# On eigenvalues of a high-dimensional spatial-sign covariance matrix 

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#### Abstract

This paper investigates limiting spectral properties of a high-dimensional sample spatial-sign covariance matrix when both the dimension and the sample size grow to infinity. The underlying population is general enough to cover the popular independent components model and the family of elliptical distributions. The first result of the paper shows that the empirical spectral distribution of a high dimensional sample spatial-sign covariance matrix converges to a generalized Marčenko-Pastur distribution. Secondly, a new central limit theorem for a class of linear spectral statistics of the covariance matrix is established under moment conditions.


## 1 Introduction

Let $\mathbf{x}_{1}, \ldots, \mathbf{x}_{n}$ be a sequence of independent and identically distributed (i.i.d.) observations from a common population $\mathbf{x} \in \mathbb{R}^{p}$ with known location vector $\mathbf{m}$ (mean or median). The sample spatial-sign covariance matrix (SSCM) is by definition

$$
\mathbf{B}_{n}=\frac{p}{n} \sum_{j=1}^{n} \frac{\left(\mathbf{x}_{j}-\mathbf{m}\right)}{\left\|\mathbf{x}_{j}-\mathbf{m}\right\|} \frac{\left(\mathbf{x}_{j}-\mathbf{m}\right)^{\prime}}{\left\|\mathbf{x}_{j}-\mathbf{m}\right\|},
$$

where $\|\cdot\|$ denotes the Euclidean norm of a vector. In Locantore et al. (1999) and Visuri et al. (2000), the authors demonstrated that the SSCM is able to mitigate the impact of extreme outliers for the purpose of robust principal components analysis. Since then, the SSCM has been widely adopted for robust statistical inference where the sample data may exhibit heavy tails, or bear tail dependence as in the case of elliptical distributions. Recent works concerning the properties of the SSCM and its applications include Magyar and Tyler (2014), Dürre et al. (2014, 2015), Li et al. (2016), Feng et al. (2016a), Feng et al. (2016b) and Chakraborty et al. (2017). Despite the popularity of the SSCM, asymptotic behaviors of its eigenvalues are not fully developed when the dimension of the population $p$ diverges to infinity along with the sample size $n$, which greatly limits its application to high-dimensional data analysis.

This paper investigates the first and second order spectrum limits of sample SSCMs under the MarčenkoPastur asymptotic regime (Marčenko and Pastur, 1967), i.e.

$$
n \rightarrow \infty, \quad p=p(n) \rightarrow \infty, \quad p / n=c_{n} \rightarrow c \in(0, \infty)
$$

which is commonly adopted in the literature of random matrix theory. The underlying population $\mathbf{x}$ considered here has a general structure,

$$
\begin{equation*}
\mathbf{x}=\mathbf{m}+w \mathbf{A}^{\frac{1}{2}} \mathbf{z}, \tag{1.1}
\end{equation*}
$$

where $\mathbf{m} \in \mathbb{R}^{p}$ is the location vector, $\mathbf{A}$ is a $p \times p$ deterministic and positive definite matrix, $w \in \mathbb{R}$ and $\mathbf{z} \in \mathbb{R}^{p}$ are two (possibly dependent) random quantities with certain moment conditions, see (2.1) for detailed model illustration. The generality of this model lies in that it encompasses the popular independent components model and the family of elliptical distributions, which will be explained in detail in Section 2.2.

The first result of the paper is a new generalized Marčenko-Pastur (MP) law for the empirical spectral distribution (ESD) of $\mathbf{B}_{n}$. The MP law was originally introduced in Marčenko and Pastur (1967) for
the limiting spectrum of sample covariance matrices (SCMs), which was then refined and extended in several works, say Yin (1986), Silverstein (1995) and Bai and Zhou (2008). With this knowledge, through a comparison between the matrix $\mathbf{B}_{n}$ and its associated $\operatorname{SCM} \mathbf{S}_{n} \triangleq \sum_{j=1}^{n} \mathbf{A}^{1 / 2} \mathbf{z}_{j} \mathbf{z}_{j}^{\prime} \mathbf{A}^{1 / 2} / n$, our result is derived by showing $\left\|\mathbf{B}_{n}-\mathbf{S}_{n}\right\|$ converges to zero, almost surely, under finite ( $4+\delta$ )-th moment condition on the components of the vector $\mathbf{z}$. As a by-product, one may draw the same "no eigenvalue" conclusion for $\mathbf{B}_{n}$ as that for $\mathbf{S}_{n}$ already established in Bai and Silverstein (1998).

The second contribution of this paper is a new CLT for general linear spectral statistics (LSSs) of $\mathbf{B}_{n}$. CLT for LSSs of large random matrices has been actively studied in recent decades in random matrix theory. Most of early works in this area concern Hermitian (symmetric) Wigner matrices. Johansson (1998) presented a CLT for LSSs of eigenvalues given their joint density for Gaussian-type random Hermitian matrices. Using the moment method, Sinai and Soshinikov (1998) derived a CLT for traces of analytic functions of Wigner-type matrices and Anderson and Zeitouni (2006) obtained a CLT for a class of band random matrices. CLT for general Wigner matrix with arbitrary entries is first derived in Bai and Yao (2005) via Stieltjes transforms establishing the explicit formula for the mean and covariance functions of the limiting Gaussian distribution of the LSSs. A related approach using Gaussian interpolation for both Wigner matrices and Wishart matrices is proposed in Lytova and Pastur (2009). As for sample covariance matrices, the earliest work dates back to Jonsson (1982) for Wishart matrices. The seminal paper Bai and Silverstein (2004) established the CLT under the independent components model, which was later extended in Pan et al. (2008) and Zheng et al. (2015). Other extensions on CLT for sample covariance matrices are recently proposed in Hu et al. (2019a) and Hu et al. (2019b) for the class of elliptical distributions.

From the technical point of view, for establishing our CLT, the structure of the sample SSCM under study is quite different from the commonly studied random matrix models in the existing literature. Although the spatial-sign (or projective) transform $(\mathbf{x}-\mathbf{m}) /\|\mathbf{x}-\mathbf{m}\|$ removes the impact from the scaling random variable $w$, it does introduce, at the same time, complex non-linear correlations among the $p$ coordinates of the transformed data through the normalization by $\|\mathbf{x}-\mathbf{m}\|$. Such new correlations make the analysis more intricate in high dimensions where $p \rightarrow \infty$. Specifically, let us compare the situation with a sample covariance matrix $\mathbf{S}_{n}=n^{-1} \sum_{j=1}^{n}\left(\mathbf{x}_{j}-\mathbf{m}\right)\left(\mathbf{x}_{j}-\mathbf{m}\right)^{\prime}$ from the independent components model (see (2.2)). Here the correlations among the coordinates of a sample vector $\mathbf{x}_{j}$ have only one source, coming from the shape matrix $\mathbf{A}$. In the case of SSCM, the correlations among the coordinates of $\left(\mathbf{x}_{j}-\mathbf{m}\right) /\left\|\mathbf{x}_{j}-\mathbf{m}\right\|$ can originate from both the shape matrix $\mathbf{A}$ and the normalization by $\left\|\mathbf{x}_{j}-\mathbf{m}\right\|$. Therefore, a main task in our analysis is to find new approaches for decoupling these two sources of correlation. To this end, by giving an asymptotic expansion of $1 /\left\|\mathbf{x}_{j}-\mathbf{m}\right\|$ to certain order, we develop some new lemmas concerning the covariance and stochastic order of certain quadratic forms, which turns out to be one of the cornerstones for establishing our new CLT (see Section A.1). Another technical innovation of the paper, compared to the classical approach in Bai and Silverstein (2004), is that we introduce a new and more straightforward method to find the limiting mean function of LSSs, see Step 3 in the proof given in Section 3.2.

The rest of the paper is organized as follows. Section 2 presents our main theoretical results including both the convergence of the ESD of $\mathbf{B}_{n}$ and the CLT for its linear spectral statistics. Proofs of these asymptotic conclusions are presented in Sections 3.1 and 3.2. Some supporting lemmas and their proofs are relegated into the Appendix.

## 2 High-dimensional theory for eigenvalues of a sample SSCM

### 2.1 Preliminary definitions

Let $\mathbf{M}_{p}$ be a $p \times p$ symmetric or Hermitian matrix with eigenvalues $\left(\lambda_{j}\right)_{1 \leq j \leq p}$. Its ESD is by definition the probability measure

$$
F^{\mathbf{M}_{p}}=\frac{1}{p} \sum_{j=1}^{p} \delta_{\lambda_{j}}
$$

where $\delta_{b}$ denotes the Dirac mass at $b$. If the ESD sequence $\left\{F^{\mathbf{M}_{p}}\right\}$ has a limit when $p \rightarrow \infty$, this limit is referred as the limiting spectral distribution (LSD). For a probability measure $G$, its Stieltjes transform is defined as

$$
m_{G}(z)=\int \frac{1}{x-z} d G(x), \quad z \in \mathbb{C}^{+}
$$

where $\mathbb{C}^{+} \equiv\{z \in \mathbb{C}: \mathfrak{I}(z)>0\}$. This definition can be extended to the whole complex plane except the support set of $G$. An inversion formula of the Stieltjes transform can be found in Bai and Silverstein (2010).

Two sequences of $\mathbb{R}^{k}$-valued random vectors $\left(\boldsymbol{\xi}_{n}\right)$ and $\left(\boldsymbol{\eta}_{n}\right)$ are called asymptotically equal in distribution, denoted as $\boldsymbol{\xi}_{n} \stackrel{d}{\sim} \boldsymbol{\eta}_{n}$, if for any Borel set $C \subset \mathbb{R}^{k}$,

$$
\mathbb{P}\left(\boldsymbol{\xi}_{n} \in C\right)-\mathbb{P}\left(\boldsymbol{\eta}_{n} \in C\right) \rightarrow 0, \quad n \rightarrow \infty
$$

### 2.2 Model assumptions

We consider a sequence of i.i.d. observations $\mathbf{x}_{1}, \ldots, \mathbf{x}_{n}$ generated from the model (1.1), which admit the following stochastic representation:

$$
\begin{equation*}
\mathbf{x}_{j}=\mathbf{m}+w_{j} \mathbf{A}^{\frac{1}{2}} \mathbf{z}_{j}, \quad j=1, \ldots, n \tag{2.1}
\end{equation*}
$$

where
(i) the location vector $\mathbf{m} \in \mathbb{R}^{p}$ is assumed to be known;
(ii) the scalar random variable $w_{j}$ is real-valued having no mass at the origin, i.e. $\mathrm{P}\left(w_{j} \neq 0\right)=1$;
(iii) the matrix $\mathbf{A} \in \mathbb{R}^{p \times p}$, referred as the shape matrix or scatter matrix of the population, is deterministic, positive definite, and normalized as $\operatorname{tr}(\mathbf{A})=p$ for the identification in the triple product $w_{j} \mathbf{A}^{1 / 2} \mathbf{z}_{j}$, since we can always move any scalar factor related to $\mathbf{A}$ into the scalar random variable $w_{j}$;
(iv) the vector $\mathbf{z}_{j}=\left(z_{1 j}, \ldots, z_{p j}\right)^{\prime} \in \mathbb{R}^{p}$ is an array of i.i.d. standardized random variables, possibly dependent of $w_{j}$.
Our main assumptions are as follows.
Assumption (a). Both the sample size $n$ and population dimension $p$ tend to infinity in such a way that $n \rightarrow \infty, p=p(n) \rightarrow \infty$ and $p / n=c_{n} \rightarrow c \in(0, \infty)$.
Assumption (b). $\quad$ The ESD $H_{p}$ of the shape matrix A has bounded support, i.e. $\operatorname{Supp}\left(H_{p}\right) \subset[a, b]$ for some $a, b \in(0, \infty)$, and converges weakly to a probability distribution $H$ as $p \rightarrow \infty$.
Assumption (c). The random variables $\left(z_{i j}\right)$ are i.i.d. and satisfy

$$
\mathbb{E}\left(z_{i j}\right)=0, \quad \mathbb{E}\left(z_{i j}^{2}\right)=1, \quad \mathbb{E}\left(z_{i j}^{4}\right)=\tau, \quad \mathbb{E}\left|z_{i j}\right|^{4+\delta}<\infty,
$$

for some $\delta>0$.

Remark 2.1 Recall that in the literature on high-dimensional SCMs, the following independent components model is routinely considered (Bai and Silverstein, 2004; Pan et al., 2008; Zheng et al., 2015; Yao et al., 2015)

$$
\begin{equation*}
\mathbf{x}_{j}=\mathbf{m}+\sigma \mathbf{A}^{\frac{1}{2}} \mathbf{z}_{j} \tag{2.2}
\end{equation*}
$$

where $\mathbf{m}, \mathbf{A}$, and $\mathbf{z}_{j}$ are the same as in model (2.1) while $\sigma$ is a positive constant. Clearly the model (2.2) is a particular case of the model (2.1) where $\left\{w_{j}\right\}$ degenerate to the constant parameter $\sigma$.

Remark 2.2 The model (2.1) contains also the family of elliptical distributions. Indeed, a generalized elliptically distributed sample $\mathbf{x}_{j}$ has the form

$$
\begin{equation*}
\mathbf{x}_{j}=\mathbf{m}+v_{j} \mathbf{A}^{\frac{1}{2}} \mathbf{u}_{j}, \tag{2.3}
\end{equation*}
$$

where $v_{j}$ is a scalar random variable, $\mathbf{u}_{j}$ is a random vector uniformly distributed on the unit sphere in $\mathbb{R}^{p}$. Let $\mathbf{u}_{j}=\mathbf{z}_{j} /\left\|\mathbf{z}_{j}\right\|$ and $w_{j}=v_{j} /\left\|\mathbf{z}_{j}\right\|$, where $\mathbf{z}_{j} \sim \mathcal{N}\left(\mathbf{0}, \mathbf{I}_{p}\right)$, we thus have

$$
\mathbf{x}_{j}=\mathbf{m}+v_{j} \mathbf{A}^{\frac{1}{2}} \mathbf{u}_{j}=\mathbf{m}+w_{j} \mathbf{A}^{\frac{1}{2}} \mathbf{z}_{j}
$$

Certainly the moment conditions in Assumption (c) are satisfied with $\tau=3$ for such standard Gaussian random vectors $\left\{\mathbf{z}_{j}\right\}$. Thus the generalized elliptical distributions described by (2.3) are also special cases of our model (2.1).

### 2.3 Sample SSCM and its limiting spectral distribution

Let $\mathbf{s}(\mathbf{y})=I_{(\mathbf{y} \neq 0)} \mathbf{y} /\|\mathbf{y}\|$ be the spatial-sign function projecting the vector $\mathbf{y}$ onto the unit sphere. Then the sample SSCM $\mathbf{B}_{n}$ formed by the sample $\left\{\mathbf{x}_{j}\right\}$ can be written as

$$
\begin{equation*}
\mathbf{B}_{n}=\frac{p}{n} \sum_{j=1}^{n} \mathbf{s}\left(\mathbf{x}_{j}-\mathbf{m}\right) \mathbf{s}\left(\mathbf{x}_{j}-\mathbf{m}\right)^{\prime} . \tag{2.4}
\end{equation*}
$$

Our first result is concerned with the convergence of the ESD $F^{\mathbf{B}_{n}}$ of the sample SSCM $\mathbf{B}_{n}$.
Theorem 2.1 Suppose that Assumptions (a)-(c) hold. Then, almost surely, the empirical spectral distribution $F^{\mathbf{B}_{n}}$ converges weakly to a probability distribution $F^{c, H}$, whose Stieltjes transform $m=m(z)$ is the unique solution to the equation

$$
\begin{equation*}
m=\int \frac{1}{t(1-c-c z m)-z} d H(t), \quad z \in \mathbb{C}^{+} \tag{2.5}
\end{equation*}
$$

in the $\operatorname{set}\left\{m \in \mathbb{C}:-(1-c) / z+c m \in \mathbb{C}^{+}\right\}$.
Theorem 2.1 demonstrates that the ESD $F^{\mathbf{B}_{n}}$ converges to the generalized MP law $F^{c, H}$ (Marčenko and Pastur, 1967) defined through the equation (2.5). Let $\underline{F}^{c, H}=c F^{c, H}+(1-c) \delta_{0}$ be the companion distribution of $F^{c, H}$ and $\underline{m}=\underline{m}(z)$ be the Stieltjes transform of $\underline{F}^{c, H}$. Then (2.5) can be rewritten as

$$
\begin{equation*}
z=-\frac{1}{\underline{m}}+c \int \frac{t}{1+t \underline{m}} d H(t), \quad z \in \mathbb{C}^{+}, \tag{2.6}
\end{equation*}
$$

see Silverstein (1995). For procedures on numerically finding the density function of $F^{c, H}$ and its support set from (2.5) or (2.6), one is referred to Bai and Silverstein (2010). The proof of this theorem is presented in Section 3.1.

### 2.4 CLT for linear spectral statistics of $B_{n}$

In this section, we study the fluctuation of LSSs of $\mathbf{B}_{n}$. Given a measurable function $f$, the LSS associated with $f$ and $\mathbf{B}_{n}$ is

$$
\begin{equation*}
\int f(x) d F^{\mathbf{B}_{n}}(x) . \tag{2.7}
\end{equation*}
$$

To centralize this statistic, we need to introduce a matrix $\mathbf{T}$ that is closely related to the shape matrix $\mathbf{A}$, i.e.

$$
\begin{equation*}
\mathbf{T}=\mathbf{A}-\frac{2}{p} \mathbf{A}^{2}-\frac{\tau-3}{p} \mathbf{A}^{\frac{1}{2}} \operatorname{diag}(\mathbf{A}) \mathbf{A}^{\frac{1}{2}}+\left(\frac{2}{p^{2}} \operatorname{tr} \mathbf{A}^{2}+\frac{\tau-3}{p^{2}} \operatorname{tr}(\mathbf{A} \circ \mathbf{A})\right) \mathbf{A} \tag{2.8}
\end{equation*}
$$

where " $\circ$ " denotes the Hadamard product of two matrices. This matrix is actually an approximation of the population $\operatorname{SSCM} \boldsymbol{\Sigma} \triangleq p \mathbb{E}(\mathbf{x}-\mathbf{m})(\mathbf{x}-\mathbf{m})^{\prime} /\|\mathbf{x}-\mathbf{m}\|^{2}$. Under certain conditions, we have the spectral norm $\|\mathbf{\Sigma}-\mathbf{T}\|=o\left(p^{-1}\right)$, see Lemma A.3. Note that for an elliptical distribution, see the model (2.3), the population $\operatorname{SSCM} \boldsymbol{\Sigma}$ and the shape matrix $\mathbf{A}$ share the same eigenvectors and their eigenvalues have a one-to-one correspondence, which can be represented through certain integrals, see Dürre et al. (2016). Our approximation is however explicit and is not restricted to elliptical distributions.

Let $\tilde{H}_{p}$ be the ESD of the matrix T defined in (2.8), $\underline{m}_{0}(z)$ be the finite-horizon proxy for the limiting Stieltjes transform $\underline{m}(z)$ in (2.6), i.e. the solution to

$$
\begin{equation*}
z=-\frac{1}{\underline{m}_{0}(z)}+c_{n} \int \frac{t}{1+t \underline{m}_{0}(z)} d \tilde{H}_{p}(t), \quad z \in \mathbb{C}^{+} \tag{2.9}
\end{equation*}
$$

This Stieltjes transform $\underline{m}_{0}(z)$ uniquely defines a distribution, denoted by $F^{c_{n}, \tilde{H}_{p}}$, through

$$
\begin{equation*}
\underline{m}_{0}(z)=-\frac{1-c_{n}}{z}+c_{n} \int \frac{1}{x-z} d F^{c_{n}, \tilde{H}_{p}}(x) . \tag{2.10}
\end{equation*}
$$

By means of this distribution, The LSS in (2.7) can be centralized as

$$
G_{n}(f) \triangleq \int f(x) d G_{n}(x)=\int f(x) d\left[F^{\mathbf{B}_{n}}(x)-F^{c_{n}, \tilde{H}_{p}}(x)\right] .
$$

We note that from (2.8), for the first order asymptotic of $G_{n}(f)$, one may replace the ESD $\tilde{H}_{p}$ of $\mathbf{T}$ by the ESD $H_{p}$ of $\mathbf{A}$ in the definition (2.9) of $\underline{m}_{0}(z)$ since the two matrices share the same LSD. However, for the second order asymptotic, the difference of $\mathbf{T}$ and $\mathbf{A}$ will contribute when the shape matrix $\mathbf{A}$ is not identity. In addition, three other auxiliary quantities defined as below will also contribute to the fluctuation of $G_{n}(f)$,

$$
\begin{align*}
& \zeta_{p}=\frac{1}{p} \operatorname{tr}[\mathbf{T} \circ \mathbf{T}], \\
& h_{p}(z)=\frac{1}{p} \operatorname{tr}\left[\mathbf{T}^{\frac{1}{2}}(\mathbf{T}-z I)^{-1} \mathbf{T}^{\frac{1}{2}} \circ \mathbf{T}\right], \\
& g_{p}(z, \tilde{z})=\frac{1}{p} \operatorname{tr}\left[\left(\mathbf{T}^{\frac{1}{2}}(\mathbf{T}-z I)^{-1} \mathbf{T}^{\frac{1}{2}}\right) \circ\left(\mathbf{T}^{\frac{1}{2}}(\mathbf{T}-\tilde{z} I)^{-1} \mathbf{T}^{\frac{1}{2}}\right)\right], \tag{2.11}
\end{align*}
$$

where $z$ and $\tilde{z}$ are two complex numbers in $\mathbb{C}^{+}$. These quantities depend not only on the eigenvalues of $\mathbf{A}$, but also on its eigenvectors.

