Delta w\, dx = \int_{\partial\Omega} \frac{\partial w}{\partial \boldsymbol{n}}\, d\sigma$,

$$\alpha = -\frac{\int_\Omega gw\, dx + \int_{\partial\Omega} hw\, d\sigma}{\int_\Omega g\, dx + \int_{\partial\Omega} h\, d\sigma}. \tag{4.8}$$

Substituting (4.8) into (4.7), it follows that

$$\int_\Omega |\nabla w|^2\, dx = \lambda \int_\Omega g w^2\, dx + \lambda \int_{\partial\Omega} h w^2\, d\sigma - \frac{\lambda \left(\int_\Omega gw\, dx + \int_{\partial\Omega} hw\, d\sigma \right)^2}{\int_\Omega g\, dx + \int_{\partial\Omega} h\, d\sigma}. \tag{4.9}$$

By use of (2.3) and (2.4), we obtain

$$\begin{aligned}
&\int_\Omega g w^2\, dx \le c_1^2 \|g^+\|_{C(\overline{\Omega})} \|\nabla w\|_{L^2(\Omega)}^2, \\
&\int_{\partial\Omega} h w^2\, d\sigma \le c_2^2 (1 + c_1^2) \|h^+\|_{C(\partial\Omega)} \|\nabla w\|_{L^2(\Omega)}^2, \\
&\left(\int_\Omega gw\, dx + \int_{\partial\Omega} hw\, d\sigma \right)^2 \le 2(c_1^2 \|g\|_{L^2(\Omega)}^2 + c_2^2 (1 + c_1^2) \|h\|_{L^2(\partial\Omega)}^2) \|\nabla w\|_{L^2(\Omega)}^2.
\end{aligned}$$

Combining (4.9) and these estimates, we have some constant $c_3 > 0$, depending only on c_1 and c_2, such that

$$\left\{ 1 - \lambda c_3 \left(\|g^+\|_{C(\overline{\Omega})} + \|h^+\|_{C(\partial\Omega)} + \frac{\|g\|_{L^2(\Omega)}^2 + \|h\|_{L^2(\partial\Omega)}^2}{|\int_\Omega g\, dx + \int_{\partial\Omega} h\, d\sigma|} \right) \right\} \|\nabla w\|_{L^2(\Omega)}^2 \le 0.$$

If we take λ such that

$$0 < \lambda \leq \left\{ c_3 \left(\|g^+\|_{C(\overline{\Omega})} + \|h^+\|_{C(\partial\Omega)} + \frac{\|g\|^2_{L^2(\Omega)} + \|h\|^2_{L^2(\partial\Omega)}}{|\int_\Omega g\,dx + \int_{\partial\Omega} h\,d\sigma|} \right) \right\}^{-1}, \tag{4.10}$$

then $\int_\Omega |\nabla w|^2\,dx = 0$, so that $w = 0$ since $w \in W$.

On the other hand, we see

$$\begin{aligned}(1-Q)[-\Delta u] &= -\frac{\lambda \int_{\partial\Omega} h(\alpha + w)\,d\sigma}{\int_\Omega g\,dx + \int_{\partial\Omega} h\,d\sigma} \left(g + \frac{1}{|\Omega|} \int_{\partial\Omega} h\,d\sigma \right), \\ (1-Q)[\lambda g u] &= \frac{\lambda \int_\Omega g(\alpha + w)\,dx}{\int_\Omega g\,dx + \int_{\partial\Omega} h\,d\sigma} \left(g + \frac{1}{|\Omega|} \int_{\partial\Omega} h\,d\sigma \right).\end{aligned}$$

If λ is in the case (4.10), then from the fact $w = 0$ it follows that

$$\lambda\alpha \left(g + \frac{1}{|\Omega|} \int_{\partial\Omega} h\,d\sigma \right) = 0, \tag{4.11}$$

where we note

$$\int_\Omega \left(g + \frac{1}{|\Omega|} \int_{\partial\Omega} h\,d\sigma \right) dx = \int_\Omega g\,dx + \int_{\partial\Omega} h\,d\sigma < 0.$$

This means that $g + (1/|\Omega|) \int_{\partial\Omega} h\,d\sigma \not\equiv 0$, and hence $\alpha = 0$ from (4.11). To sum up, we obtain $u = \alpha + w = 0$. This assertion ensures that (4.10) implies (2.6), since $\lambda_1(g,h)$ is minimal among the positive eigenvalues of (1.1) with $c = 0$ by virtue of (2.5). The proof of Theorem 2.2 is now complete.