Theorem 2.2 Suppose that Assumptions (a)-(c) hold with $\delta=1$. Let $f_{1}, \ldots, f_{k}$ be $k$ functions analytic on an open set that includes the interval

$$
I_{c}=\left[\liminf _{p \rightarrow \infty} \lambda_{\min }^{\mathbf{T}} \delta_{(0,1)}(c)(1-\sqrt{c})^{2}, \quad \underset{p \rightarrow \infty}{\limsup } \lambda_{\max }^{\mathbf{T}}(1+\sqrt{c})^{2}\right] .
$$

Also let

$$
\mathbf{Y}_{n}=p\left\{G_{n}\left(f_{1}\right), \ldots, G_{n}\left(f_{k}\right)\right\}
$$

be the vector of $k$ normalized LSSs with respect to $f_{1}, \ldots, f_{k}$. Then $\mathbf{Y}_{n}$ is asymptotically equal in distribution to a $k$-dimensional Gaussian random vector $\xi_{n}=\left(\xi_{n 1}, \ldots, \xi_{n k}\right)$ with mean function

$$
\mathbb{E}\left(\xi_{n j}\right)=-\frac{1}{2 \pi \mathrm{i}} \oint_{C_{1}} f_{j}(z)\left[\mu_{1}(z)+(\tau-3) \mu_{2}(z)\right] d z
$$

where

$$
\begin{aligned}
\mu_{1}(z)= & \int \frac{c_{n}\left(\underline{m}_{0}^{\prime}(z) t\right)^{2} d \tilde{H}_{p}(t)}{\underline{m}_{0}(z)\left(1+\underline{m}_{0}(z) t\right)^{3}}-\int \frac{2 \underline{m}_{0}^{\prime}(z)\left(1+z \underline{m}_{0}(z)\right) t^{2} d \tilde{H}_{p}(t)}{\left(1+\underline{m}_{0}(z) t\right)^{2}} \\
& +\int \frac{\left(\operatorname{tr}\left(\mathbf{A}^{2} / p\right) t-t^{2}\right) d \tilde{H}_{p}(t)}{1+\underline{m}_{0}(z) t} \int \frac{2 c_{n} \underline{m}_{0}(z) \underline{m}_{0}^{\prime}(z) t d \tilde{H}_{p}(t)}{\left(1+\underline{m}_{0}(z) t\right)^{2}}, \\
\mu_{2}(z)= & \frac{c_{n} \underline{m}_{0}^{\prime}(z)}{\underline{m}_{0}^{2}(z)} g_{p, z}^{\prime}\left(\frac{-1}{\underline{m}_{0}(z)}, \frac{-1}{\underline{m}_{0}(z)}\right)+\zeta_{p} \int \frac{\left(1+z \underline{m}_{0}(z)\right) t \underline{m}_{0}^{\prime}(z) d \tilde{H}_{p}(t)}{\left(1+\underline{m}_{0}(z) t\right)^{2}} \\
& -\frac{\left(1+z \underline{m}_{0}(z)\right) \underline{m}_{0}^{\prime}(z)}{\underline{m}_{0}^{2}(z)} h_{p}^{\prime}\left(\frac{-1}{\underline{m}_{0}(z)}\right)-\int \frac{c_{n} \underline{m}_{0}^{\prime}(z) t d \tilde{H}_{p}(t)}{\left(1+\underline{m}_{0}(z) t\right)^{2}} h_{p}\left(\frac{-1}{\underline{m}_{0}(z)}\right),
\end{aligned}
$$

and covariance function

$$
\operatorname{Cov}\left(\xi_{n j}, \xi_{n \ell}\right)=-\frac{1}{4 \pi^{2}} \oint_{C_{1}} \oint_{C_{2}} f_{j}(z) f_{\ell}(\tilde{z})\left[\sigma_{1}(z, \tilde{z})+(\tau-3) \sigma_{2}(z, \tilde{z})\right] d z d \tilde{z}
$$

where

$$
\begin{aligned}
\sigma_{1}(z, \tilde{z})= & \frac{2 \partial^{2}}{\partial z \partial \tilde{z}}\left[\log \frac{\underline{m}_{0}(z)-\underline{m}_{0}(\tilde{z})}{\underline{m}_{0}(z) \underline{m}_{0}(\tilde{z})(z-\tilde{z})}+\left(\frac{\operatorname{tr}\left(\mathbf{T}^{2}\right)}{p c_{n}}+\frac{1}{c_{n} \underline{m}_{0}(z)}+\frac{1}{c_{n} \underline{m}_{0}(\tilde{z})}\right)\right. \\
& \left.\times\left(1+z \underline{m}_{0}(z)\right)\left(1+\tilde{z} \underline{m}_{0}(\tilde{z})\right)-z \underline{m}_{0}(z)-\tilde{z} \underline{m}_{0}(\tilde{z})-2\right], \\
\sigma_{2}(z, \tilde{z})= & \frac{\partial^{2}}{\partial z \partial \tilde{z}}\left[c_{n} g_{p}\left(\frac{-1}{\underline{m}_{0}(z)}, \frac{-1}{\underline{m}_{0}(\tilde{z})}\right)+\frac{\zeta_{p}}{c_{n}}\left(1+z \underline{m}_{0}(z)\right)\left(1+\tilde{z} \underline{m}_{0}(\tilde{z})\right)\right. \\
& \left.-\left(1+z \underline{m}_{0}(z)\right) h_{p}\left(\frac{-1}{\underline{m}_{0}(\tilde{z})}\right)-\left(1+\tilde{z} \underline{m}_{0}(\tilde{z})\right) h_{p}\left(\frac{-1}{\underline{m}_{0}(z)}\right)\right] .
\end{aligned}
$$

The contours $C_{1}$ and $C_{2}$ are non-overlapping, closed, counter-clockwise orientated in the complex plane and enclosing the interval $I_{c}$.

Remark 2.3 Theorem 2.2 approximates the distribution of $\mathbf{Y}_{n}$ by that of a Gaussian random vector $\boldsymbol{\xi}_{n}$. However, this approximating vector $\boldsymbol{\xi}_{n}$ may not converge in distribution, that is, the sequence of $\left\{\left(\zeta_{p}, h_{p}(z), g_{p}(z, \tilde{z})\right)\right\}$ which determines the mean and covariance functions of $\xi_{n}$ may not have a limit as $(p, n) \rightarrow \infty$. In addition, the convergence of $H_{p}$ in Assumption (b) is not required and the convergence of $c_{n}$ in Assumption (a) can be weaken to $0<\liminf c_{n} \leq \limsup c_{n}<\infty$. The proof of this theorem is presented in Section 3.2.

Remark 2.4 Theorem 2.2 contains the CLT for LSSs of high dimensional correlation matrices when the population mean is assumed known (Gao et al., 2017). To see this, consider the simplest case that $\mathbf{m}=\mathbf{0}$, $w_{j} \equiv 1$ and $\mathbf{A}=\mathbf{I}_{p}$ in (2.1), then the sample SSCM under study can be written as

$$
\mathbf{B}_{n}=\frac{p}{n} \sum_{j=1}^{n} \frac{\mathbf{z}_{j}}{\left\|\mathbf{z}_{j}\right\|} \frac{\mathbf{z}_{j}^{\prime}}{\left\|\mathbf{z}_{j}\right\|}=\frac{p}{n}\left(\frac{\mathbf{z}_{1}}{\left\|\mathbf{z}_{1}\right\|}, \cdots, \frac{\mathbf{z}_{n}}{\left\|\mathbf{z}_{n}\right\|}\right)\left(\frac{\mathbf{z}_{1}}{\left\|\mathbf{z}_{1}\right\|}, \cdots, \frac{\mathbf{z}_{n}}{\left\|\mathbf{z}_{n}\right\|}\right)^{\prime} .
$$

Denote its companion matrix as

$$
\begin{equation*}
\underline{\mathbf{B}}_{n}=\frac{p}{n}\left(\frac{\mathbf{z}_{1}}{\left\|\mathbf{z}_{1}\right\|}, \cdots, \frac{\mathbf{z}_{n}}{\left\|\mathbf{z}_{n}\right\|}\right)^{\prime}\left(\frac{\mathbf{z}_{1}}{\left\|\mathbf{z}_{1}\right\|}, \cdots, \frac{\mathbf{z}_{n}}{\left\|\mathbf{z}_{n}\right\|}\right), \tag{2.12}
\end{equation*}
$$

which shares the same non-zero eigenvalues as $\mathbf{B}_{n}$. Thus the result in Theorem 2.2 gives the CLT for LSSs of $\underline{\mathbf{B}}_{n}$. Now let's denote the data matrix as $\mathbf{Z}=\left(\mathbf{z}_{1}, \ldots, \mathbf{z}_{n}\right)=\left(\mathbf{v}_{1}, \ldots, \mathbf{v}_{p}\right)^{\prime}$, where $\mathbf{z}_{j}$ is the $j$-th column ( $j$-th observation) and $\mathbf{v}_{j}^{\prime}$ is the $j$-th row ( $j$-th coordinate) of $\mathbf{Z}$. Moreover, the table $\mathbf{Z}$ consists of independent and identically distributed entries across both the rows and columns so permuting the entries in $\mathbf{Z}$ will not change its distribution. The correlation matrix $\mathbf{R}_{n}$ associated with the data set $\mathbf{Z}$ can be expressed as

$$
\begin{equation*}
\mathbf{R}_{n}=\left(\frac{\mathbf{v}_{1}}{\left\|\mathbf{v}_{1}\right\|}, \cdots, \frac{\mathbf{v}_{p}}{\left\|\mathbf{v}_{p}\right\|}\right)^{\prime}\left(\frac{\mathbf{v}_{1}}{\left\|\mathbf{v}_{1}\right\|}, \cdots, \frac{\mathbf{v}_{p}}{\left\|\mathbf{v}_{p}\right\|}\right) \tag{2.13}
\end{equation*}
$$

which has the same structure (up to a constant factor) as $\underline{\mathbf{B}}_{n}$ in (2.12) by interchanging the roles of $p$ and $n$. Therefore in the case of $\mathbf{A}=\mathbf{I}_{p}$, the CLT for LSSs of $\mathbf{R}_{n}$ is readily derived from an application of Theorem 2.2 to the matrix $\underline{\mathbf{B}}_{n}$.

### 2.5 Example

As an illustration, we exhibit the CLT for a widely used LSS which is the second moment of the eigenvalues of $\mathbf{B}_{n}$, denoted by

$$
\hat{\beta}_{2}=\frac{1}{p} \operatorname{tr}\left(\mathbf{B}_{n}^{2}\right)
$$

We consider the case where the population shape matrix $\mathbf{A}$ is diagonal. In this case, the matrix $\mathbf{T}$ in (2.8) can be simplified as

$$
\mathbf{T}=\mathbf{A}-\frac{\tau-1}{p} \mathbf{A}^{2}+\frac{\tau-1}{p^{2}} \operatorname{tr} \mathbf{A}^{2} \cdot \mathbf{A}
$$

and the three auxiliary quantities in (2.11) become

$$
\zeta_{p}=\int t^{2} d \tilde{H}_{p}(t), \quad h_{p}(z)=\int \frac{t^{2}}{t-z} d \tilde{H}_{p}(t), \quad g_{p}(z, \tilde{z})=\int \frac{t^{2}}{(t-z)(t-\tilde{z})} d \tilde{H}_{p}(t)
$$

which are only functions of the eigenvalues of $\mathbf{A}$. Let

$$
\alpha_{k, p}=\frac{1}{p} \operatorname{tr}\left(\mathbf{T}^{k}\right)=\int t^{k} d \tilde{H}_{p}(t)
$$

By the relations in (2.9) and (2.10), the centering term for the statistic $\hat{\beta}_{2}$ is

$$
\beta_{2, p} \triangleq \int x^{2} d F^{c_{n}, \tilde{H}_{p}}(x)=\alpha_{2, p}+c_{n}
$$

The approximating mean and covariance of $p\left[\hat{\beta}_{2}-\beta_{2, p}\right]$ can be figured out through the residue theorem. For illustration, we calculate the integral corresponding to the first term in $\mu_{1}(z)$, that is,

$$
\begin{equation*}
-\frac{1}{2 \pi \mathrm{i}} \oint_{\mathcal{C}_{1}} z^{2} \int \frac{c_{n}\left(\underline{\underline{m}}_{0}^{\prime}(z) t\right)^{2} d \tilde{H}_{p}(t)}{\underline{m}_{0}(z)\left(1+\underline{m}_{0}(z) t\right)^{3}} d z \tag{2.14}
\end{equation*}
$$

Taking derivatives with respect to $z$ on both sides of (2.9), we obtain

$$
\underline{m}_{0}^{\prime}(z)=\left(\frac{1}{\underline{m}_{0}^{2}(z)}-c_{n} \int \frac{t^{2}}{\left(1+t \underline{m}_{0}(z)\right)^{2}} d \tilde{H}_{p}(t)\right)^{-1}
$$

It then follows that

$$
\begin{aligned}
\text { (2.14) } & =\int-\frac{1}{2 \pi \mathrm{i}} \oint_{C_{1}} z^{2} \frac{c_{n} \underline{m}_{0}^{\prime}(z) t^{2}}{\underline{m}_{0}(z)\left(1+\underline{m}_{0}(z) t\right)^{3}} d \underline{m}_{0}(z) d \tilde{H}_{p}(t) \\
& =\int-\frac{1}{2 \pi \mathrm{i}} \oint_{C_{1}} \frac{c_{n} t^{2}\left(z \underline{m}_{0}(z)\right)^{2}}{\underline{m}_{0}(z)\left(1+\underline{m}_{0}(z) t\right)^{3}}\left(1-c_{n} \int \frac{u^{2} \underline{\underline{m}}_{0}^{2}(z)}{\left(1+u \underline{m}_{0}(z)\right)^{2}} d \tilde{H}_{p}(u)\right)^{-1} d \underline{m}_{0}(z) d \tilde{H}_{p}(t) \\
& =-\int\left\{\left.\frac{c_{n} t^{2}\left(z \underline{m}_{0}(z)\right)^{2}}{\left(1+\underline{m}_{0}(z) t\right)^{3}}\left(1-c_{n} \int \frac{u^{2} \underline{m}_{0}^{2}(z)}{\left(1+u \underline{m}_{0}(z)\right)^{2}} d \tilde{H}_{p}(u)\right)^{-1}\right|_{\underline{m}_{0}(z)=0}\right\} d \tilde{H}_{p}(t) \\
& =-c_{n} \alpha_{2, p} .
\end{aligned}
$$

Similar procedure can be repeated to find the values of the remaining contour integrals. As a result and by Theorem 2.2, the distribution of $p\left[\hat{\beta}_{2}-\beta_{2, p}\right]$ is asymptotically equivalent to the Gaussian distribution $N\left(\mu, \sigma^{2}\right)$, where the mean and variance parameters are given by

$$
\begin{aligned}
\mu & =-c_{n} \alpha_{2, p} \\
\sigma^{2} & =8 c_{n}\left(\alpha_{2, p}^{3}-2 \alpha_{2, p} \alpha_{3, p}+\alpha_{4, p}\right)+4 c_{n}^{2} \alpha_{2, p}^{2}+4 c_{n}(\tau-3)\left(\alpha_{2, p}^{3}-2 \alpha_{2, p} \alpha_{3, p}+\alpha_{4, p}\right)
\end{aligned}
$$

## 3 Proofs of the main results

This section presents the proofs of Theorem 2.1 and Theorem 2.2. In all the proofs, we assume the location vector $\mathbf{m}=\mathbf{0}$, otherwise, it can be directly subtracted from the sample $\left\{\mathbf{x}_{j}\right\}$. We will denote by $K$ some constants appearing in inequalities that can vary from place to place.

### 3.1 Proof of Theorem 2.1

Let $g_{j}=p /\left(\mathbf{z}_{j}^{\prime} \mathbf{A} \mathbf{z}_{j}\right)$ for $j=1, \ldots, n$, and denote

$$
\mathbf{Z}=\left(z_{i j}\right), \quad \mathbf{G}=\operatorname{diag}\left(g_{1}, \ldots, g_{n}\right), \quad \mathbf{B}_{n}=\frac{1}{n} \mathbf{A}^{\frac{1}{2}} \mathbf{Z} \mathbf{G} \mathbf{Z}^{\prime} \mathbf{A}^{\frac{1}{2}}, \quad \mathbf{S}_{n}=\frac{1}{n} \mathbf{A}^{\frac{1}{2}} \mathbf{Z} \mathbf{Z}^{\prime} \mathbf{A}^{\frac{1}{2}}
$$

Under Assumptions (a)-(c), the generalized MP law holds true for the sample covariance matrix $\mathbf{S}_{n}$ (Silverstein, 1995). Thus it's sufficient to show

$$
\begin{equation*}
\left\|\mathbf{B}_{n}-\mathbf{S}_{n}\right\| \xrightarrow{\text { a.s. }} 0 . \tag{3.1}
\end{equation*}
$$

To this end, with the moment conditions in Assumption (c), we shall truncate the variables $\left(z_{i j}\right)$ at $n^{2 / \gamma}$ for some $\gamma \in(4,4+\delta]$. Some relevant quantities are denoted as below. For $i=1, \ldots, p$ and $j=1, \ldots, n$,

$$
\hat{z}_{i j}=z_{i j} I\left(\left|z_{i j}\right|^{\gamma} \leq n^{2}\right), \quad \hat{\mathbf{z}}_{j}=\left(\hat{z}_{1 j}, \ldots, \hat{z}_{p j}\right)^{\prime}, \quad \hat{g}_{j}=p /\left(\hat{\mathbf{z}}_{j}^{\prime} \mathbf{A} \hat{\mathbf{z}}_{j}\right),
$$

$$
\widehat{\mathbf{Z}}=\left(\hat{z}_{i j}\right), \quad \widehat{\mathbf{G}}=\operatorname{diag}\left(\hat{g}_{1}, \ldots, \hat{g}_{n}\right), \quad \widehat{\mathbf{B}}_{n}=\frac{1}{n} \mathbf{A}^{\frac{1}{2}} \widehat{\mathbf{Z}} \widehat{\mathbf{G}} \widehat{\mathbf{Z}}^{\prime} \mathbf{A}^{\frac{1}{2}}, \quad \widehat{\mathbf{S}}_{n}=\frac{1}{n} \mathbf{A}^{\frac{1}{2}} \widehat{\mathbf{Z} \mathbf{Z}^{\prime}} \mathbf{A}^{\frac{1}{2}} .
$$

Note that for the truncated variables $\left(\hat{z}_{i j}\right)$, the following results hold automatically

$$
\begin{equation*}
\left|\mathbb{E} \hat{z}_{i j}\right|=o\left(n^{-2+2 / \gamma}\right), \mathbb{E}\left(\hat{z}_{i j}^{2}\right)=1+o\left(n^{-2+4 / \gamma}\right), \mathbb{E}\left(\hat{z}_{i j}^{4}\right)=\tau+o(1), \mathbb{E}\left|\hat{z}_{i j}\right|^{\gamma}<\infty,\left|\hat{z}_{i j}\right|^{\gamma}<n^{2}, \tag{3.2}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{k} 2^{k} \mathbb{E}\left|z_{i j}\right|^{\gamma / 2} I\left(\left|z_{i j}\right|>2^{2 k / \gamma}\right)<\infty . \tag{3.3}
\end{equation*}
$$