5 Proof of Theorem 2.3

This section is devoted to the proof of Theorem 2.3. For the existence it is essential to construct suitable super and subsolutions of (1.1) with $c > 0$ by using the principal positive eigenfunction $\phi_1(\lambda)$ of (2.1) corresponding to the principal eigenvalue $\mu_1(\lambda)$. Here $\phi_1(\lambda)$ is normalized as $\|\phi_1(\lambda)\|_{C(\overline{\Omega})} = 1$.

Now we recall that $\phi_1(\lambda) > 0$ in $\overline{\Omega}$, and that $\mu_1(\lambda) < 0$ for $\lambda > \lambda_1(g,h)$ from Theorem 2.1. For $\lambda > \lambda_1(g,h)$ and $\varepsilon > 0$, direct calculations give us that

$$-\Delta(\varepsilon\phi_1(\lambda)) - \lambda(g - c\varepsilon\phi_1(\lambda))\varepsilon\phi_1(\lambda) = \lambda\varepsilon\phi_1(\lambda) \left(\frac{\mu_1(\lambda)}{\lambda} + c\varepsilon\phi_1(\lambda) \right)$$

$$\begin{cases} \leq \lambda\varepsilon\phi_1(\lambda) \left(\dfrac{\mu_1(\lambda)}{\lambda} + c\varepsilon \right) & \text{in } \Omega, \\ \geq \lambda\varepsilon\phi_1(\lambda) \left(\dfrac{\mu_1(\lambda)}{\lambda} + c\varepsilon \min\limits_{\overline{\Omega}} \phi_1(\lambda) \right) & \text{in } \Omega. \end{cases}$$

This verifies the assertion that

$$\frac{1}{c}\left(-\frac{\mu_1(\lambda)}{\lambda}\right)\phi_1(\lambda), \qquad \frac{1}{c\min_{\overline{\Omega}}\phi_1(\lambda)}\left(-\frac{\mu_1(\lambda)}{\lambda}\right)\phi_1(\lambda)$$

are a subsolution and a supersolution of (1.1) with $c > 0$, respectively. By the super and subsolution method (see [3], [10]), there exists a positive solution $u \in C^{2+\theta}(\overline{\Omega})$ of (1.1) with $c > 0$ such that

$$\frac{1}{c}\left(-\frac{\mu_1(\lambda)}{\lambda}\right)\phi_1(\lambda) \le u \le \frac{1}{c\min_{\overline{\Omega}}\phi_1(\lambda)}\left(-\frac{\mu_1(\lambda)}{\lambda}\right)\phi_1(\lambda) \quad \text{in } \overline{\Omega}. \tag{5.1}$$

The uniqueness of positive solutions is a direct consequence of [10, Theorem 4.6.3].

Next we use the generalized Picone identity to verify (2.7), following the line of arguments in Berestycki, Capuzzo-Dolcetta and Nirenberg [4]. By integration by parts we have, for the unique positive solution u,

$$\begin{aligned}
&\int_\Omega \left(\frac{u}{\phi_1(\lambda)}\right)\sum_{j=1}^N \frac{\partial}{\partial x_j}\left(\phi_1(\lambda)^2\frac{\partial}{\partial x_j}\left(\frac{u}{\phi_1(\lambda)}\right)\right)dx \\
&= -\int_\Omega \phi_1(\lambda)^2\left|\nabla\left(\frac{u}{\phi_1(\lambda)}\right)\right|^2 dx + \int_{\partial\Omega}\frac{u}{\phi_1(\lambda)}\left(\frac{\partial u}{\partial \boldsymbol{n}}\phi_1(\lambda) - u\frac{\partial\phi_1(\lambda)}{\partial\boldsymbol{n}}\right)d\sigma \\
&= -\int_\Omega \phi_1(\lambda)^2\left|\nabla\left(\frac{u}{\phi_1(\lambda)}\right)\right|^2 dx \le 0.
\end{aligned}$$