From (3.3) and similar arguments as in the proof of Lemma 5.12 in Bai and Silverstein (2010), we have

$$
\begin{equation*}
\mathrm{P}\left(\widehat{\mathbf{B}}_{n} \neq \mathbf{B}_{n}, \text { i.o. }\right)=\mathrm{P}\left(\widehat{\mathbf{S}}_{n} \neq \mathbf{S}_{n}, \text { i.o. }\right)=0 \tag{3.4}
\end{equation*}
$$

Next we will prove that for any $\varepsilon>0$ and $k \geq 2$,

$$
\begin{equation*}
\mathrm{P}\left(\left\|\widehat{\mathbf{B}}_{n}-\widehat{\mathbf{S}}_{n}\right\|>\varepsilon\right) \leq K \varepsilon^{-k}\left(n^{-\frac{k}{2}+1}+n^{-\frac{k(\gamma-4)}{\gamma}}\right) \tag{3.5}
\end{equation*}
$$

Notice that the spectral norm of the difference between $\widehat{\mathbf{B}}_{n}$ and $\widehat{\mathbf{S}}_{n}$ can be bounded by

$$
\begin{equation*}
\left\|\widehat{\mathbf{B}}_{n}-\widehat{\mathbf{S}}_{n}\right\| \leq\|\mathbf{A}\| \frac{\left\|\widehat{\mathbf{Z} \mathbf{Z}^{\prime}}\right\|}{n} \max _{1 \leq j \leq n}\left|\hat{g}_{j}-1\right| . \tag{3.6}
\end{equation*}
$$

From Bai and Silverstein (1998), almost surely, the spectral norm $\left\|\widehat{\mathbf{Z Z}}{ }^{\prime}\right\| / n$ is bounded for all large $n$. Thus, we only need to control the convergence rate of $\max _{j}\left|\hat{g}_{j}-1\right|$ or $\max _{j}\left|1 / \hat{g}_{j}-1\right|$. By Markov's inequality, for any $\varepsilon>0$ and $k \geq 2$, we have

$$
\begin{equation*}
\mathrm{P}\left(\max _{j}\left|\frac{1}{\hat{g}_{j}}-1\right|>\varepsilon\right) \leq n p^{-k} \varepsilon^{-k} \mathbb{E}\left|\hat{\mathbf{z}}_{1}^{\prime} \mathbf{A} \hat{\mathbf{z}}_{1}-p\right|^{k} \tag{3.7}
\end{equation*}
$$

To bound the expectation in (3.7), we divide it into three parts

$$
\mathbb{E}\left|\hat{\mathbf{z}}_{1}^{\prime} \mathbf{A} \hat{\mathbf{z}}_{1}-p\right|^{k} \leq K \mathbb{E}\left|\hat{\mathbf{z}}_{1}^{\prime} \mathbf{A} \hat{\mathbf{z}}_{1}-\tilde{\mathbf{z}}_{1}^{\prime} \mathbf{A} \tilde{\mathbf{z}}_{1}\right|^{k}+K \mathbb{E}\left|\tilde{\mathbf{z}}_{1}^{\prime} \mathbf{A} \tilde{\mathbf{z}}_{1}-\mathbb{E} \tilde{\mathbf{z}}_{1}^{\prime} \mathbf{A} \tilde{\mathbf{z}}_{1}\right|^{k}+K\left|\mathbb{E} \tilde{\mathbf{z}}_{1}^{\prime} \mathbf{A} \tilde{\mathbf{z}}_{1}-p\right|^{k}
$$

From (3.2) and the boundedness of $\|\mathbf{A}\|$, the first term can be controlled by

$$
\begin{align*}
\mathbb{E}\left|\hat{\mathbf{z}}_{1}^{\prime} \mathbf{A} \hat{\mathbf{z}}_{1}-\tilde{\mathbf{z}}_{1}^{\prime} \mathbf{A} \tilde{\mathbf{z}}_{1}\right|^{k} & \leq K \mathbb{E}\left|\tilde{\mathbf{z}}_{1}^{\prime} \mathbf{A} \mathbb{E} \hat{\mathbf{z}}_{1}\right|^{k}+K\left|\mathbb{E} \hat{\mathbf{z}}_{1}^{\prime} \mathbf{A} \mathbb{E} \hat{\mathbf{z}}_{1}\right|^{k} \\
& \leq K \mathbb{E}^{\frac{1}{2}}\left|\tilde{\mathbf{z}}_{1}^{\prime} \tilde{\mathbf{z}}_{1}\right|^{k}\left|\mathbb{E} \hat{\mathbf{z}}_{1}^{\prime} \mathbb{E} \hat{\mathbf{z}}_{1}\right|^{\frac{k}{2}}+K n^{(-3+4 / \gamma) k} \\
& \leq K\left[\mathbb{E}\left|\tilde{\mathbf{z}}_{1}^{\prime} \tilde{\mathbf{z}}_{1}-\mathbb{E} \tilde{\mathbf{z}}_{1}^{\prime} \tilde{\mathbf{z}}_{1}\right|^{k}+\left|\mathbb{E} \tilde{\mathbf{z}}_{1}^{\prime} \tilde{\mathbf{z}}_{1}\right|^{k}\right]^{\frac{1}{2}} n^{(-3 / 2+2 / \gamma) k}+K n^{(-3+4 / \gamma) k} \\
& \leq K \mathbb{E}^{\frac{1}{2}}\left|\tilde{\mathbf{z}}_{1}^{\prime} \tilde{\mathbf{z}}_{1}-\mathbb{E} \tilde{\mathbf{z}}_{1}^{\prime} \tilde{\mathbf{z}}_{1}\right|^{k} n^{(-3 / 2+2 / \gamma) k}+K n^{(-3+4 / \gamma) k} \\
& \leq K n^{-1 / 2+(-3 / 2+4 / \gamma) k}+K n^{(-1+2 / \gamma) k}, \tag{3.8}
\end{align*}
$$

where the last inequality is from Lemma A.1. Again from this lemma, the second term is bounded by

$$
\begin{equation*}
\mathbb{E}\left|\tilde{\mathbf{z}}_{1}^{\prime} \mathbf{A} \tilde{\mathbf{z}}_{1}-\mathbb{E} \tilde{\mathbf{z}}_{1}^{\prime} \mathbf{A} \tilde{\mathbf{z}}_{1}\right|^{k} \leq K\left(n^{k / 2}+n \mathbb{E}\left|\tilde{z}_{11}\right|^{2 k}\right) \leq K\left(n^{k / 2}+n^{-1+4 k / \gamma}\right) . \tag{3.9}
\end{equation*}
$$

For the third one, we have from (3.2)

$$
\begin{equation*}
\left|\mathbb{E} \tilde{\mathbf{z}}_{1}^{\prime} \mathbf{A} \tilde{\mathbf{z}}_{1}-p\right|^{k}=p^{k}\left|\operatorname{Var}\left(\hat{\mathbf{z}}_{11}\right)-1\right|^{k} \leq K n^{(-1+4 / \gamma) k} \tag{3.10}
\end{equation*}
$$

Collecting the results in (3.8)-(3.10) yields

$$
\begin{equation*}
\mathbb{E}\left|\hat{\mathbf{z}}_{1}^{\prime} \mathbf{A} \hat{\mathbf{z}}_{1}-p\right|^{k} \leq K\left(n^{k / 2}+n^{-1+4 k / \gamma}\right), \tag{3.11}
\end{equation*}
$$

which together with (3.6) and (3.7) give the result in (3.5). Hence, the conclusion of the theorem follows from (3.1), (3.4), and (3.5) with some large $k$.

### 3.2 Proof of Theorem 2.2

### 3.2.1 Sketch of the proof

Following the truncation step in the proof of Theorem 2.1, we now centralize the truncated variables $\left(\hat{z}_{i j}\right)$. Some quantities are denoted as below.

$$
\begin{aligned}
& \tilde{z}_{i j}=\hat{z}_{i j}-\mathbb{E}\left(\hat{z}_{i j}\right), \quad \tilde{\mathbf{z}}_{j}=\left(\tilde{z}_{1 j}, \ldots, \tilde{z}_{p j}\right)^{\prime}, \quad \tilde{g}_{j}=p /\left(\tilde{\mathbf{z}}_{j}^{\prime} \mathbf{A}_{j}\right), \quad \widetilde{\mathbf{G}}=\operatorname{diag}\left(\tilde{g}_{1}, \ldots, \tilde{g}_{n}\right), \\
& \widetilde{\mathbf{Z}}=\left(\tilde{z}_{i j}\right), \quad \widetilde{\mathbf{B}}_{n}=\frac{1}{n} \mathbf{A}^{\frac{1}{2}} \widetilde{\mathbf{Z}} \widetilde{\mathbf{G}}^{\prime} \mathbf{A}^{\frac{1}{2}}, \quad \overline{\mathbf{B}}_{n}=\frac{1}{n} \mathbf{A}^{\frac{1}{2}} \widetilde{\mathbf{Z}} \widetilde{\mathbf{G}}^{\prime} \mathbf{A}^{\frac{1}{2}}, \quad \widetilde{\mathbf{S}}_{n}=\frac{1}{n} \mathbf{A}^{\frac{1}{2}} \widetilde{\mathbf{Z}} \widetilde{\mathbf{Z}}^{\prime} \mathbf{A}^{\frac{1}{2}} .
\end{aligned}
$$

Similar to the derivation of (3.5), one may show that

$$
\begin{equation*}
\max \left\{\mathrm{P}\left(\left\|\widetilde{\mathbf{B}}_{n}-\widetilde{\mathbf{S}}_{n}\right\|>\varepsilon\right), \mathrm{P}\left(\left\|\overline{\mathbf{B}}_{n}-\widetilde{\mathbf{S}}_{n}\right\|>\varepsilon\right)\right\} \leq K \varepsilon^{-k}\left(n^{-\frac{k}{2}+1}+n^{-\frac{k(\gamma-4)}{\gamma}}\right) . \tag{3.12}
\end{equation*}
$$

It thus follows from Bai and Silverstein (1998) that, almost surely, $\lim \sup _{n}\left\|\widehat{\mathbf{B}}_{n}\right\|, \lim \sup _{n}\left\|\widetilde{\mathbf{B}}_{n}\right\|$, and $\lim \sup _{n}\left\|\overline{\mathbf{B}}_{n}\right\|$ are all bounded.

Let $F^{\mathbf{B}_{n}}, F^{\widehat{\mathbf{B}}_{n}}, F^{\overline{\mathbf{B}}_{n}}$, and $F^{\widetilde{\mathbf{B}}_{n}}$ be the ESDs of the matrices $\mathbf{B}_{n}, \widehat{\mathbf{B}}_{n}, \overline{\mathbf{B}}_{n}$, and $\widetilde{\mathbf{B}}_{n}$, respectively. Then, for each function $f_{j}(x)$, we have from (3.4)

$$
\begin{equation*}
p\left|\int f_{j}(x) d F^{\mathbf{B}_{n}}-\int f_{j}(x) d F^{\widehat{\mathbf{B}}_{n}}\right| \xrightarrow{\text { a.s. }} 0 . \tag{3.13}
\end{equation*}
$$

By Corollary A. 37 in Bai and Silverstein (2010), it holds that

$$
\begin{align*}
p\left|\int f_{j}(x) d F^{\widehat{\mathbf{B}}_{n}}-\int f_{j}(x) d F^{\overline{\mathbf{B}}_{n}}\right| & \leq K_{j} \sum_{k=1}^{p}\left|\lambda_{k}^{\widehat{\mathbf{B}}_{n}}-\lambda_{k}^{\overline{\mathbf{B}}_{n}}\right|  \tag{3.14}\\
& \leq 2 K_{j}\left[c_{n} \operatorname{tr} \mathbf{A}^{\frac{1}{2}}(\widehat{\mathbf{Z}}-\widetilde{\mathbf{Z}}) \widehat{\mathbf{G}}(\widehat{\mathbf{Z}}-\widetilde{\mathbf{Z}})^{\prime} \mathbf{A}^{\frac{1}{2}}\left(\left\|\widehat{\mathbf{B}}_{n}\right\|+\left\|\overline{\mathbf{B}}_{n}\right\|\right)\right]^{1 / 2} .
\end{align*}
$$

where $K_{j}$ is an upper bound on $\left|f_{j}^{\prime}(x)\right|$ and $\lambda_{k}^{\mathbf{B}}$ denotes the $k$-th largest eigenvalue of the matrix $\mathbf{B}$. By (3.2) and (3.7), one may get

$$
\left|\operatorname{tr} \mathbf{A}^{\frac{1}{2}}(\widehat{\mathbf{Z}}-\widetilde{\mathbf{Z}}) \widehat{\mathbf{G}}(\widehat{\mathbf{Z}}-\widetilde{\mathbf{Z}})^{\prime} \mathbf{A}^{\frac{1}{2}}\right| \leq\|\mathbf{A}\| \max _{j}\left|\hat{g}_{j}\right| \operatorname{tr}\left(\widehat{\mathbb{E}} \widehat{\mathbf{Z}} \mathbb{E} \widehat{\mathbf{Z}}^{\prime}\right) \xrightarrow{\text { a.s. }} 0,
$$

and thus (3.14) is $o_{\text {a.s. }}(1)$. Moreover, from (3.7) and (3.8), applying Markov's inequality, we have also

$$
\begin{align*}
p\left|\int f_{j}(x) d F^{\overline{\mathbf{B}}_{n}}-\int f_{j}(x) d F^{\widetilde{\mathbf{B}}_{n}}\right| & \leq K_{j} p\left\|\overline{\mathbf{B}}_{n}-\widetilde{\mathbf{B}}_{n}\right\| \\
& \leq K_{j} p\|\mathbf{A}\| \cdot\left\|\frac{1}{n} \widetilde{\mathbf{Z}} \widetilde{\mathbf{Z}}^{\prime}\right\| \max _{j}\left|\hat{g}_{j}-\tilde{g}_{j}\right| \xrightarrow{\text { a.s. }} 0 . \tag{3.15}
\end{align*}
$$

Collecting (3.13), (3.14), and (3.15), we get

$$
\begin{equation*}
p\left|\int f_{j}(x) d F^{\mathbf{B}_{n}}-\int f_{j}(x) d F^{\widetilde{\mathbf{B}}_{n}}\right| \xrightarrow{\text { a.s. }} 0 . \tag{3.16}
\end{equation*}
$$

Therefore, it is sufficient to prove the theorem by replacing the matrix $\mathbf{B}_{n}$ with its truncated and centralized version $\widetilde{\mathbf{B}}_{n}$, or equivalently, we assume

$$
\begin{equation*}
\mathbb{E}\left(z_{11}\right)=0, \quad \mathbb{E}\left(z_{11}^{2}\right)=1, \quad \mathbb{E}\left(z_{11}^{4}\right)=\tau+o(1), \quad \mathbb{E}\left(\left|z_{11}\right|^{\gamma}\right)<\infty, \quad \max _{i, j}\left|z_{i j}\right|^{\gamma}<n^{2} \tag{3.17}
\end{equation*}
$$

with $\gamma=5$ for the proof of the theorem. Note that we assume $\mathbb{E}\left(z_{11}^{2}\right)=1$ without rescaling $\left(\tilde{z}_{i j}\right)$ since the sample spatial-sign vectors $\left\{\mathbf{A}^{1 / 2} \mathbf{z}_{j} /\left\|\mathbf{A}^{1 / 2} \mathbf{z}_{j}\right\|\right\}$ are all self-normalized.

Next we define a rectangular contour enclosing the interval $I_{c}=\left[s_{l}, s_{r}\right]$,

$$
\begin{equation*}
s_{l}=\liminf _{p \rightarrow \infty} \lambda_{\min }^{\mathrm{T}}(1-\sqrt{c})^{2} I_{(0,1)}(c) \quad \text { and } \quad s_{r}=\limsup _{p \rightarrow \infty} \lambda_{\max }^{\mathrm{T}}(1+\sqrt{c})^{2} \tag{3.18}
\end{equation*}
$$

and thus enclosing all supports of the LSDs $\left\{F^{c_{n}, \tilde{H}_{p}}\right\}$. Choosing two numbers $x_{l}<x_{r}$ such that $\left[s_{l}, s_{r}\right] \subset$ $\left(x_{l}, x_{r}\right)$ and letting $v_{0}>0$ be arbitrary, then the contour can be described as

$$
C=\left\{x \pm i v_{0}: x \in\left[x_{l}, x_{r}\right]\right\} \cup\left\{x+i v: x \in\left\{x_{r}, x_{l}\right\}, v \in\left[-v_{0}, v_{0}\right]\right\} .
$$

Denote

$$
m_{n}(z)=\int \frac{1}{x-z} d F^{\mathbf{B}_{n}}(x), \quad m_{0}(z)=\int \frac{1}{x-z} d F^{c_{n}, \tilde{H}_{p}}(x), \quad \underline{m}_{n}(z)=-\frac{1-c_{n}}{z}+c_{n} m_{n}(z),
$$

We then define a random process on $C$ as

$$
M_{n}(z)=p\left[m_{n}(z)-m_{0}(z)\right]=n\left[\underline{m}_{n}(z)-\underline{m}_{0}(z)\right], \quad z \in C,
$$

where $\underline{m}_{0}(z)$ is defined in (2.10). From Cauchy's integral formula, for any $k$ analytic functions $\left(f_{\ell}\right)$ and complex numbers $\left(a_{\ell}\right)$, we have

$$
\sum_{\ell=1}^{k} p a_{\ell} \int f_{\ell}(x) d G_{n}(x)=-\sum_{\ell=1}^{k} \frac{a_{\ell}}{2 \pi \mathrm{i}} \oint_{C} f_{\ell}(z) M_{n}(z) d z
$$

when all sample eigenvalues fall in the interval $\left(x_{l}, x_{r}\right)$, which holds with probability $1-o\left(n^{-s}\right)$ for any $s>0$. That is,

$$
\begin{equation*}
\mathrm{P}\left(\left\|\mathbf{B}_{n}\right\|>x_{r}\right)=o\left(n^{-s}\right) \quad \text { and } \quad \mathrm{P}\left(\lambda_{\min }^{\mathbf{B}_{n}}<x_{l}\right)=o\left(n^{-s}\right), \forall s>0 \tag{3.19}
\end{equation*}
$$

which follows from (3.12) and a similar conclusion for $\mathbf{S}_{n}$ (Bai and Silverstein, 2004). In order to deal with the small probability event where some eigenvalues are outside the interval $\left(x_{l}, x_{r}\right)$ in finite dimensional situations, Bai and Silverstein (2004) suggested truncating $M_{n}(z)$ as, for $z=x+\mathrm{i} v \in C$,

$$
\widehat{M}_{n}(z)= \begin{cases}M_{n}(z) & z \in C_{n}, \\ M_{n}\left(x+\mathrm{i} n^{-1} \varepsilon_{n}\right) & x \in\left\{x_{l}, x_{r}\right\} \quad \text { and } \quad v \in\left[0, n^{-1} \varepsilon_{n}\right] \\ M_{n}\left(x-\mathrm{i} n^{-1} \varepsilon_{n}\right) & x \in\left\{x_{l}, x_{r}\right\} \quad \text { and } \quad v \in\left[-n^{-1} \varepsilon_{n}, 0\right]\end{cases}
$$

where $C_{n}=\left\{x \pm \mathrm{i} v_{0}: x \in\left[x_{l}, x_{r}\right]\right\} \cup\left\{x \pm \mathrm{i} v: x \in\left\{x_{l}, x_{r}\right\}, v \in\left[n^{-1} \varepsilon_{n}, v_{0}\right]\right\}$, a regularized version of $C$ excluding a small segment near the real line, and the positive sequence ( $\varepsilon_{n}$ ) decreases to zero satisfying $\varepsilon_{n}>n^{-a}$ for some $a \in(0,1)$. From this and (3.19), one may thus find

$$
\oint_{C} f_{\ell}(z) M_{n}(z) d z=\oint_{C} f_{\ell}(z) \widehat{M}_{n}(z) d z+o_{p}(1)
$$

for every $\ell \in\{1, \ldots, k\}$. Hence, the proof of Theorem 2.2 can be completed by verifying the convergence of $\widehat{M}_{n}(z)$ on $C$ as stated in the following lemma.

Lemma 3.1 In addition to Assumptions (a)-(c), suppose that the conditions in (3.17) hold with $\gamma=5$. We have

$$
\widehat{M}_{n}(z) \stackrel{d}{=} M_{0}(z)+o_{p}(1), \quad z \in C,
$$

where the random process $M_{0}(z)$ is a two-dimensional Gaussian process. The mean function is

$$
\begin{equation*}
\mathbb{E} M_{0}(z)=\mu_{1}(z)+(\tau-3) \mu_{2}(z) \tag{3.20}
\end{equation*}
$$

and the covariance function is

$$
\operatorname{Cov}\left(M_{0}(z), M_{0}(\tilde{z})\right)=\sigma_{1}(z, \tilde{z})+(\tau-3) \sigma_{2}(z, \tilde{z})
$$

### 3.2.2 Proof of Lemma 3.1

Some quantities are listed below which will be used frequently throughout this proof.

$$
\begin{aligned}
& \mathbf{s}_{j}=\mathbf{s}\left(\mathbf{x}_{j}\right), \quad \mathbf{r}_{j}=\sqrt{p / n} \mathbf{s}_{j}, \quad \boldsymbol{\Sigma}=n \mathbb{E} \mathbf{r}_{1} \mathbf{r}_{1}^{\prime}, \\
& \mathbf{D}(z)=\mathbf{B}_{n}-z I, \quad \mathbf{D}_{j}(z)=\mathbf{D}(z)-\mathbf{r}_{j} \mathbf{r}_{j}^{\prime}, \quad \mathbf{D}_{i j}(z)=\mathbf{D}(z)-\mathbf{r}_{i} \mathbf{r}_{i}^{\prime}-\mathbf{r}_{j} \mathbf{r}_{j}^{\prime},(i \neq j), \\
& \varepsilon_{j}(z)=\mathbf{r}_{j}^{\prime} \mathbf{D}_{j}^{-1}(z) \mathbf{r}_{j}-\frac{1}{n} \operatorname{tr} \boldsymbol{\Sigma \mathbf { D } _ { j } ^ { - 1 } ( z ) , \quad \gamma _ { j } ( z ) = \mathbf { r } _ { j } ^ { \prime } \mathbf { D } _ { j } ^ { - 1 } ( z ) \mathbf { r } _ { j } - \frac { 1 } { n } \mathbb { E } \operatorname { t r } \boldsymbol { \Sigma } \mathbf { D } _ { j } ^ { - 1 } ( z ) ,} \\
& \delta_{j}(z)=\mathbf{r}_{j}^{\prime} \mathbf{D}_{j}^{-2}(z) \mathbf{r}_{j}-\frac{1}{n} \operatorname{tr} \boldsymbol{\Sigma \mathbf { D } _ { j } ^ { - 2 } ( z ) ,} \\
& \beta_{j}(z)=\frac{1}{1+\mathbf{r}_{j}^{\prime} \mathbf{D}_{j}^{-1}(z) \mathbf{r}_{j}}, \quad \beta_{j k}(z)=\frac{1}{1+\mathbf{r}_{j}^{\prime} \mathbf{D}_{k j}^{-1}(z) \mathbf{r}_{j}}, \\
& \bar{\beta}_{j}(z)=\frac{1}{1+n^{-1} \operatorname{tr} \boldsymbol{\Sigma} \mathbf{D}_{j}^{-1}(z)}, \quad \bar{\beta}_{j k}(z)=\frac{1}{1+n^{-1} \operatorname{tr} \mathbf{\Sigma} \mathbf{D}_{k j}^{-1}(z)}, \\
& b_{n}(z)=\frac{1}{1+n^{-1} \mathbb{E} \operatorname{tr} \boldsymbol{\Sigma} \mathbf{D}_{j}^{-1}(z)}, \quad \bar{b}_{n}(z)=\frac{1}{1+n^{-1} \mathbb{E} \operatorname{tr} \boldsymbol{\Sigma} \mathbf{D}_{k j}^{-1}(z)} .
\end{aligned}
$$

Note that by Lemma A.3, the matrix $\boldsymbol{\Sigma}$ and $\mathbf{T}$ defined in (3.17) are asymptotically equivalent and the last six quantities are bounded in absolute value by $|z| / v$ for any $z=u+i v \in \mathbb{C}^{+}$. Now we split $\widehat{M}_{n}(z)$ into two parts as

$$
\begin{aligned}
\widehat{M}_{n}(z) & =p\left[m_{n}(z)-\mathbb{E} m_{n}(z)\right]+p\left[\mathbb{E} m_{n}(z)-m_{0}(z)\right] \\
& :=M_{n}^{(1)}(z)+M_{n}^{(2)}(z) .
\end{aligned}
$$

Hence, the convergence of $\widehat{M}_{n}(z)$ can be obtained through the following three steps.
Step 1: Finite dimensional convergence of $M_{n}^{(1)}(z)$. Let $z_{1}, \ldots, z_{q}$ be any $q$ complex numbers on $C_{n}$, this step approximates joint distribution of

$$
\begin{equation*}
\left[M_{n}^{(1)}\left(z_{1}\right), \ldots, M_{n}^{(1)}\left(z_{q}\right)\right] \tag{3.21}
\end{equation*}
$$

through martingale CLT (Billingsley, 2008). Beyond the techniques used in Bai and Silverstein (2004), a particularly important problem is to find new approaches to deal with the non-linear correlation structure among the entries of $\mathbf{s}\left(\mathbf{x}_{j}\right)$. And such non-linear correlation is actually introduced by the spatial-sign transform of the data, to be precise, the norm $\left\|\mathbf{x}_{j}\right\|$ that appears in the denominator of $\mathbf{s}\left(\mathbf{x}_{j}\right)$. To this end, by giving an asymptotic expansion of $\mathbf{s}\left(\mathbf{x}_{j}\right)$, we develop Lemma A. 2 concerning the covariance of certain quadratic forms, which turns out to be one of the cornerstones for establishing our new CLT.

Step 2: Tightness of $M_{n}^{(1)}(z)$ on $C_{n}$. We illustrate in this step the basic idea for proving the tightness. As shown in (3.19), the probability of extreme eigenvalues falling outside the contour $C$ can be well controlled. By virtue of this and Lemma A.4, the tightness can be obtained following similar arguments in Bai and Silverstein (2004).

Step 3: Convergence of $M_{n}^{(2)}(z)$. In this final step, we approximate the quantity $M_{n}^{(2)}(z)$. In parallel with Step 1, dealing with the nonlinear effects as shown in Lemma A. 2 is the main focus in this part. As will be seen, such nonlinear effects will contribute several new terms to the mean of $\xi_{n}$.

Step 1: Finite dimensional convergence of $M_{n}^{(1)}(z)$ in distribution.

Let $\mathbb{E}_{0}(\cdot)$ denote expectation and $\mathbb{E}_{j}(\cdot)$ denote conditional expectation with respect to the $\sigma$-field generated by $\mathbf{r}_{1}, \ldots, \mathbf{r}_{j}, j=1, \ldots, n$. From the martingale decomposition and the identity

$$
\begin{equation*}
\mathbf{D}^{-1}(z)-\mathbf{D}_{j}^{-1}(z)=-\mathbf{D}_{j}^{-1}(z) \mathbf{r}_{j} \mathbf{r}_{j}^{\prime} \mathbf{D}_{j}^{-1}(z) \beta_{j}(z) \tag{3.22}
\end{equation*}
$$

we get

$$
\begin{align*}
M_{n}^{(1)}(z) & =\sum_{j=1}^{n}\left(\mathbb{E}_{j}-\mathbb{E}_{j-1}\right) \operatorname{tr}\left[\mathbf{D}^{-1}(z)-\mathbf{D}_{j}^{-1}(z)\right] \\
& =\sum_{j=1}^{n}\left(\mathbb{E}_{j}-\mathbb{E}_{j-1}\right) \frac{d \log \left(\beta_{j}(z) / \bar{\beta}_{j}(z)\right)}{d z}, \\
& =\frac{d}{d z} \sum_{j=1}^{n}\left(\mathbb{E}_{j}-\mathbb{E}_{j-1}\right) \log \left[1-\bar{\beta}_{j}(z) \varepsilon_{j}(z)+\bar{\beta}_{j}(z) \beta_{j}(z) \varepsilon_{j}^{2}(z)\right] \tag{3.23}
\end{align*}
$$

where the last equality is from the identity $\beta_{j}(z)=\bar{\beta}_{j}(z)-\bar{\beta}_{j}^{2}(z) \varepsilon_{j}(z)+\bar{\beta}_{j}^{2}(z) \beta_{j}(z) \varepsilon_{j}^{2}(z)$. From Lemma A. 4 and the boundedness of $\beta_{j}(z)$ and $\bar{\beta}_{j}(z)$, we have

$$
\mathbb{E}\left|\sum_{j=1}^{n}\left(\mathbb{E}_{j}-\mathbb{E}_{j-1}\right) \bar{\beta}_{j}(z) \beta_{j}(z) \varepsilon_{j}^{2}(z)\right|^{2} \leq K n \mathbb{E}\left|\varepsilon_{j}(z)\right|^{4} \rightarrow 0
$$

Thus applying Taylor's expansion to the log function in (3.23), one may conclude

$$
\begin{aligned}
M_{n}^{(1)}(z) & =-\frac{d}{d z} \sum_{j=1}^{n}\left(\mathbb{E}_{j}-\mathbb{E}_{j-1}\right) \bar{\beta}_{j}(z) \varepsilon_{j}(z)+o_{p}(1) \\
& =-\frac{d}{d z} \sum_{j=1}^{n} \mathbb{E}_{j} \bar{\beta}_{j}(z) \varepsilon_{j}(z)+o_{p}(1)
\end{aligned}
$$

Therefore, we turn to consider the martingale difference sequence

$$
Y_{n j}(z):=\frac{d}{d z} \mathbb{E}_{j} \bar{\beta}_{j}(z) \varepsilon_{j}(z), j=1, \ldots, n .
$$

The Lyapunov condition for this sequence is guaranteed by the fact that

$$
\begin{aligned}
\sum_{j=1}^{n} \mathbb{E}\left|Y_{n j}(z)\right|^{4} & =\sum_{j=1}^{n} \mathbb{E}\left|\mathbb{E}_{j}\left(\delta_{j}(z) \bar{\beta}_{j}(z)-\varepsilon_{j}(z) \bar{\beta}_{j}^{2}(z) \frac{1}{n} \operatorname{tr} \boldsymbol{\Sigma} \mathbf{D}_{j}^{-2}(z)\right)\right|^{4} \\
& \leq K \sum_{j=1}^{n}\left(\frac{|z|^{4} \mathbb{E}\left|\delta_{j}(z)\right|^{4}}{v^{4}}+\frac{|z|^{8} p^{4} \mathbb{E}\left|\varepsilon_{j}(z)\right|^{4}}{v^{16} n^{4}}\right) \rightarrow 0
\end{aligned}
$$

where the convergence is from Lemma A.4.
We next consider the sum $\sigma_{n}(z, \tilde{z}) \triangleq \sum_{j=1}^{n} \mathbb{E}_{j-1}\left[Y_{n j}(z) Y_{n j}(\tilde{z})\right]$, for $z \neq \tilde{z} \in\left\{z_{1}, \ldots, z_{w}\right\}$. Notice that

$$
\begin{equation*}
\mathbb{E}\left|\bar{\beta}_{j}(z)-b_{n}(z)\right| \leq \frac{K}{n} \mathbb{E}\left|\operatorname{tr} \boldsymbol{\Sigma} \mathbf{D}_{j}^{-1}(z)-\mathbb{E} \operatorname{tr} \boldsymbol{\Sigma} \mathbf{D}_{j}^{-1}\right| \rightarrow 0 \quad \text { and } \quad b_{n}(z)+z \underline{m}_{0}(z) \rightarrow 0 \tag{3.24}
\end{equation*}
$$

which follow from Bai and Silverstein (2004) and Lemma A.3, and thus we have

$$
\sigma_{n}(z, \tilde{z})=\frac{\partial^{2}}{\partial z \partial \tilde{z}} z \tilde{z} \underline{m}_{0}(z) \underline{m}_{0}(\tilde{z}) \sum_{j=1}^{n} \mathbb{E}_{j-1}\left(\mathbb{E}_{j} \varepsilon_{j}(z) \mathbb{E}_{j} \varepsilon_{j}(\tilde{z})\right)+o_{p}(1)
$$

Moreover, applying Lemma A. 2 to the above conditional expectations, one may get

$$
\begin{aligned}
& z \tilde{z} \underline{m}_{0}(z) \underline{m}_{0}(\tilde{z}) \sum_{j=1}^{n} \mathbb{E}_{j-1}\left(\mathbb{E}_{j} \varepsilon_{j}(z) \mathbb{E}_{j} \varepsilon_{j}(\tilde{z})\right) \\
= & 2 T_{1}+\frac{2}{p} \operatorname{tr}\left(\mathbf{T}^{2}\right) T_{2}-2 T_{3}-2 T_{4}+(\tau-3)\left(T_{5}+T_{6}-T_{7}-T_{8}\right)+o(1)
\end{aligned}
$$

where

$$
\begin{aligned}
& T_{1}=\frac{z \tilde{z} \underline{m}_{0}(z) \underline{m}_{0}(\tilde{z})}{n^{2}} \sum_{j=1}^{n} \operatorname{tr}\left[\mathbb{E}_{j} \mathbf{T D}_{j}^{-1}(z) \mathbb{E}_{j} \mathbf{T D}_{j}^{-1}(\tilde{z})\right], \\
& T_{2}=\frac{z \tilde{z} \underline{m}_{0}(z) \underline{m}_{0}(\tilde{z})}{p n^{2}} \sum_{j=1}^{n} \operatorname{tr}\left[\mathbb{E}_{j} \mathbf{T} \mathbf{D}_{j}^{-1}(z)\right] \operatorname{tr}\left[\mathbb{E}_{j} \mathbf{T D} \mathbf{D}_{j}^{-1}(\tilde{z})\right], \\
& T_{3}=\frac{z \tilde{z} \underline{m}_{0}(z) \underline{m}_{0}(\tilde{z})}{p n^{2}} \sum_{j=1}^{n} \operatorname{tr}\left[\mathbb{E}_{j} \mathbf{T}^{2} \mathbf{D}_{j}^{-1}(z)\right] \operatorname{tr}\left[\mathbb{E}_{j} \mathbf{T} \mathbf{D}_{j}^{-1}(\tilde{z})\right], \\
& T_{4}=\frac{z \tilde{z} \underline{m}_{0}(z) \underline{m}_{0}(\tilde{z})}{p n^{2}} \sum_{j=1}^{n} \operatorname{tr}\left[\mathbb{E}_{j} \mathbf{T} \mathbf{D}_{j}^{-1}(z)\right] \operatorname{tr}\left[\mathbb{E}_{j} \mathbf{T}^{2} \mathbf{D}_{j}^{-1}(\tilde{z})\right], \\
& T_{5}=\frac{z \tilde{z} \underline{m}_{0}(z) \underline{m}_{0}(\tilde{z})}{n^{2}} \sum_{j=1}^{n} \operatorname{tr}\left[\mathbb{E}_{j}\left(\mathbf{T}^{\frac{1}{2}} \mathbf{D}_{j}^{-1}(z) \mathbf{T}^{\frac{1}{2}}\right) \circ \mathbb{E}_{j}\left(\mathbf{T}^{\frac{1}{2}} \mathbf{D}_{j}^{-1}(\tilde{z}) \mathbf{T}^{\frac{1}{2}}\right)\right], \\
& T_{6}=\frac{z \tilde{z} \underline{m}_{0}(z) \underline{m}_{0}(\tilde{z})}{p^{2} n^{2}} \sum_{j=1}^{n} \operatorname{tr}\left[\mathbb{E}_{j} \mathbf{T D}_{j}^{-1}(z)\right] \operatorname{tr}\left[\mathbb{E}_{j} \mathbf{T D}_{j}^{-1}(\tilde{z})\right] \operatorname{tr}[\mathbf{T} \circ \mathbf{T}], \\
& T_{7}=\frac{z \tilde{z} \underline{m}_{0}(z) \underline{m}_{0}(\tilde{z})}{p n^{2}} \sum_{j=1}^{n} \operatorname{tr}\left[\mathbb{E}_{j} \mathbf{T D} \mathbf{D}_{j}^{-1}(z)\right] \operatorname{tr}\left[\mathbb{E}_{j}\left(\mathbf{T}^{\frac{1}{2}} \mathbf{D}_{j}^{-1}(\tilde{z}) \mathbf{T}^{\frac{1}{2}}\right) \circ \mathbf{T}\right], \\
& T_{8}=\frac{z \tilde{z} \underline{m}_{0}(z) \underline{m}_{0}(\tilde{z})}{p n^{2}} \sum_{j=1}^{n} \operatorname{tr}\left[\mathbb{E}_{j}\left(\mathbf{T}^{\frac{1}{2}} \mathbf{D}_{j}^{-1}(z) \mathbf{T}^{\frac{1}{2}}\right) \circ \mathbf{T}\right] \operatorname{tr}\left[\mathbb{E}_{j} \mathbf{T} \mathbf{D}_{j}^{-1}(\tilde{z})\right] .
\end{aligned}
$$

Following similar steps as in Bai and Silverstein (2004) and Hu et al. (2019a), applying Lemma A. 3 and Lemma A.4, we obtain

$$
\begin{aligned}
T_{1} & =\log \frac{\underline{m}_{0}(z)-\underline{m}_{0}(\tilde{z})}{\underline{m}_{0}(z) \underline{m}_{0}(\tilde{z})(z-\tilde{z})}+o_{p}(1), \\
T_{2}=\frac{T_{6}}{\zeta_{p}} & =c_{n} \int \frac{t \underline{m}_{0}(z) d \tilde{H}_{p}(t)}{1+t \underline{m}_{0}(z)} \int \frac{t \underline{m}_{0}(\tilde{z}) d \tilde{H}_{p}(t)}{1+t \underline{m}_{0}(\tilde{z})}+o_{p}(1) \\
& =\frac{\left[1+z \underline{m}_{0}(z)\right]\left[1+\tilde{z} \underline{m}_{0}(\tilde{z})\right]}{c_{n}}+o_{p}(1) .
\end{aligned}
$$

Notice that statistics $T_{3}$ and $T_{4}$ will reduce to $T_{2}$ if $\mathbf{T}^{2}$ is replaced with $\mathbf{T}$. By this, we have

$$
\begin{aligned}
T_{3} & =c_{n} \int \frac{t^{2} \underline{m}_{0}(z) d \tilde{H}_{p}(t)}{1+t \underline{m}_{0}(z)} \int \frac{t \underline{m}_{0}(\tilde{z}) d \tilde{H}_{p}(t)}{1+t \underline{m}_{0}(\tilde{z})}+o_{p}(1) \\
& =\left[1-\frac{1+z \underline{m}_{0}(z)}{c_{n} \underline{m}_{0}(z)}\right]\left[1+\tilde{z} \underline{m}_{0}(\tilde{z})\right]+o_{p}(1)
\end{aligned}
$$

$$
\begin{aligned}
T_{4} & =c_{n} \int \frac{t \underline{m}_{0}(z) d \tilde{H}_{p}(t)}{1+t \underline{m}_{0}(z)} \int \frac{t^{2} \underline{m}_{0}(\tilde{z}) d \tilde{H}_{p}(t)}{1+t \underline{m}_{0}(\tilde{z})}+o_{p}(1) \\
& =\left[1-\frac{1+\tilde{z} \underline{\underline{m}}_{0}(\tilde{z})}{c_{n} \underline{m}_{0}(\tilde{z})}\right]\left[1+z \underline{\underline{m}}_{0}(z)\right]+o_{p}(1)
\end{aligned}
$$

For the terms $T_{5}, T_{7}$, and $T_{8}$, following similar procedure as in Pan et al. (2008) for proving their Theorem 1.4, using Lemma A.3, Lemma A.4, and Theorem 2.1, one may get

$$
\begin{aligned}
T_{5} & =\frac{1}{n} \operatorname{tr}\left[\left(\mathbf{T}^{\frac{1}{2}}\left(\underline{m}_{0}^{-1}(z) \mathbf{I}+\mathbf{T}\right)^{-1} \mathbf{T}^{\frac{1}{2}}\right) \circ\left(\mathbf{T}^{\frac{1}{2}}\left(\underline{m}_{0}^{-1}(\tilde{z}) \mathbf{I}+\mathbf{T}\right)^{-1} \mathbf{T}^{\frac{1}{2}}\right)\right]+o_{p}(1) \\
& =c_{n} g_{p}\left(\frac{-1}{\underline{m}_{0}(z)}, \frac{-1}{m_{0}(\tilde{z})}\right)+o_{p}(1), \\
T_{7} & =\frac{1}{p n} \operatorname{tr}\left[\mathbf{T}\left(\underline{m}_{0}^{-1}(z) \mathbf{I}+\mathbf{T}\right)^{-1}\right] \operatorname{tr}\left[\left(\mathbf{T}^{\frac{1}{2}}\left(\underline{m}_{0}^{-1}(\tilde{z}) \mathbf{I}+\mathbf{T}\right)^{-1} \mathbf{T}^{\frac{1}{2}}\right) \circ \mathbf{T}\right]+o_{p}(1) \\
& =h_{p}\left(\frac{-1}{\underline{m}_{0}(\tilde{z})}\right)\left[1+z \underline{m}_{0}(z)\right]+o_{p}(1), \\
T_{8} & =\frac{1}{p n} \operatorname{tr}\left[\left(\mathbf{T}^{\frac{1}{2}}\left(\underline{m}_{0}^{-1}(z) \mathbf{I}+\mathbf{T}\right)^{-1} \mathbf{T}^{\frac{1}{2}}\right) \circ \mathbf{T}\right] \operatorname{tr}\left[\mathbf{T}\left(\underline{m}_{0}^{-1}(\tilde{z}) \mathbf{I}+\mathbf{T}\right)^{-1}\right]+o_{p}(1) \\
& =h_{p}\left(\frac{-1}{\underline{m}_{0}(z)}\right)\left[1+\tilde{z} \underline{m}_{0}(\tilde{z})\right]+o_{p}(1) .
\end{aligned}
$$