By a direct calculation, we have

$$\int_\Omega \left(\frac{u}{\phi_1(\lambda)}\right)\sum_{j=1}^N \frac{\partial}{\partial x_j}\left(\phi_1(\lambda)^2\frac{\partial}{\partial x_j}\left(\frac{u}{\phi_1(\lambda)}\right)\right)dx = \lambda c\int_\Omega u^3\,dx + \mu_1(\lambda)\int_\Omega u^2\,dx.$$

It follows that

$$\lambda c\int_\Omega u^3\,dx + \mu_1(\lambda)\int_\Omega u^2\,dx \le 0. \tag{5.2}$$

By Hölder's inequality, we obtain

$$\int_\Omega u^3\,dx \le c^{-1}\left(-\frac{\mu_1(\lambda)}{\lambda}\right)\int_\Omega u^2\,dx \le c^{-1}\left(-\frac{\mu_1(\lambda)}{\lambda}\right)\left(\int_\Omega u^3\,dx\right)^{2/3}|\Omega|^{1/3},$$

which implies (2.7).

The consideration of the limiting behavior (2.8) is based on the combination of (5.1) and elliptic regularity arguments. In fact, by elliptic regularity, it follows that

$$\phi_1(\lambda) \longrightarrow 1 \quad \text{in } C^2(\overline{\Omega}) \text{ as } \lambda \to 0. \tag{5.3}$$

From (3.4) we deduce

$$-\frac{\mu_1(\lambda)}{\lambda} \longrightarrow -\mu_1'(0) = \frac{\int_\Omega g\,dx + \int_{\partial\Omega} h\,d\sigma}{|\Omega|} \quad \text{as } \lambda \to 0. \tag{5.4}$$

Hence if $\int_\Omega g\,dx + \int_{\partial\Omega} h\,d\sigma \geq 0$, then (5.1), (5.3) and (5.4) provide us with the assertion that as $\lambda \downarrow 0$,

$$u_\lambda(x) \longrightarrow \frac{\int_\Omega g\,dx + \int_{\partial\Omega} h\,d\sigma}{c|\Omega|} \quad \text{for each } x \in \overline{\Omega}. \tag{5.5}$$

Since we see from (5.1), (5.3) and (5.4) that u_λ is bounded in $C(\overline{\Omega})$ near $\lambda = 0$, it follows from (5.5) that, by elliptic regularity,

$$u_\lambda \longrightarrow \frac{\int_\Omega g\,dx + \int_{\partial\Omega} h\,d\sigma}{c|\Omega|} \quad \text{in } C^2(\overline{\Omega}) \text{ as } \lambda \downarrow 0.$$

If $\int_\Omega g\,dx + \int_{\partial\Omega} h\,d\sigma < 0$, then $-\mu_1(\lambda)/\lambda \to 0$ as $\lambda \to \lambda_1(g,h)$ since $\mu_1(\lambda_1(g,h)) = 0$. Therefore, by the same argument as above, (5.1) gives us that

$$u_\lambda \longrightarrow 0 \quad \text{in } C^2(\overline{\Omega}) \text{ as } \lambda \downarrow \lambda_1(g,h).$$

Finally, to verify the nonexistence result, we have only to recall from Theorem 2.1 that $\mu_1(\lambda) \geq 0$ for $0 < \lambda \leq \lambda_1(g,h)$ when $\int_\Omega g\,dx + \int_{\partial\Omega} h\,d\sigma < 0$. In fact, if there exists a positive solution u of (1.1) with $c > 0$ for some $0 < \lambda \leq \lambda_1(g,h)$, then this is contradictory for (5.2). The proof of Theorem 2.3 is now complete.

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