Collecting the above results we get

$$
(3.21) \stackrel{d}{=}\left[M_{0}^{(1)}\left(z_{1}\right), \ldots, M_{0}^{(1)}\left(z_{q}\right)\right]+o_{p}(1),
$$

where $\left[M_{0}^{(1)}\left(z_{1}\right), \ldots, M_{0}^{(1)}\left(z_{q}\right)\right]$ is a $q$-dimensional zero-mean Gaussian random vector with covariance function

$$
\operatorname{Cov}\left[M_{0}^{(1)}(z), M_{0}^{(1)}(\tilde{z})\right]=\sigma_{1}(z, \tilde{z})+(\tau-3) \sigma_{2}(z, \tilde{z})
$$

Step 2: Tightness of $M_{n}^{(1)}(z)$. The tightness can be established by verifying the moment condition (12.51) of Billingsley (1968):

$$
\begin{equation*}
\sup _{n, z_{1}, z_{2} \in C_{n}} \frac{\mathbb{E}\left|M_{n}^{(1)}\left(z_{1}\right)-M_{n}^{(1)}\left(z_{2}\right)\right|^{2}}{\left|z_{1}-z_{2}\right|^{2}}<\infty . \tag{3.25}
\end{equation*}
$$

By (3.19) and arguments in Bai and Silverstein (2004), the moments of $\mathbf{D}^{-1}(z), \mathbf{D}_{j}^{-1}(z)$ and $\mathbf{D}_{i j}^{-1}(z)$ are uniformly bounded in $n$ and $z \in C_{n}$, that is, for any positive $k$,

$$
\begin{equation*}
\max \left\{\mathbb{E}\left\|\mathbf{D}^{-1}(z)\right\|^{k}, \mathbb{E}\left\|\mathbf{D}_{j}^{-1}(z)\right\|^{k}, \mathbb{E}\left\|\mathbf{D}_{i j}^{-1}(z)\right\|^{k}\right\} \leq K \tag{3.26}
\end{equation*}
$$

By such boundedness, the inequality in Lemma A. 4 can be extended to

$$
\begin{equation*}
\left|\mathbb{E}\left[a(v) \prod_{l=1}^{k}\left(\mathbf{r}^{\prime} \mathbf{B}_{l}(v) \mathbf{r}-\frac{1}{n} \operatorname{tr} \boldsymbol{\Sigma} \mathbf{B}_{l}(v)\right)\right]\right| \leq K n^{-1-k(\gamma-4) / \gamma}, \quad k \geq 2 . \tag{3.27}
\end{equation*}
$$

The matrices $\mathbf{B}_{l}(v)$ in (3.27) are independent of $\mathbf{r}$ and

$$
\max \left\{|a(v)|,\left\|\mathbf{B}_{l}(v)\right\|\right\} \leq K\left[1+p^{s} I\left(\left\|\mathbf{B}_{n}\right\| \geq x_{r} \text { or } \lambda_{\min }^{\tilde{\mathbf{B}}} \leq x_{l}\right)\right]
$$

for some positive $s$, where $\tilde{\mathbf{B}}$ denotes $\mathbf{B}_{n}=\sum \mathbf{r}_{j} \mathbf{r}_{j}^{\prime}, \mathbf{B}_{j}=\sum_{k \neq j} \mathbf{r}_{k} \mathbf{r}_{k}^{\prime}$, or $\mathbf{B}_{i j}=\sum_{k \neq i, j} \mathbf{r}_{k} \mathbf{r}_{k}^{\prime}$. Finally, following similar procedure as in Section 3 of Bai and Silverstein (2004), and applying Lemma A.3, Lemma A. 4 together with (3.19), (3.26), and (3.27), one may verify (3.25). The details are thus omitted.

Step 3: Convergence of $M_{n}^{(2)}(z)$. To finish the proof, it is enough to show that the sequence of $M_{n}^{(2)}(z)$ is bounded and equicontinuous, and is equal to the mean function (3.20) asymptotically. The boundedness and equicontinuity can be verified following the arguments in Bai and Silverstein (2004). We next propose a novel method to approximate $M_{n}^{(2)}(z)$, which is quite different from the idea in Bai and Silverstein (2004). This new procedure is more straightforward and easier to follow. Before the proof, we first list some results that will be used in this part:

$$
\begin{align*}
& \sup _{z \in C_{n}} \mathbb{E}\left|\varepsilon_{j}(z)\right|^{k} \leq K n^{-1-k(\gamma-4) / \gamma}, \quad \sup _{z \in C_{n}} \mathbb{E}\left|\gamma_{j}(z)\right|^{k} \leq K n^{-1-k(\gamma-4) / \gamma},  \tag{3.28}\\
& \sup _{n, z \in \mathcal{C}_{n}}\left|b_{n}(z)+z \underline{m}_{0}(z)\right| \rightarrow 0, \quad \sup _{n, z \in C_{n}}\left\|z \mathbf{I}-b_{n}(z) \boldsymbol{\Sigma}\right\|^{-1}<\infty, \\
& \sup _{n, z \in C_{n}} \mathbb{E}\left|\operatorname{tr} \mathbf{D}^{-1}(z) \mathbf{M}-\mathbb{E} \operatorname{tr} \mathbf{D}^{-1}(z) \mathbf{M}\right|^{2} \leq K\|\mathbf{M}\|^{2}, \tag{3.29}
\end{align*}
$$

where $k \geq 2$ and $\mathbf{M}$ is a nonrandom $p \times p$ matrix. These results can be verified step by step following the discussions in Bai and Silverstein (2004) and we omit the details.

Writing $\mathbf{V}(z)=z I-b_{n}(z) \boldsymbol{\Sigma}$, we decompose $M_{n}^{(2)}(z)$ in two ways:

$$
\begin{aligned}
& M_{n}^{(2)}(z)=\left[p \mathbb{E} m_{n}(z)+\operatorname{tr} \mathbf{V}^{-1}(z)\right]-\left[\operatorname{tr} \mathbf{V}^{-1}(z)+p m_{0}(z)\right]:=S_{n}(z)-T_{n}(z), \\
& M_{n}^{(2)}(z)=\left[n \mathbb{E} \underline{m}_{n}(z)+n b_{n}(z) / z\right]-\left[n b_{n}(z) / z+n \underline{m}_{0}(z)\right]:=\underline{S}_{n}(z)-\underline{T}_{n}(z) .
\end{aligned}
$$

Notice that by Lemma A.3,

$$
\begin{aligned}
T_{n}(z) & =p \int \frac{d \tilde{H}_{p}(t)}{z-b_{n}(z) t}-p \int \frac{d \tilde{H}_{p}(t)}{z+\underline{m}_{0}(z) t}+o(1) \\
& =p\left[b_{n}(z)+z \underline{m}_{0}(z)\right] \int \frac{t d \tilde{H}_{p}(t)}{\left(z-b_{n}(z) t\right)\left(z+z \underline{m}_{0}(z) t\right)}+o(1) \\
& =c_{n} \underline{T}_{n}(z) \int \frac{t d \tilde{H}_{p}(t)}{\left(z-b_{n}(z) t\right)\left(1+\underline{m}_{0}(z) t\right)}+o(1) .
\end{aligned}
$$

From this and the convergence in (3.29), we have

$$
\begin{equation*}
\frac{M_{n}^{(2)}(z)-S_{n}(z)}{M_{n}^{(2)}(z)-\underline{S}_{n}(z)}=\frac{T_{n}(z)}{\underline{T}_{n}(z)}=\frac{c_{n}}{z} \int \frac{t d \tilde{H}_{p}(t)}{\left(1+\underline{m}_{0}(z) t\right)^{2}}+o(1) . \tag{3.31}
\end{equation*}
$$

Our next task is to study the convergence of $S_{n}(z)$ and $\underline{S}_{n}(z)$. For simplicity of notation, we suppress the expression $z$ in the sequel when it is served as independent variables of some functions. All expressions and convergence statements hold uniformly for $z \in C_{n}$.

We first simplify the expression of $S_{n}$. Using the identity $\mathbf{r}_{j}^{\prime} \mathbf{D}^{-1}=\mathbf{r}_{j}^{\prime} \mathbf{D}_{j}^{-1} \beta_{j}$, we have

$$
\begin{align*}
S_{n} & =\mathbb{E} \operatorname{tr}\left(\mathbf{D}^{-1}+\mathbf{V}^{-1}\right)=\mathbb{E} \operatorname{tr}\left[\mathbf{V}^{-1}\left(\sum_{j=1}^{n} \mathbf{r}_{j} \mathbf{r}_{j}^{\prime}-b_{n} \boldsymbol{\Sigma}\right) \mathbf{D}^{-1}\right] \\
& =n \mathbb{E} \beta_{1} \mathbf{r}_{1}^{\prime} \mathbf{D}_{1}^{-1} \mathbf{V}^{-1} \mathbf{r}_{1}-b_{n} \mathbb{E} \operatorname{tr} \boldsymbol{\Sigma} \mathbf{D}^{-1} \mathbf{V}^{-1} . \tag{3.32}
\end{align*}
$$

From (3.22) and $\beta_{1}=b_{n}-b_{n} \beta_{1} \gamma_{1}$,

$$
\mathbb{E} \operatorname{tr} \mathbf{V}^{-1} \boldsymbol{\Sigma}\left(\mathbf{D}_{1}^{-1}-\mathbf{D}^{-1}\right)=\mathbb{E} \operatorname{tr} \mathbf{V}^{-1} \boldsymbol{\Sigma} \mathbf{D}_{1}^{-1} \mathbf{r}_{1} \mathbf{r}_{1}^{\prime} \mathbf{D}_{1}^{-1} \beta_{1}
$$

$$
=b_{n} \mathbb{E}\left(1-\beta_{1} \gamma_{1}\right) \mathbf{r}_{1}^{\prime} \mathbf{D}_{1}^{-1} \mathbf{V}^{-1} \boldsymbol{\Sigma} \mathbf{D}_{1}^{-1} \mathbf{r}_{1}
$$

where $\left|\mathbb{E} \beta_{1} \gamma_{1} \mathbf{r}_{1}^{\prime} \mathbf{D}_{1}^{-1} \mathbf{V}^{-1} \boldsymbol{\Sigma} \mathbf{D}_{1}^{-1} \mathbf{r}_{1}\right| \leq K n^{-1 / 2}$. From this and (3.32), we get

$$
S_{n}=n \mathbb{E} \beta_{1} \mathbf{r}_{1}^{\prime} \mathbf{D}_{1}^{-1} \mathbf{V}^{-1} \mathbf{r}_{1}-b_{n} \mathbb{E} \operatorname{tr} \boldsymbol{\Sigma} \mathbf{D}_{1}^{-1} \mathbf{V}^{-1}+\frac{1}{n} b_{n}^{2} \mathbb{E} \operatorname{tr} \mathbf{D}_{1}^{-1} \mathbf{V}^{-1} \boldsymbol{\Sigma} \mathbf{D}_{1}^{-1} \boldsymbol{\Sigma}+o(1)
$$

Then plugging $\beta_{1}=b_{n}-b_{n}^{2} \gamma_{1}+b_{n}^{3} \gamma_{1}^{2}-\beta_{1} b_{n}^{3} \gamma_{1}^{3}$ into the first term in the above equation, we obtain

$$
\begin{aligned}
n \mathbb{E} \beta_{1} \mathbf{r}_{1}^{\prime} \mathbf{D}_{1}^{-1} \mathbf{V}^{-1} \mathbf{r}_{1} & =b_{n} \mathbb{E} \operatorname{tr} \mathbf{D}_{1}^{-1} \mathbf{V}^{-1} \boldsymbol{\Sigma}-n b_{n}^{2} \mathbb{E} \gamma_{1} \mathbf{r}_{1}^{\prime} \mathbf{D}_{1}^{-1} \mathbf{V}^{-1} \mathbf{r}_{1} \\
& +n b_{n}^{3} \mathbb{E} \gamma_{1}^{2} \mathbf{r}_{1}^{\prime} \mathbf{D}_{1}^{-1} \mathbf{V}^{-1} \mathbf{r}_{1}-n b_{n}^{3} \mathbb{E} \beta_{1} \gamma_{1}^{3} \mathbf{r}_{1}^{\prime} \mathbf{D}_{1}^{-1} \mathbf{V}^{-1} \mathbf{r}_{1} .
\end{aligned}
$$

Note that, from (3.27), (3.28), and (3.30),

$$
\begin{aligned}
\mathbb{E} \gamma_{1} \mathbf{r}_{1}^{\prime} \mathbf{D}_{1}^{-1} \mathbf{V}^{-1} \mathbf{r}_{1}= & \mathbb{E}\left[\mathbf{r}_{1}^{\prime} \mathbf{D}_{1}^{-1} \mathbf{r}_{1}-\frac{1}{n} \operatorname{tr} \mathbf{D}_{1}^{-1} \boldsymbol{\Sigma}\right]\left[\mathbf{r}_{1}^{\prime} \mathbf{D}_{1}^{-1} \mathbf{V}^{-1} \mathbf{r}_{1}\right. \\
& \left.-\frac{1}{n} \operatorname{tr} \mathbf{D}_{1}^{-1} \mathbf{V}^{-1} \boldsymbol{\Sigma}\right]+\frac{1}{n^{2}} \operatorname{Cov}\left(\operatorname{tr} \mathbf{D}_{1}^{-1} \boldsymbol{\Sigma}, \operatorname{tr} \mathbf{D}_{1}^{-1} \mathbf{V}^{-1} \boldsymbol{\Sigma}\right) \\
& =\mathbb{E}\left[\mathbf{r}_{1}^{\prime} \mathbf{D}_{1}^{-1} \mathbf{r}_{1}-\frac{1}{n} \operatorname{tr} \mathbf{D}_{1}^{-1} \boldsymbol{\Sigma}\right]\left[\mathbf{r}_{1}^{\prime} \mathbf{D}_{1}^{-1} \mathbf{V}^{-1} \mathbf{r}_{1}\right. \\
& \left.-\frac{1}{n} \operatorname{tr} \mathbf{D}_{1}^{-1} \mathbf{V}^{-1} \boldsymbol{\Sigma}\right]+o\left(\frac{1}{n}\right), \\
\mathbb{E} \gamma_{1}^{2} \mathbf{r}_{1}^{\prime} \mathbf{D}_{1}^{-1} \mathbf{V}^{-1} \mathbf{r}_{1} & =\mathbb{E} \gamma_{1}^{2}\left[\mathbf{r}_{1}^{\prime} \mathbf{D}_{1}^{-1} \mathbf{V}^{-1} \mathbf{r}_{1}-\frac{1}{n} \operatorname{tr} \mathbf{D}_{1}^{-1} \mathbf{V}^{-1} \boldsymbol{\Sigma}\right] \\
& +\frac{1}{n} \operatorname{Cov}\left(\gamma_{1}^{2}, \operatorname{tr} \mathbf{D}_{1}^{-1} \mathbf{V}^{-1} \boldsymbol{\Sigma}\right)+\frac{1}{n} \mathbb{E} \gamma_{1}^{2} E \operatorname{tr} \mathbf{D}_{1}^{-1} \mathbf{V}^{-1} \boldsymbol{\Sigma} \\
& =\frac{1}{n} \mathbb{E} \gamma_{1}^{2} \mathbb{E} \operatorname{tr} \mathbf{D}_{1}^{-1} \mathbf{V}^{-1} \boldsymbol{\Sigma}+o\left(\frac{1}{n}\right), \\
\mathbb{E} \beta_{1} \gamma_{1}^{3} \mathbf{r}_{1}^{\prime} \mathbf{D}_{1}^{-1} \mathbf{V}^{-1} \mathbf{r}_{1} & =o\left(\frac{1}{n}\right) .
\end{aligned}
$$

We thus arrive at

$$
\begin{aligned}
S_{n} & =-n b_{n}^{2} \mathbb{E}\left[\mathbf{r}_{1}^{\prime} \mathbf{D}_{1}^{-1} \mathbf{r}_{1}-\frac{1}{n} \operatorname{tr} \mathbf{D}_{1}^{-1} \boldsymbol{\Sigma}\right]\left[\mathbf{r}_{1}^{\prime} \mathbf{D}_{1}^{-1} \mathbf{V}^{-1} \mathbf{r}_{1}-\frac{1}{n} \operatorname{tr} \mathbf{D}_{1}^{-1} \mathbf{V}^{-1} \boldsymbol{\Sigma}\right] \\
& +b_{n}^{3} \mathbb{E} \gamma_{1}^{2} \mathbb{E} \operatorname{tr} \mathbf{D}_{1}^{-1} \mathbf{V}^{-1} \boldsymbol{\Sigma}+\frac{1}{n} b_{n}^{2} \mathbb{E} \operatorname{tr} \mathbf{D}_{1}^{-1} \mathbf{V}^{-1} \boldsymbol{\Sigma} \mathbf{D}_{1}^{-1} \boldsymbol{\Sigma}+o(1) .
\end{aligned}
$$

On the other hand, by the identity $\mathbf{r}_{j}^{\prime} \mathbf{D}^{-1}=\mathbf{r}_{j}^{\prime} \mathbf{D}_{j}^{-1} \beta_{j}$, we have

$$
p+z \operatorname{tr} \mathbf{D}^{-1}=\operatorname{tr}\left(\mathbf{B}_{n} \mathbf{D}^{-1}\right)=\sum_{j=1}^{n} \beta_{j} \mathbf{r}_{j}^{\prime} \mathbf{D}_{j}^{-1} \mathbf{r}_{j}=n-\sum_{j=1}^{n} \beta_{j}
$$

which implies $n z \underline{m}_{n}=-\sum_{j=1}^{n} \beta_{j}$. From this, together with $\beta_{1}=b_{n}-b_{n}^{2} \gamma_{1}+b_{n}^{3} \gamma_{1}^{2}-\beta_{1} b_{n}^{3} \gamma_{1}^{3}$ and (3.27), we get

$$
\underline{S}_{n}=-\frac{n}{z} \mathbb{E}\left(\beta_{1}-b_{n}\right)=-\frac{n}{z} b_{n}^{3} \mathbb{E} \gamma_{1}^{2}+o(1)
$$

Applying Lemma A. 2 to the simplified $S_{n}$ and $\underline{S}_{n}$, and then replacing $\mathbf{D}_{j}$ with $\mathbf{D}$ in the derived results yield

$$
\begin{aligned}
S_{n}= & -\frac{b_{n}^{2}}{n}\left[\mathbb{E} \operatorname{tr} \mathbf{D}^{-1} \mathbf{T} \mathbf{D}^{-1} \mathbf{V}^{-1} \mathbf{T}+\frac{2}{p}\left(\frac{1}{n} \operatorname{tr} \mathbf{T}^{2} \mathbb{E} \operatorname{tr} \mathbf{T} \mathbf{D}^{-1} \operatorname{tr} \mathbf{T} \mathbf{D}^{-1} \mathbf{V}^{-1}\right.\right. \\
& \left.\left.-\mathbb{E} \operatorname{tr} \mathbf{T}^{2} \mathbf{D}^{-1} \operatorname{tr} \mathbf{T} \mathbf{D}^{-1} \mathbf{V}^{-1}-\mathbb{E} \operatorname{tr} \mathbf{T} \mathbf{D}^{-1} \operatorname{tr} \mathbf{T}^{2} \mathbf{D}^{-1} \mathbf{V}^{-1}\right)\right] \\
+ & \frac{2 b_{n}^{3}}{n^{2}}\left[\mathbb{E} \operatorname{tr} \mathbf{D}^{-1} \mathbf{T} \mathbf{D}^{-1} \mathbf{T}+\frac{1}{p}\left(\frac{1}{n} \operatorname{tr} \mathbf{T}^{2} \mathbb{E} \operatorname{tr} \mathbf{T D ^ { - 1 } \operatorname { t r } \mathbf { T D }}{ }^{-1}-2 \mathbb{E} \operatorname{tr} \mathbf{T}^{2} \mathbf{D}^{-1} \operatorname{tr} \mathbf{T D ^ { - 1 }}\right)\right] \\
& \cdot \mathbb{E} \operatorname{tr} \mathbf{D}^{-1} \mathbf{V}^{-1} \mathbf{T}-\frac{(\tau-3) b_{n}^{2}}{n}\left[\mathbb{E} \operatorname{tr}\left[\left(\mathbf{T}^{\frac{1}{2}} \mathbf{D}^{-1} \mathbf{T}^{\frac{1}{2}}\right) \circ\left(\mathbf{T}^{\frac{1}{2}} \mathbf{D}^{-1} \mathbf{V}^{-1} \mathbf{T}^{\frac{1}{2}}\right)\right]\right. \\
& +\frac{1}{p^{2}} \mathbb{E} \operatorname{tr}\left(\mathbf{D}^{-1} \mathbf{T}\right) \operatorname{tr}\left(\mathbf{D}^{-1} \mathbf{V}^{-1} \mathbf{T}\right) \operatorname{tr}[\mathbf{T} \circ \mathbf{T}] \\
& \left.-\frac{1}{p} \mathbb{E} \operatorname{tr}\left(\mathbf{D}^{-1} \mathbf{T}\right) \operatorname{tr}\left[\left(\mathbf{T}^{\frac{1}{2}} \mathbf{D}^{-1} \mathbf{V}^{-1} \mathbf{T}^{\frac{1}{2}}\right) \circ \mathbf{T}\right]-\frac{1}{p} \mathbb{E} \operatorname{tr}\left(\mathbf{D}^{-1} \mathbf{V}^{-1} \mathbf{T}\right) \operatorname{tr}\left[\left(\mathbf{T}^{\frac{1}{2}} \mathbf{D}^{-1} \mathbf{T}^{\frac{1}{2}}\right) \circ \mathbf{T}\right]\right] \\
& +\frac{(\tau-3) b_{n}^{3}}{n^{2}}\left[\mathbb{E} \operatorname{tr}\left[\left(\mathbf{T}^{\frac{1}{2}} \mathbf{D}^{-1} \mathbf{T}^{\frac{1}{2}}\right) \circ\left(\mathbf{T}^{\frac{1}{2}} \mathbf{D}^{-1} \mathbf{T}^{\frac{1}{2}}\right)\right]+\frac{1}{p^{2}} \mathbb{E} \operatorname{tr}^{2}\left(\mathbf{D}^{-1} \mathbf{T}\right) \operatorname{tr}[\mathbf{T} \circ \mathbf{T}]\right. \\
& \left.-\frac{2}{p} \mathbb{E} \operatorname{tr}\left(\mathbf{D}^{-1} \mathbf{T}\right) \operatorname{tr}\left[\left(\mathbf{T}^{\frac{1}{2}} \mathbf{D}^{-1} \mathbf{T}^{\frac{1}{2}}\right) \circ \mathbf{T}\right]\right] \mathbb{E} \mathbf{D}^{-1} \mathbf{V}^{-1} \mathbf{T}+o(1), \\
S_{n}= & \frac{-2 b_{n}^{3}}{z n}\left[\mathbb{E} \operatorname{tr} \mathbf{D}^{-1} \mathbf{T} \mathbf{D}^{-1} \mathbf{T}+\frac{1}{p}\left(\frac{1}{p} \operatorname{tr} \mathbf{T}^{2} \mathbb{E} \operatorname{tr} \mathbf{T D} \mathbf{D}^{-1} \operatorname{tr} \mathbf{T D} \mathbf{D}^{-1}-2 \mathbb{E} \operatorname{tr} \mathbf{T}^{2} \mathbf{D}^{-1} \operatorname{tr} \mathbf{T} \mathbf{D}^{-1}\right)\right] \\
& -\frac{(\tau-3) b_{n}^{3}}{z n}\left[\mathbb{E} \operatorname{tr}\left[\left(\mathbf{T}^{\frac{1}{2}} \mathbf{D}^{-1} \mathbf{T}^{\frac{1}{2}}\right) \circ\left(\mathbf{T}^{\frac{1}{2}} \mathbf{D}^{-1} \mathbf{T}^{\frac{1}{2}}\right)\right]+\frac{1}{p^{2}} \mathbb{E} \operatorname{tr}^{2}\left(\mathbf{D}^{-1} \mathbf{T}\right) \operatorname{tr}[\mathbf{T} \circ \mathbf{T}]\right. \\
& \left.-\frac{2}{p} \mathbb{E} \operatorname{tr}\left(\mathbf{D}^{-1} \mathbf{T}\right) \operatorname{tr}\left[\left(\mathbf{T}^{\frac{1}{2}} \mathbf{D}^{-1} \mathbf{T}^{\frac{1}{2}}\right) \circ \mathbf{T}\right]\right]+o(1) .
\end{aligned}
$$

To study the convergence of $S_{n}$ and $\underline{S}_{n}$, we need to figure out the difference between $\mathbf{D}^{-1}$ and $\mathbf{V}^{-1}$. Write

$$
\begin{equation*}
\mathbf{D}^{-1}+\mathbf{V}^{-1}=b_{n} \tilde{\mathbf{R}}_{1}+\tilde{\mathbf{R}}_{2}+\tilde{\mathbf{R}}_{3} \tag{3.33}
\end{equation*}
$$

where

$$
\begin{gathered}
\tilde{\mathbf{R}}_{1}=\sum_{j=1}^{n} \mathbf{V}^{-1}\left(\mathbf{r}_{j} \mathbf{r}_{j}^{\prime}-n^{-1} \boldsymbol{\Sigma}\right) \mathbf{D}_{j}^{-1}, \quad \tilde{\mathbf{R}}_{2}=\sum_{j=1}^{n} \mathbf{V}^{-1} \mathbf{r}_{j} \mathbf{r}_{j}^{\prime} \mathbf{D}_{j}^{-1}\left(\beta_{j}-b_{n}\right) \\
\tilde{\mathbf{R}}_{3}=\frac{1}{n} \sum_{j=1}^{n} b_{n} \mathbf{V}^{-1} \boldsymbol{\Sigma}\left(\mathbf{D}_{j}^{-1}-\mathbf{D}^{-1}\right)
\end{gathered}
$$

From Bai and Silverstein (2004) we have, for any $p \times p$ matrix M,

$$
\begin{equation*}
\left|\mathbb{E} \operatorname{tr} \tilde{\mathbf{R}}_{2} \mathbf{M}\right| \leq n^{1 / 2} K\left(\mathbb{E}\|\mathbf{M}\|^{4}\right)^{1 / 4} \text { and }\left|\operatorname{tr} \tilde{\mathbf{R}}_{3} \mathbf{M}\right| \leq K\left(\mathbb{E}\|\mathbf{M}\|^{2}\right)^{1 / 2} \tag{3.34}
\end{equation*}
$$

and, for nonrandom matrix $\mathbf{M}$,

$$
\begin{equation*}
\left|\mathbb{E} \operatorname{tr} \tilde{\mathbf{R}}_{1} \mathbf{M}\right| \leq n^{1 / 2} K\|\mathbf{M}\| . \tag{3.35}
\end{equation*}
$$

Taking a step further, for $\mathbf{M}$ nonrandom, we write

$$
\begin{equation*}
\operatorname{tr} \tilde{\mathbf{R}}_{1} \boldsymbol{\Sigma} \mathbf{D}^{-1} \mathbf{M}=\tilde{R}_{11}+\tilde{R}_{12}+\tilde{R}_{13} \tag{3.36}
\end{equation*}
$$

where

$$
\begin{aligned}
& \tilde{R}_{11}=\operatorname{tr} \sum_{j=1}^{n} \mathbf{V}^{-1} \mathbf{r}_{j} \mathbf{r}_{j}^{\prime} \mathbf{D}_{j}^{-1} \boldsymbol{\Sigma}\left(\mathbf{D}^{-1}-\mathbf{D}_{j}^{-1}\right) \mathbf{M}, \\
& \tilde{R}_{12}=\operatorname{tr} \sum_{j=1}^{n} \mathbf{V}^{-1}\left(\mathbf{r}_{j} \mathbf{r}_{j}^{\prime}-n^{-1} \boldsymbol{\Sigma}\right) \mathbf{D}_{j}^{-1} \boldsymbol{\Sigma} \mathbf{D}_{j}^{-1} \mathbf{M}, \\
& \tilde{R}_{13}=-\frac{1}{n} \operatorname{tr} \sum_{j=1}^{n} \mathbf{V}^{-1} \boldsymbol{\Sigma} \mathbf{D}_{j}^{-1} \boldsymbol{\Sigma}\left(\mathbf{D}^{-1}-\mathbf{D}_{j}^{-1}\right) \mathbf{M} .
\end{aligned}
$$

It's clear that $\mathbb{E} \tilde{R}_{12}=0$ and moreover, using (3.26), (3.27) and (3.30), we get

$$
\begin{align*}
\left|\mathbb{E} \tilde{R}_{13}\right| & \leq K\|\mathbf{M}\|,  \tag{3.37}\\
\mathbb{E} \tilde{R}_{11} & =-n \mathbb{E} \beta_{1} \mathbf{r}_{1} \mathbf{D}_{1}^{-1} \boldsymbol{\Sigma} \mathbf{D}_{1}^{-1} \mathbf{r}_{1} \mathbf{r}_{1}^{\prime} \mathbf{D}_{1}^{-1} \mathbf{M} \mathbf{V}^{-1} \mathbf{r}_{1} \\
& =-b_{n} n^{-1} \mathbb{E}\left(\operatorname{tr} \mathbf{D}_{1}^{-1} \boldsymbol{\Sigma} \mathbf{D}_{1}^{-1} \mathbf{T}\right)\left(\operatorname{tr} \mathbf{D}_{1}^{-1} \mathbf{M} \mathbf{V}^{-1} \boldsymbol{\Sigma}\right)+o(1) \\
& =-b_{n} n^{-1} \mathbb{E}\left(\operatorname{tr} \mathbf{D}^{-1} \boldsymbol{\Sigma} \mathbf{D}^{-1} \boldsymbol{\Sigma}\right)\left(\operatorname{tr} \mathbf{D}^{-1} \mathbf{M} \mathbf{V}^{-1} \boldsymbol{\Sigma}\right)+o(1) \\
& =-b_{n} n^{-1} \mathbb{E}\left(\operatorname{tr} \mathbf{D}^{-1} \boldsymbol{\Sigma} \mathbf{D}^{-1} \boldsymbol{\Sigma}\right) \mathbb{E}\left(\operatorname{tr} \mathbf{D}^{-1} \mathbf{M} \mathbf{V}^{-1} \boldsymbol{\Sigma}\right)+o(1) . \tag{3.38}
\end{align*}
$$

Applying (3.24), (3.33)-(3.38), and Lemma A.3, one may approximate each components of $S_{n}$ and $\underline{S}_{n}$. Specifically, we have

$$
\begin{aligned}
& \frac{1}{n} \mathbb{E} \operatorname{tr} \mathbf{D}^{-1} \mathbf{T}^{k}=-\int \frac{c_{n} t^{k} d \tilde{H}_{p}(t)}{z\left(1+\underline{m}_{0} t\right)}+o(1), \\
& \frac{1}{n} \mathbb{E} \operatorname{tr} \mathbf{D}^{-1} \mathbf{V}^{-1} \mathbf{T}^{k}=-\int \frac{c_{n} t^{k} d \tilde{H}_{p}(t)}{z^{2}\left(1+\underline{m}_{0} t\right)^{2}}+o(1), \\
& \frac{1}{n} \mathbb{E} \operatorname{tr} \mathbf{D}^{-1} \mathbf{T D}^{-1} \mathbf{T} \\
= & -\frac{1}{n} \mathbb{E} \operatorname{tr} \mathbf{V}^{-1} \mathbf{T} \mathbf{D}^{-1} \mathbf{T}-\frac{b_{n}^{2}}{n^{2}} \mathbb{E} \operatorname{tr} \mathbf{D}^{-1} \mathbf{T} \mathbf{D}^{-1} \mathbf{T E} \operatorname{tr} \mathbf{V}^{-1} \mathbf{T} \mathbf{D}^{-1} \mathbf{T}+o(1) \\
= & -\frac{1}{n} \mathbb{E} \operatorname{tr} \mathbf{V}^{-1} \mathbf{T D} \mathbf{D}^{-1} \mathbf{T}\left[1+\frac{b_{n}^{2}}{n} \mathbb{E} \operatorname{tr} \mathbf{V}^{-1} \mathbf{T D} \mathbf{D}^{-1} \mathbf{T}\right]^{-1}+o(1), \\
= & \int \frac{c_{n} t^{2} d \tilde{H}_{p}(t)}{z^{2}\left(1+\underline{m}_{0} t\right)^{2}}\left[1-\int \frac{c_{n} \underline{m_{0}^{2}} t^{2} d \tilde{H}_{p}(t)}{\left(1+\underline{m}_{0} t\right)^{2}}\right]^{-1}+o(1), \\
& \frac{1}{n} \mathbb{E} \operatorname{tr} \mathbf{D}^{-1} \mathbf{T D}^{-1} \mathbf{V}^{-1} \mathbf{T} \\
= & -\frac{1}{n} \mathbb{E} \operatorname{tr} \mathbf{V}^{-1} \mathbf{T D}^{-1} \mathbf{V}^{-1} \mathbf{T}\left[1+\frac{b_{n}^{2}}{n} \mathbb{E} \operatorname{tr} \mathbf{D}^{-1} \mathbf{T D} \mathbf{D}^{-1} \mathbf{T}\right]+o(1) \\
= & -\frac{1}{n} \mathbb{E} \operatorname{tr} \mathbf{V}^{-1} \mathbf{T D}^{-1} \mathbf{V}^{-1} \mathbf{T}\left[1+\frac{b_{n}^{2}}{n} \mathbb{E} \operatorname{tr} \mathbf{V}^{-1} \mathbf{T D} \mathbf{D}^{-1} \mathbf{T}\right]+o(1), \\
= & \int \frac{c_{n} t^{2} d \tilde{H}_{p}(t)}{z^{3}\left(1+\underline{m}_{0} t\right)^{3}}\left[1-\int \frac{c_{n} \underline{m}_{0}^{2} t^{2} d \tilde{H}_{p}(t)}{\left(1+\underline{m}_{0} t\right)^{2}}\right]^{-1}+o(1) .
\end{aligned}
$$

Combining the above results, we obtain

$$
T_{n} / \underline{T}_{n}=\int \frac{c_{n} t d \tilde{H}_{p}(t)}{z\left(1+\underline{m}_{0} t\right)^{2}}+o(1)
$$

$$
\begin{aligned}
& S_{n}-\underline{S}_{n} T_{n} / \underline{T}_{n} \\
= & -\int \frac{c_{n} \underline{\underline{m}}_{0}^{2} t^{2} d \tilde{H}_{p}(t)}{z\left(1+\underline{m}_{0} t\right)^{3}}\left[1-\int \frac{c_{n} \underline{m}_{0}^{2} t^{2} d \tilde{H}_{p}(t)}{\left(1+\underline{m}_{0} t\right)^{2}}\right]^{-1} \\
& -\frac{2 c_{n} \underline{m}_{0}^{2}}{z}\left[\int \frac{\left(\alpha_{2} t-t^{2}\right) d \tilde{H}_{p}(t)}{1+\underline{m}_{0} t} \int \frac{t d \tilde{H}_{p}(t)}{\left(1+\underline{m}_{0} t\right)^{2}}-\int \frac{t d \tilde{H}_{p}(t)}{1+\underline{m}_{0} t} \int \frac{t^{2} d \tilde{H}_{p}(t)}{\left(1+\underline{m}_{0} t\right)^{2}}\right] \\
& -c_{n}(\tau-3)\left\{\frac{1}{z \underline{m}_{0}} g_{p, z}^{\prime}\left(\frac{-1}{\underline{m}_{0}}, \frac{-1}{\underline{m}_{0}}\right)+\zeta_{p} \int \frac{t \underline{m}_{0} d \tilde{H}_{p}(t)}{1+\underline{m}_{0} t} \int \frac{t \underline{m}_{0} d \tilde{H}_{p}(t)}{z\left(1+\underline{m}_{0} t\right)^{2}}\right. \\
& \left.-\left[\int \frac{t d \tilde{H}_{p}(t)}{z\left(1+\underline{m}_{0} t\right)} h_{p}^{\prime}\left(\frac{-1}{\underline{m}_{0}}\right)+\int \frac{t \underline{m}_{0} d \tilde{H}_{p}(t)}{z\left(1+\underline{m}_{0} t\right)^{2}} h_{p}\left(\frac{-1}{\underline{m}_{0}}\right)\right]\right\}+o(1) .
\end{aligned}
$$

Therefore, from (3.31) and the identities

$$
\left[1-\int \frac{c_{n} t d \tilde{H}_{p}(t)}{z\left(1+\underline{m}_{0} t\right)^{2}}\right]^{-1}=-z \underline{m}_{0}\left[1-\int \frac{c_{n} \underline{m}_{0}^{2} t^{2} d \tilde{H}_{p}(t)}{\left(1+\underline{m}_{0} t\right)^{2}}\right]^{-1}=-\frac{z \underline{m}_{0}^{\prime}}{\underline{m}_{0}}
$$

we obtain

$$
M_{n}^{(2)}(z)=\frac{S_{n}-\underline{S}_{n} T_{n} / \underline{T}_{n}}{1-T_{n} / \underline{T}_{n}}=\mu_{1}(z)+(\tau-3) \mu_{2}(z)+o(1) .
$$

The proof is complete.

## A Additional lemmas

In this Appendix, we present some lemmas and their proofs, which will be used in the proof of our main theorems.

## A. 1 Lemmas

Lemma A. 1 (Lemma 2.7 in Bai and Silverstein (1998)) For $\mathbf{z}=\left(z_{1}, \ldots, z_{p}\right)^{\prime}$ i.i.d. standardized entries, $\mathbf{C} p \times p$ matrix (complex) we have for any $k \geq 2$

$$
\mathbb{E}\left|\mathbf{z}^{\prime} \mathbf{C z}-\operatorname{tr} \mathbf{C}\right|^{k} \leq K\left[\left(\mathbb{E}\left|z_{1}\right|^{4} \operatorname{tr} \mathbf{C} \mathbf{C}^{*}\right)^{\frac{k}{2}}+\mathbb{E}\left|z_{1}\right|^{2 k} \operatorname{tr}\left(\mathbf{C} \mathbf{C}^{*}\right)^{\frac{k}{2}}\right],
$$

where $K$ is a constant depending only on $k$.
Lemma A. 2 Suppose that Assumptions (a)-(c) and (3.17) hold. Let $\mathbf{z}=\left(z_{1}, \ldots, z_{p}\right)^{\prime}$ be a random vector with i.i.d. standardized entries, $\mathbf{r}=\sqrt{p / n} \mathbf{A}^{1 / 2} \mathbf{z} /\left\|\mathbf{A}^{1 / 2} \mathbf{z}\right\|$, and $\boldsymbol{\Sigma}=n \mathbb{E} \mathbf{r r}^{\prime}$. Then for any $p \times p$ complex matrices $\mathbf{C}$ and $\tilde{\mathbf{C}}$ with bounded spectral norms,

$$
\begin{aligned}
& n^{2} \mathbb{E}\left(\mathbf{r}^{\prime} \mathbf{C r}-\frac{1}{n} \operatorname{tr} \boldsymbol{\Sigma} \mathbf{C}\right)\left(\mathbf{r}^{\prime} \tilde{\mathbf{C}} \mathbf{r}-\frac{1}{n} \operatorname{tr} \boldsymbol{\Sigma} \tilde{\mathbf{C}}\right) \\
= & \operatorname{tr} \mathbf{T C T} \tilde{\mathbf{C}}+\operatorname{tr} \mathbf{T} \mathbf{C T} \tilde{\mathbf{C}}^{\prime}+\frac{2}{p^{2}} \operatorname{tr} \mathbf{T}^{2} \operatorname{tr} \mathbf{T C} \operatorname{tr} \mathbf{T} \tilde{\mathbf{C}}-\frac{2}{p} \operatorname{tr} \mathbf{T}^{2} \mathbf{C} \operatorname{tr} \mathbf{T} \tilde{\mathbf{C}}-\frac{2}{p} \operatorname{tr} \mathbf{T C} \operatorname{tr} \mathbf{T}^{2} \tilde{\mathbf{C}} \\
& +(\tau-3)\left\{\operatorname{tr}\left[\left(\mathbf{T}^{\frac{1}{2}} \mathbf{C} \mathbf{T}^{\frac{1}{2}}\right) \circ\left(\mathbf{T}^{\frac{1}{2}} \tilde{\mathbf{C}} \mathbf{T}^{\frac{1}{2}}\right)\right]+\frac{1}{p^{2}} \operatorname{tr} \mathbf{C T} \operatorname{tr} \tilde{\mathbf{C}} \mathbf{T} \operatorname{tr}[\mathbf{T} \circ \mathbf{T}]\right. \\
& \left.-\frac{1}{p} \operatorname{tr} \mathbf{C T} \operatorname{tr}\left[\left(\mathbf{T}^{\frac{1}{2}} \tilde{\mathbf{C}} \mathbf{T}^{\frac{1}{2}}\right) \circ \mathbf{T}\right]-\frac{1}{p} \operatorname{tr} \tilde{\mathbf{C}} \mathbf{T} \operatorname{tr}\left[\left(\mathbf{T}^{\frac{1}{2}} \mathbf{C} \mathbf{T}^{\frac{1}{2}}\right) \circ \mathbf{T}\right]\right\}+o(p) .
\end{aligned}
$$

Lemma A. 3 Suppose that Assumptions (a)-(c) and (3.17) hold with $\gamma=5$. We have

$$
\|\boldsymbol{\Sigma}-\mathbf{T}\|=o\left(p^{-1}\right)
$$

where $\boldsymbol{\Sigma}$ is defined in Lemma A. 2 and $\mathbf{T}$ is given in (2.8).
Lemma A. 4 Under the assumptions in Lemma A.2, for any $k \geq 2$,

$$
\begin{align*}
\mathbb{E}\left|\mathbf{r}^{\prime} \mathbf{C r}-\frac{1}{n} \operatorname{tr} \boldsymbol{\Sigma} \mathbf{C}\right|^{k} & \leq K n^{-k}\left[\mathbb{E}\left|z_{1}\right|^{2 k} \operatorname{tr}(\mathbf{C} \boldsymbol{\Sigma})^{k}+\left(\mathbb{E}\left|z_{1}\right|^{4} \operatorname{tr}(\mathbf{C} \boldsymbol{\Sigma})^{2}\right)^{\frac{k}{2}}+\|\left.\mathbf{C} \boldsymbol{\Sigma}\right|^{k}\left(p^{\frac{k}{2}} \mathbb{E}^{\frac{k}{2}}\left|z_{1}\right|^{4}+p \mathbb{E}\left|z_{1}\right|^{2 k}\right)\right] \\
& \leq K\left(n^{-\frac{k}{2}}+n^{-1-\frac{k(v-4)}{\gamma}}\right) \tag{A.1}
\end{align*}
$$

where $K$ is a constant depending only on $k$.

## A. 2 Proof of Lemma A. 2

Denote $\mathbf{W}=\mathbf{A}^{\frac{1}{2}} \mathbf{C A}{ }^{\frac{1}{2}}, \mathbf{U}=\mathbf{A}^{\frac{1}{2}} \tilde{\mathbf{C}} \mathbf{A}^{\frac{1}{2}}$, and $s=\mathbf{z}^{\prime} \mathbf{A z} / p$. We consider the product of the quadratic form $n^{2} \mathbf{r}^{\prime} \mathbf{C r} \mathbf{r}^{\prime} \tilde{\mathbf{C}} \mathbf{r}=\mathbf{z}^{\prime} \mathbf{W} \mathbf{z z} \mathbf{z}^{\prime} \mathbf{U z} / s^{2}$. From Lemma A.1, the fact $\operatorname{tr} \mathbf{A}=p$, and the conditions in (3.17), it holds that

$$
\begin{equation*}
\mathbb{E}|s-1|^{k} \leq K\left(p^{-\frac{k}{2}}+p^{-1-\frac{k(\gamma-4)}{\gamma}}\right), \quad k \geq 2 . \tag{A.2}
\end{equation*}
$$

By the identity

$$
\frac{1}{s^{2}}=2-s^{2}+\left(1-s^{2}\right)^{2}+s^{-2}\left(1-s^{2}\right)^{3}
$$

and the inequality

$$
\mathbb{E}\left(\mathbf{z}^{\prime} \mathbf{W} \mathbf{z z}^{\prime} \mathbf{U z}\right)\left(s^{-2}\left(1-s^{2}\right)^{3}\right) \leq K p^{2} \mathbb{E}|1-s|^{3}=o(p),
$$

we have

$$
\begin{equation*}
\mathbb{E} \mathbf{r}^{\prime} \mathbf{C r} r^{\prime} \tilde{\mathbf{C}} \mathbf{r}=\mathbb{E}\left(\mathbf{z}^{\prime} \mathbf{W} \mathbf{z z}^{\prime} \mathbf{U z}\right)\left(6-8 s+3 s^{2}\right)+o(p) \tag{A.3}
\end{equation*}
$$

Therefore, the main task in the following is to derive the limits for the three terms $\mathbb{E} \mathbf{z}^{\prime} \mathbf{W} \mathbf{z z}{ }^{\prime} \mathbf{U z}, \mathbb{E z z}^{\prime} \mathbf{W z z}{ }^{\prime} \mathbf{U z} s$ and $\mathbb{E} \mathbf{z}^{\prime} \mathbf{W z z} \mathbf{U z} s^{2}$ up to the order $O(p)$.

For the first term $\mathbb{E} \mathbf{z}^{\prime} \mathbf{W} \mathbf{z z} \mathbf{z}^{\prime} \mathbf{U z}$, we have

$$
\mathbb{E}^{\prime} \mathbf{W} \mathbf{z z} \mathbf{U z}=\mathbb{E} \sum_{i, j, k, \ell} z_{i} z_{j} z_{k} z_{\ell} \mathbf{W}_{i j} \mathbf{U}_{k \ell}
$$

Since all the $p$ components $z_{i}$ are independent and standardized, with mean zero, variance one and finite fourth moment, the terms that will contribute are the ones with their indexes either can be glued together or divided into two groups, i.e. $i=j=k=\ell$, or $i=j \neq k=\ell$, or $i=k \neq j=\ell$ or $i=\ell \neq j=k$. All the four cases together gives

$$
\begin{equation*}
\mathbb{E} \mathbf{z}^{\prime} \mathbf{W} \mathbf{z z} \mathbf{U} \mathbf{z}=\operatorname{tr} \mathbf{W} \operatorname{tr} \mathbf{U}+\operatorname{tr} \mathbf{W} \mathbf{U}+\operatorname{tr} \mathbf{W}^{\prime} \mathbf{U}+(\tau-3) \sum_{i} \mathbf{W}_{i i} \mathbf{U}_{i i}+o(p) \tag{A.4}
\end{equation*}
$$

For the second term $\mathbb{E} \mathbf{z}^{\prime} \mathbf{W} \mathbf{z z} \mathbf{z}^{\prime} \mathbf{U z} s$, we have

$$
\begin{equation*}
\mathbb{E} \mathbf{z}^{\prime} \mathbf{W} \mathbf{z z} \mathbf{z}^{\prime} \mathbf{U} s=\frac{1}{p} \mathbb{E} \sum_{i, j, k, \ell, s, u} z_{i} z_{j} z_{k} z_{\ell} z_{s} z_{u} \mathbf{W}_{i j} \mathbf{U}_{k \ell} \mathbf{A}_{s u} \tag{A.5}
\end{equation*}
$$

The terms that will contribute up to order $O(p)$ are in $\sum_{(2)}$ and $\sum_{(3)}$, where the index $(\cdot)$ denotes the number of distinct integers in the set $\{i, j, k, \ell, s, u\}$. It can be checked that the following three cases should be counted in $\sum_{(2)}$ (all have the form of the product of two traces)
case 1: $i=j \neq k=\ell=s=u$,
case 2: $k=\ell \neq i=j=s=u$,
case 3: $s=u \neq i=j=k=\ell$,
while in $\sum_{(3)}$ the following four cases should be taken into account,
case 1: $k=s \neq \ell=u \neq i=j$ and $k=u \neq \ell=s \neq i=j$,
case 2: $i=s \neq j=u \neq k=\ell$ and $i=u \neq j=s \neq k=\ell$,
case 3: $i=\ell \neq j=k \neq s=u$ and $i=k \neq j=\ell \neq s=u$,
case 4: $i=j \neq k=\ell \neq s=u$.
Combining the contribution of each cases in $\sum_{(2)}$ and $\sum_{(3)}$, we have

$$
\begin{aligned}
& \text { case } 1=\frac{\tau+o(1)}{p} \sum_{i \neq k} \mathbf{W}_{i i} \mathbf{U}_{k k} \mathbf{A}_{k k}+\frac{2}{p} \sum_{i \neq k \neq \ell} \mathbf{W}_{i i} \mathbf{U}_{k \ell} \mathbf{A}_{\ell k} \\
&=\frac{\tau-2}{p} \sum_{i \neq k} \mathbf{W}_{i i} \mathbf{U}_{k k} \mathbf{A}_{k k}+\frac{2}{p} \sum_{i \neq k} \mathbf{W}_{i i}(\mathbf{U} \mathbf{A})_{k k}+o(p) \\
&=\frac{\tau-2}{p} \operatorname{tr} \mathbf{W} \sum_{k} \mathbf{U}_{k k} \mathbf{A}_{k k}+\frac{2}{p} \operatorname{tr} \mathbf{W} \operatorname{tr}(\mathbf{U} \mathbf{A})+o(p), \\
& \text { case } 2=\frac{\tau+o(1)}{p} \sum_{i \neq k} \mathbf{W}_{i i} \mathbf{U}_{k k} \mathbf{A}_{i i}+\frac{2}{p} \sum_{i \neq j \neq k} \mathbf{W}_{i j} \mathbf{U}_{k k} \mathbf{A}_{j i} \\
&=\frac{\tau-2}{p} \sum_{i \neq k} \mathbf{W}_{i i} \mathbf{A}_{i i} \mathbf{U}_{k k}+\frac{2}{p} \sum_{i \neq k} \mathbf{U}_{k k}(\mathbf{W} \mathbf{A})_{i i}+o(p) \\
&=\frac{\tau-2}{p} \operatorname{tr} \mathbf{U} \sum_{i} \mathbf{W}_{i i} \mathbf{A}_{i i}+\frac{2}{p} \operatorname{tr} \mathbf{U} \operatorname{tr}(\mathbf{W} \mathbf{A})+o(p), \\
& \text { case 3 }=\frac{\tau+o(1)}{p} \sum_{s \neq i} \mathbf{W}_{i i} \mathbf{U}_{i i} \mathbf{A}_{s s}+\frac{1}{p} \sum_{i \neq j \neq s} \mathbf{W}_{i j} \mathbf{U}_{j i} \mathbf{A}_{s s}+\frac{1}{p} \sum_{i \neq j \neq s} \mathbf{W}_{i j} \mathbf{U}_{j i}^{*} \mathbf{A}_{s s} \\
&=\frac{\tau-2}{p} \sum_{s \neq i} \mathbf{W}_{i i} \mathbf{U}_{i i} \mathbf{A}_{s s}+\frac{1}{p} \sum_{i \neq s} \mathbf{A}_{s s}(\mathbf{W} \mathbf{U})_{i i}+\frac{1}{p} \sum_{i \neq s} \mathbf{A}_{s s}\left(\mathbf{W} \mathbf{U}^{*}\right)_{i i}+o(p) \\
&=\frac{\tau-2}{p} \operatorname{tr} \mathbf{A} \sum_{i} \mathbf{W}_{i i} \mathbf{U}_{i i}+\frac{1}{p} \operatorname{tr} \mathbf{A} \operatorname{tr}(\mathbf{W} \mathbf{U})+\frac{1}{p} \operatorname{tr} \mathbf{A} \operatorname{tr}\left(\mathbf{W} \mathbf{U}^{*}\right)+o(p), \\
& \text { case 4 }=\frac{1}{p} \sum_{i \neq k \neq s} \mathbf{W}_{i i} \mathbf{U}_{k k} \mathbf{A}_{s s} \\
&=\frac{1}{p} \operatorname{tr} \mathbf{W} \operatorname{tr} \mathbf{U} \operatorname{tr} \mathbf{A}-\frac{1}{p} \operatorname{tr} \mathbf{A} \sum_{i} \mathbf{W}_{i i} \mathbf{U}_{i i}-\frac{1}{p} \operatorname{tr} \mathbf{U} \sum_{i} \mathbf{W}_{i i} \mathbf{A}_{i i} \\
&-\frac{1}{p} \operatorname{tr} \mathbf{W} \sum_{i} \mathbf{A}_{i i} \mathbf{U}_{i i}+o(p), \\
&
\end{aligned}
$$

which further gives

$$
\begin{aligned}
\mathbb{E} \mathbf{z}^{\prime} \mathbf{W} \mathbf{z z} \mathbf{Z}^{\prime} \mathbf{z} s & =\text { case } 1+\text { case } 2+\text { case } 3+\text { case } 4+o(p) \\
& =\frac{1}{p} \operatorname{tr} \mathbf{W} \operatorname{tr} \mathbf{U} \operatorname{tr} \mathbf{A}+\frac{2}{p} \operatorname{tr} \mathbf{W} \operatorname{tr}(\mathbf{U A})+\frac{2}{p} \operatorname{tr} \mathbf{U} \operatorname{tr}(\mathbf{W A})
\end{aligned}
$$

$$
\begin{align*}
& +\frac{1}{p} \operatorname{tr} \mathbf{A} \operatorname{tr}(\mathbf{W} \mathbf{U})+\frac{1}{p} \operatorname{tr} \mathbf{A} \operatorname{tr}\left(\mathbf{W} \mathbf{U}^{*}\right)+\frac{\tau-3}{p} \operatorname{tr} \mathbf{W} \sum_{k} \mathbf{U}_{k k} \mathbf{A}_{k k} \\
& +\frac{\tau-3}{p} \operatorname{tr} \mathbf{U} \sum_{i} \mathbf{W}_{i i} \mathbf{A}_{i i}+\frac{\tau-3}{p} \operatorname{tr} \mathbf{A} \sum_{i} \mathbf{W}_{i i} \mathbf{U}_{i i}+o(p) . \tag{A.6}
\end{align*}
$$

Finally, for the third term $\mathbb{E} \mathbf{z}^{\prime} \mathbf{W z z}^{\prime} \mathbf{U z} s^{2}$, we have

$$
\mathbb{E}^{\prime} \mathbf{W} \mathbf{z z} \mathbf{U z}^{\prime}=\frac{1}{p^{2}} \mathbb{E} \sum_{i, j, k, \ell, s, u, m, b} z_{i} z_{j} z_{k} z_{\ell} z_{s} z_{u} z_{m} z_{b} \mathbf{W}_{i j} \mathbf{U}_{k \ell} \mathbf{A}_{s u} \mathbf{A}_{m b}
$$

The terms that will make the main contribution up to order $O(p)$ are in $\sum_{(3)}$ and $\sum_{(4)}$. For example, when considering $\sum_{(1)}$, we have

$$
\sum_{(1)}=\mathbb{E} \sum_{i} \frac{1}{p^{2}} z_{i}^{8} \mathbf{W}_{i i} \mathbf{U}_{i i} \mathbf{A}_{i i}^{2}=O\left(p^{1-4(\gamma-4) / \gamma}\right)=o(p)
$$

by using the assumptions in (3.17). Similar technique can be applied for dealing with the terms in $\sum_{(2)}$ and get the $o(p)$ bound, thus can be neglected. For terms in $\sum_{(3)}$ and $\sum_{(4)}$, we list in the following all the cases that should be counted, which are all up to order $O(p)$. For $\sum_{(3)}$, we have six cases
case 1: $i=j \neq k=\ell \neq s=u=m=b$,
case 2: $i=j=s=u \neq k=\ell \neq m=b$,
case 3: $i=j=m=b \neq k=\ell \neq s=u$,
case 4: $k=\ell=m=b \neq i=j \neq s=u$,
case 5: $i=j=k=\ell \neq s=u \neq m=b$, case 6: $k=\ell=s=u \neq i=j \neq m=b$, while in $\sum_{(4)}$, we have seven cases
case $1: i=j \neq k=\ell \neq u=m \neq s=b$ and $i=j \neq k=\ell \neq s=m \neq u=b$, case 2: $i=s \neq j=u \neq k=\ell \neq m=b$ and $i=u \neq j=s \neq k=\ell \neq m=b$, case 3: $i=m \neq j=b \neq u=s \neq k=\ell$ ang $i=b \neq j=m \neq k=\ell \neq s=u$, case 4: $k=m \neq \ell=b \neq i=j \neq s=u$ and $k=b \neq \ell=m \neq i=j \neq s=u$, case 5: $i=k \neq j=\ell \neq s=u \neq m=b$ and $i=\ell \neq j=k \neq s=u \neq m=b$, case 6: $k=s \neq \ell=u \neq i=j \neq m=b$ and $k=u \neq \ell=s \neq i=j \neq m=b$, case 7: $i=j \neq k=\ell \neq s=u \neq m=b$.
Combining the above, we have

$$
\begin{aligned}
\text { case } 1 & =\frac{2}{p^{2}} \sum_{i \neq k \neq m \neq s} \mathbf{W}_{i i} \mathbf{U}_{k k} \mathbf{A}_{m s} \mathbf{A}_{m s}+\frac{\tau+o(1)}{p^{2}} \sum_{i \neq k \neq s} \mathbf{W}_{i i} \mathbf{U}_{k k} \mathbf{A}_{s s}^{2} \\
& =\frac{2}{p^{2}} \sum_{i \neq k \neq s} \mathbf{W}_{i i} \mathbf{U}_{k k}(\mathbf{A} \mathbf{A})_{s s}+\frac{\tau-2}{p^{2}} \sum_{i \neq k \neq s} \mathbf{W}_{i i} \mathbf{U}_{k k} \mathbf{A}_{s s}^{2}+o(p) \\
& =\frac{2}{p^{2}} \operatorname{tr} \mathbf{A}^{2} \operatorname{tr} \mathbf{W} \operatorname{tr} \mathbf{U}+\frac{\tau-2}{p^{2}} \operatorname{tr} \mathbf{W} \operatorname{tr} \mathbf{U} \sum_{s} \mathbf{A}_{s s}^{2}+o(p), \\
\text { case } 2 & =\frac{2}{p^{2}} \sum_{i \neq j \neq k \neq m} \mathbf{W}_{i j} \mathbf{U}_{k k} \mathbf{A}_{i j} \mathbf{A}_{m m}+\frac{\tau+o(1)}{p^{2}} \sum_{i \neq k \neq m} \mathbf{W}_{i i} \mathbf{U}_{k k} \mathbf{A}_{i i} \mathbf{A}_{m m} \\
& =\frac{2}{p^{2}} \sum_{i \neq k \neq m}(\mathbf{W A})_{i i} \mathbf{U}_{k k} \mathbf{A}_{m m}+\frac{\tau-2}{p^{2}} \sum_{i \neq k \neq m} \mathbf{W}_{i i} \mathbf{U}_{k k} \mathbf{A}_{i i} \mathbf{A}_{m m}+o(p) \\
& =\frac{2}{p^{2}} \operatorname{tr}(\mathbf{W A}) \operatorname{tr} \mathbf{U} \operatorname{tr} \mathbf{A}+\frac{\tau-2}{p^{2}} \operatorname{tr} \mathbf{U} \operatorname{tr} \mathbf{A} \sum_{i} \mathbf{W}_{i i} \mathbf{A}_{i i}+o(p),
\end{aligned}
$$

$$
\begin{aligned}
\text { case } 3= & \text { case 2, } \\
\text { case } 4= & \frac{2}{p^{2}} \sum_{k \neq \ell \neq i \neq s} \mathbf{W}_{i i} \mathbf{U}_{k \ell} \mathbf{A}_{s s} \mathbf{A}_{k \ell}+\frac{\tau}{p^{2}} \sum_{k \neq i \neq s} \mathbf{W}_{i i} \mathbf{U}_{k k} \mathbf{A}_{k k} \mathbf{A}_{s s} \\
= & \frac{2}{p^{2}} \sum_{k \neq i \neq s}(\mathbf{U A})_{k k} \mathbf{W}_{i i} \mathbf{A}_{s s}+\frac{\tau-2}{p^{2}} \sum_{k \neq i \neq s} \mathbf{W}_{i i} \mathbf{U}_{k k} \mathbf{A}_{s s} \mathbf{A}_{k k}+o(p) \\
= & \frac{2}{p^{2}} \operatorname{tr}(\mathbf{U A}) \operatorname{tr} \mathbf{W} \operatorname{tr} \mathbf{A}+\frac{\tau-2}{p^{2}} \operatorname{tr} \mathbf{W} \operatorname{tr} \mathbf{A} \sum_{k} \mathbf{U}_{k k} \mathbf{A}_{k k}+o(p), \\
\text { case } 5= & \frac{1}{p^{2}} \sum_{i \neq j \neq s \neq m} \mathbf{W}_{i j} \mathbf{U}_{i j} \mathbf{A}_{s s} \mathbf{A}_{m m}+\frac{1}{p^{2}} \sum_{i \neq j \neq s \neq m} \mathbf{W}_{i j} \mathbf{U}_{j i} \mathbf{A}_{s s} \mathbf{A}_{m m} \\
& +\frac{\tau}{p^{2}} \sum_{i \neq s \neq m} \mathbf{W}_{i i} \mathbf{U}_{i i} \mathbf{A}_{s s} \mathbf{A}_{m m} \\
= & \frac{1}{p^{2}} \sum_{i \neq s \neq m}\left(\mathbf{W} \mathbf{U}_{i i} \mathbf{A}_{s s} \mathbf{A}_{m m}+\frac{1}{p^{2}} \sum_{i \neq s \neq m}\left(\mathbf{W} \mathbf{U}^{*}\right)_{i i} \mathbf{A}_{s s} \mathbf{A}_{m m}\right. \\
& +\frac{\tau-2}{p^{2}} \sum_{i \neq s \neq m} \mathbf{W}_{i i} \mathbf{U}_{i i} \mathbf{A}_{s s} \mathbf{A}_{m m}+o(p) \\
= & \frac{1}{p^{2}} \operatorname{tr}(\mathbf{W} \mathbf{U})(\operatorname{tr} \mathbf{A})^{2}+\frac{1}{p^{2}} \operatorname{tr}\left(\mathbf{W} \mathbf{U}^{*}\right)(\operatorname{tr} \mathbf{A})^{2}+\frac{\tau-2}{p^{2}}(\operatorname{tr} \mathbf{A})^{2} \sum_{i} \mathbf{W}_{i i} \mathbf{U}_{i i}+o(p), \\
\text { case 6=}= & \operatorname{case} 4, \\
\text { case 7=}= & \frac{1}{p^{2}} \sum_{i \neq k \neq s \neq m} \mathbf{W}_{i i} \mathbf{U}_{k k} \mathbf{A}_{s s} \mathbf{A}_{m m} \\
= & \frac{1}{p^{2}} \operatorname{tr} \mathbf{W} \operatorname{tr} \mathbf{U}(\operatorname{tr} \mathbf{A})^{2}-\frac{1}{p^{2}} \operatorname{tr} \mathbf{W} \operatorname{tr} \mathbf{U} \sum_{s} \mathbf{A}_{s s}^{2}-\frac{2}{p^{2}} \operatorname{tr} \mathbf{W} \operatorname{tr} \mathbf{A} \sum_{s} \mathbf{A}_{s s} \mathbf{U}_{s s} \\
& -\frac{1}{p^{2}}(\operatorname{tr} \mathbf{A})^{2} \sum_{i} \mathbf{W}_{i i} \mathbf{U}_{i i}-\frac{2}{p^{2}} \operatorname{tr} \mathbf{U} \operatorname{tr} \mathbf{A} \sum_{i} \mathbf{W}_{i i} \mathbf{A}_{i i}+o(p),
\end{aligned}
$$

which finally leads to

## $\mathbb{E}\left(\mathbf{z}^{\prime} \mathbf{W} \mathbf{z z}^{\prime} \mathbf{U z}\right) s^{2}$

$$
\begin{align*}
= & \frac{1}{p^{2}} \operatorname{tr} \mathbf{W} \operatorname{tr} \mathbf{U}(\operatorname{tr} \mathbf{A})^{2}+\frac{2}{p^{2}} \operatorname{tr} \mathbf{A}^{2} \operatorname{tr} \mathbf{W} \operatorname{tr} \mathbf{U}+\frac{4}{p^{2}} \operatorname{tr}(\mathbf{W} \mathbf{A}) \operatorname{tr} \mathbf{U} \operatorname{tr} \mathbf{A} \\
& +\frac{4}{p^{2}} \operatorname{tr}(\mathbf{U A}) \operatorname{tr} \mathbf{W} \operatorname{tr} \mathbf{A}+\frac{1}{p^{2}} \operatorname{tr}(\mathbf{W} \mathbf{U})(\operatorname{tr} \mathbf{A})^{2}+\frac{1}{p^{2}} \operatorname{tr}\left(\mathbf{W} \mathbf{U}^{*}\right)(\operatorname{tr} \mathbf{A})^{2} \\
& +\frac{\tau-3}{p^{2}} \operatorname{tr} \mathbf{W} \operatorname{tr} \mathbf{U} \sum_{s} \mathbf{A}_{s s}^{2}+\frac{2 \tau-6}{p^{2}} \operatorname{tr} \mathbf{U} \operatorname{tr} \mathbf{A} \sum_{i} \mathbf{W}_{i i} \mathbf{A}_{i i} \\
& +\frac{2 \tau-6}{p^{2}} \operatorname{tr} \mathbf{W} \operatorname{tr} \mathbf{A} \sum_{k} \mathbf{U}_{k k} \mathbf{A}_{k k}+\frac{\tau-3}{p^{2}}(\operatorname{tr} \mathbf{A})^{2} \sum_{i} \mathbf{W}_{i i} \mathbf{U}_{i i}+o(p) . \tag{A.7}
\end{align*}
$$

Collecting (A.3), (A.4), (A.6), and (A.7), we have

$$
\begin{aligned}
& n^{2} \mathbb{E} \mathbf{r}^{\prime} \mathbf{C r r} \\
= & (\tau-3) \sum_{i} \mathbf{W}_{i i} \mathbf{U}_{i i}+\operatorname{tr} \mathbf{W} \operatorname{tr} U+\operatorname{tr}(\mathbf{W} \mathbf{U})+\operatorname{tr}\left(\mathbf{W}^{\prime} \mathbf{U}\right) \\
& +\frac{6}{p^{2}} \operatorname{tr} \mathbf{A}^{2} \operatorname{tr} \mathbf{W} \operatorname{tr} \mathbf{U}-\frac{4}{p} \operatorname{tr}(\mathbf{W A}) \operatorname{tr} \mathbf{U}-\frac{4}{p} \operatorname{tr}(\mathbf{U A}) \operatorname{tr} \mathbf{W}
\end{aligned}
$$

$$
\begin{align*}
& +\frac{3(\tau-3)}{p^{2}} \operatorname{tr} \mathbf{W} \operatorname{tr} \mathbf{U} \sum_{s} \mathbf{A}_{s s}^{2}-\frac{2(\tau-3)}{p} \operatorname{tr} \mathbf{W} \sum_{k} \mathbf{U}_{k k} \mathbf{A}_{k k} \\
& -\frac{2(\tau-3)}{p} \operatorname{tr} \mathbf{U} \sum_{i} \mathbf{W}_{i i} \mathbf{A}_{i i}+o(p) \tag{A.8}
\end{align*}
$$

On the other hand, using the identity

$$
\frac{1}{s}=2-s+(1-s)^{2}+s^{-1}(1-s)^{3}
$$

and the inequality (A.2), we can derive

$$
\begin{equation*}
n \mathbb{E} \mathbf{r}^{\prime} \mathbf{C r}=\mathbb{E} \frac{1}{s} \mathbf{z}^{\prime} \mathbf{W} \mathbf{z}=\mathbb{E} \mathbf{z}^{\prime} \mathbf{W} \mathbf{z}\left(3-3 s+s^{2}\right)+o(1) \tag{A.9}
\end{equation*}
$$

It is trivial to have

$$
\begin{equation*}
\mathbb{E} \mathbf{z}^{\prime} \mathbf{W} \mathbf{z}=\operatorname{tr} \mathbf{W} \tag{A.10}
\end{equation*}
$$

and by applying (A.4) and (A.6) again,

$$
\begin{align*}
\mathbb{E} \mathbf{z}^{\prime} \mathbf{W} \mathbf{z} s= & \frac{\tau-3}{p} \sum_{i} \mathbf{W}_{i i} \mathbf{A}_{i i}+\operatorname{tr} \mathbf{W}+\frac{1}{p} \operatorname{tr}(\mathbf{W} \mathbf{A})+\frac{1}{p} \operatorname{tr}\left(\mathbf{W}^{*} \mathbf{A}\right),  \tag{A.11}\\
\mathbb{E} \mathbf{z}^{\prime} \mathbf{W} \mathbf{z} s^{2}= & \operatorname{tr} \mathbf{W}+\frac{2}{p^{2}} \operatorname{tr} \mathbf{W} \operatorname{tr}\left(\mathbf{A}^{2}\right)+\frac{4}{p} \operatorname{tr}(\mathbf{W} \mathbf{A})+\frac{2(\tau-3)}{p} \sum_{i} \mathbf{W}_{i i} \mathbf{A}_{i i}  \tag{A.12}\\
& +\frac{\tau-3}{p^{2}} \operatorname{tr} \mathbf{W} \sum_{i} \mathbf{A}_{i i}^{2}+o(1) .
\end{align*}
$$

Collecting (A.9)-(A.12) leads to

$$
\begin{align*}
n \mathbb{E} \mathbf{r}^{\prime} \mathbf{C r}= & \operatorname{tr} \mathbf{W}+\frac{\tau-3}{p^{2}} \operatorname{tr} \mathbf{W} \sum_{i} \mathbf{A}_{i i}^{2}+\frac{2}{p^{2}} \operatorname{tr} \mathbf{W} \operatorname{tr} \mathbf{A}^{2}-\frac{\tau-3}{p} \sum_{i} \mathbf{W}_{i i} \mathbf{A}_{i i} \\
& -\frac{2}{p} \operatorname{tr}(\mathbf{W} \mathbf{A})+o(1) . \tag{A.13}
\end{align*}
$$

Therefore, combining (A.8)-(A.13), we have reached

$$
\begin{align*}
& n^{2} \mathbb{E}\left(\mathbf{r}^{\prime} \mathbf{C r}-\frac{1}{n} \operatorname{tr} \boldsymbol{\Sigma} \mathbf{C}\right)\left(\mathbf{r}^{\prime} \tilde{\mathbf{C}} \mathbf{r}-\frac{1}{n} \operatorname{tr} \boldsymbol{\Sigma} \tilde{\mathbf{C}}\right)  \tag{A.14}\\
= & n^{2} \mathbb{E} \mathbf{r}^{\prime} \mathbf{C r r} \\
= & \operatorname{tr}\left[\left(\mathbf{W}^{\prime}+\mathbf{C} \mathbf{r}\right) \mathbf{n}\right]+\frac{2}{p^{2}} \operatorname{Er} \mathbf{r}^{\prime} \mathbf{C r} \mathbf{A}^{2} \mathbf{\mathbf { A } ^ { \prime }} \tilde{\mathbf{C}} \mathbf{r} \mathbf{W} \operatorname{tr} \mathbf{U}-\frac{2}{p} \operatorname{tr}(\mathbf{W} \mathbf{A}) \operatorname{tr} \mathbf{U} \\
& -\frac{2}{p} \operatorname{tr}(\mathbf{U A}) \operatorname{tr} \mathbf{W}+(\tau-3) \operatorname{tr}(\mathbf{W} \circ \mathbf{U})+\frac{\tau-3}{p^{2}} \operatorname{tr} \mathbf{W} \operatorname{tr} \mathbf{U} \operatorname{tr}(\mathbf{A} \circ \mathbf{A}) \\
& -\frac{\tau-3}{p} \operatorname{tr} \mathbf{W} \operatorname{tr}(\mathbf{U} \circ \mathbf{A})-\frac{\tau-3}{p} \operatorname{tr} \mathbf{U} \operatorname{tr}(\mathbf{W} \circ \mathbf{A})+o(p) .
\end{align*}
$$

The proof is then complete by replacing all the matrix $\mathbf{A}$ with $\mathbf{T}$.

## A. 3 Proof of Lemma A. 3

Using the identity

$$
\frac{1}{s}=2-s+(1-s)^{2}+s^{-1}(1-s)^{3}
$$

we have

$$
\begin{equation*}
\boldsymbol{\Sigma}=\mathbb{E} \frac{1}{s} \mathbf{A}^{\frac{1}{2}} \mathbf{z Z}^{\prime} \mathbf{A}^{\frac{1}{2}}=\mathbb{E} \mathbf{A}^{\frac{1}{2}} \mathbf{z Z}^{\prime} \mathbf{A}^{\frac{1}{2}}\left(2-s+(1-s)^{2}+s^{-1}(1-s)^{3}\right) \tag{A.15}
\end{equation*}
$$

where $s=\mathbf{z}^{\prime} \mathbf{A z} / p$. First we show that

$$
\begin{equation*}
\left\|\mathbb{E} \mathbf{A}^{\frac{1}{2}} \mathbf{Z Z}^{\prime} \mathbf{A}^{\frac{1}{2}} s^{-1}(1-s)^{3}\right\|=o\left(p^{-1}\right) \tag{A.16}
\end{equation*}
$$

Define an event $A=\{|s-1|>1 / 2\}$ then, by Markov's inequality and (A.2), we have $\mathrm{P}(A)=o\left(n^{-s}\right)$ for any $s>0$. Therefore,

$$
\begin{aligned}
\left\|\mathbb{E} \mathbf{A}^{\frac{1}{2}} \mathbf{z z}^{\prime} \mathbf{A}^{\frac{1}{2}} s^{-1}(1-s)^{3}\right\| & \leq K\left\|\mathbb{E} \mathbf{z z} s^{\prime} s^{-1}(1-s)^{3} I(A)\right\|+K\left\|\mathbb{E} \mathbf{z z}^{\prime} s^{-1}(1-s)^{3} I\left(A^{c}\right)\right\| \\
& \leq K\left\|\mathbb{E} \mathbf{z z}|1-s|^{3}\right\|+o\left(n^{-s}\right) .
\end{aligned}
$$

Applying Hölder's inequality and (A.2), we have

$$
\begin{aligned}
\left\|\mathbb{E} \mathbf{z z} \mathbf{z}^{\prime}|1-s|^{3}\right\| & =\max _{\alpha \in \mathbb{R}^{p},\|\alpha\|=1} \mathbb{E} \alpha^{\prime} \mathbf{z z ^ { \prime }} \alpha|1-s|^{3} \leq \max _{\alpha \in \mathbb{R}^{p},\|\alpha\|=1} \mathbb{E}\left|\mathbf{z}^{\prime} \alpha \alpha^{\prime} \mathbf{z}-1 \| 1-s\right|^{3}+\mathbb{E}|1-s|^{3} \\
& \leq \max _{\alpha \in \mathbb{R}^{p},\|\alpha\|=1} \mathbb{E}^{\frac{1}{2}}\left|\mathbf{z}^{\prime} \alpha \alpha^{\prime} \mathbf{z}-1\right|^{2} \mathbb{E}^{\frac{1}{2}}|1-s|^{6}+o\left(p^{-1}\right),
\end{aligned}
$$

which is $o\left(p^{-1}\right)$ from (A.2) and the fact $\mathbb{E}\left|\mathbf{z}^{\prime} \alpha \alpha^{\prime} \mathbf{z}-1\right|^{2}=O(1)$. Therefore, (A.16) is verified, which together with (A.15) give

$$
\begin{equation*}
\boldsymbol{\Sigma}=\mathbb{E} \mathbf{A}^{\frac{1}{2}} \mathbf{z z}^{\prime} \mathbf{A}^{\frac{1}{2}}\left(2-s+(1-s)^{2}\right)+o\left(p^{-1}\right)=\mathbf{A}^{\frac{1}{2}}\left[\mathbb{E} \mathbf{z z ^ { \prime }}\left(3-3 s+s^{2}\right)\right] \mathbf{A}^{\frac{1}{2}}+o\left(p^{-1}\right) \tag{A.17}
\end{equation*}
$$

where (and in the following) the " $o\left(p^{-1}\right)$ " is in terms of spectral norm.
Next, we deal with the terms $\mathbb{E} \mathbf{z z}{ }^{\prime} s$ and $\mathbb{E} z z^{\prime} s^{2}$. For $\mathbb{E} \mathbf{z z} s$, we have its $(i, j)$-th entry given by
which gives

$$
\mathbb{E} \mathbf{z z}^{\prime} s=\mathbf{I}_{p}+\frac{2}{p} \mathbf{A}+\frac{1}{p}(\tau-3+o(1)) \operatorname{diag}(\mathbf{A})
$$

and

$$
\begin{equation*}
\mathbb{E} \mathbf{A}^{\frac{1}{2}} \mathbf{z} \mathbf{z}^{\prime} \mathbf{A}^{\frac{1}{2}} s=\mathbf{A}+\frac{2}{p} \mathbf{A}^{2}+\frac{1}{p}(\tau-3) \mathbf{A}^{\frac{1}{2}} \operatorname{diag}(\mathbf{A}) \mathbf{A}^{\frac{1}{2}}+o\left(p^{-1}\right) \tag{A.18}
\end{equation*}
$$

For the term $\mathbb{E} \mathbf{z z} s^{2} s^{2}$, similar to the derivation of (A.5), we have its $(i, j)$-th entry is given by

$$
\left[\mathbb{E} \mathbf{z z} \mathbf{z}^{\prime} s^{2}\right]_{(i, j)}=\frac{1}{p^{2}} \mathbb{E} z_{i} z_{j} \sum_{k, \ell, s, u} z_{k} z_{\ell} z_{s} z_{u} \mathbf{A}_{k \ell} \mathbf{A}_{s u}
$$

$$
=\left\{\begin{array}{ll}
\frac{4}{p} \mathbf{A}_{i j}-\frac{4}{p^{2}} \mathbf{A}_{i i} \mathbf{A}_{i j}-\frac{4}{p^{2}} \mathbf{A}_{i j} \mathbf{A}_{j j}+o\left(p^{-2}\right) & i \neq j \\
\frac{1}{p^{2}}(\tau-3) \sum_{k} \mathbf{A}_{k k}^{2}+\frac{2}{p}(\tau-1) \mathbf{A}_{i i}+\frac{2}{p^{2}} \operatorname{tr} \mathbf{A}^{2}+1+o\left(p^{-1}\right) & i=j
\end{array} .\right.
$$

Therefore, we get

$$
\mathbb{E} \mathbf{z z ^ { \prime }} s^{2}=\frac{4}{p} \mathbf{A}+\frac{1}{p^{2}}(\tau-3) \operatorname{tr}(\mathbf{A} \circ \mathbf{A}) \mathbf{I}_{p}+\mathbf{I}_{p}+\frac{2}{p}(\tau-3) \operatorname{diag}(\mathbf{A})+\frac{2}{p^{2}} \operatorname{tr} \mathbf{A}^{2} \cdot \mathbf{I}_{p}+o\left(p^{-1}\right)
$$

which further gives that

$$
\begin{equation*}
\mathbb{E} \mathbf{A}^{\frac{1}{2}} \mathbf{z} \mathbf{Z}^{\prime} \mathbf{A}^{\frac{1}{2}} s^{2}=\frac{4}{p} \mathbf{A}^{2}+\frac{1}{p^{2}}(\tau-3) \operatorname{tr}(\mathbf{A} \circ \mathbf{A}) \mathbf{A}+\mathbf{A}+\frac{2}{p}(\tau-3) \mathbf{A}^{\frac{1}{2}} \operatorname{diag}(\mathbf{A}) \mathbf{A}^{\frac{1}{2}}+\frac{2}{p^{2}} \operatorname{tr} \mathbf{A}^{2} \cdot \mathbf{A}+o\left(p^{-1}\right) \tag{A.19}
\end{equation*}
$$

Collecting (A.17), (A.18), and (A.19), we obtain

$$
\boldsymbol{\Sigma}=\mathbf{A}-\frac{\tau-3}{p} \mathbf{A} \operatorname{diag}(\mathbf{A}) \mathbf{A}^{\prime}-\frac{2}{p} \mathbf{A}^{2}+\left(\frac{\tau-3}{p^{2}} \operatorname{tr}(\mathbf{A} \circ \mathbf{A})+\frac{2}{p^{2}} \operatorname{tr} \mathbf{A}^{2}\right) \mathbf{A}+o\left(p^{-1}\right)
$$

The proof is thus complete.

## A. 4 Proof of Lemma A. 4

This lemma can be obtained from similar arguments for the proof of Lemma 6 in Morales-Jimenez et al. (2019). We omit the details.

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