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On Electronic Specific Heat of Ferromagnetic Rare Earth Metals*)

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On the basis of the *sf* exchange model and by the method of Green functions, a many-body theoretical formulation is given to the problem of exchange enhancement of electronic specific heat in a ferromagnetic metal. By introducing the spin wave description of localized spins, the *sf* Hamiltonian is reduced to a form similar to the coupled electron-phonon system, and Dyson equations are solved by the Migdal approximation. For the same reason as is known in the electron-phonon case, we expect, independently of the detail of the model, the exchange enhancement of electronic specific heat may amount to hundred percents.

§ 1. Introduction

As is well known by now, the thermal mass m_t of the conduction electron in a metal is enhanced by the coupling with phonons as $m_t = m_t^0(1+\lambda_p)$, where m_t^0 is the band theoretical value and the order of magnitude of λ_p is between 0.1 and 1. In analogy with this phonon enhancement, one may expect a similar enhancement arising from the coupling with magnons in a magnetically ordered metal.

For instance, in comparison with Cu, Phillips and Mattheiss[1] suggested a rather large mass enhancement in Ni. The electron-magnon interaction in transition metals may be studied theoretically by assuming a contact interaction potential between *d* electrons and applying the simple ladder approximation to the spin wave mode of collective motion. Along this line, Berk and Schrieffer,[2] and Doniach and Engelsberg[3] have pointed out that even the specific heat of a paramagnetic system can be enhanced by the critical spin fluctuation if the system is near a magnetic ordering point. From the many-body theoretical point of view, however, the consistency of the calculation is still open to question.[4]

The exchange enhancement of electronic specific heat can be investigated also by the so-called *sd* (or *sf*) exchange model, in which conduction electrons are interacting with the system of localized spins through an exchange type coupling. When the system of localized spins is in some ordered state, the

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exchange scattering of the conduction electron may be described as emission and absorption of magnons. Its analogy with the electron-phonon case was clearly noticed by Kondo[5] when he showed that the anomalous T-linear specific heat of Cu+Mn alloys may be accounted for as the exchange enhancement of the electronic part. He obtained $m_t = m_t^0(1+\lambda_m)$, where the enhancement factor λ_m is of the order of unity, depending neither on the magnitude of the *sd* exchange integral, nor on the impurity concentration. This is rather surprising and may even be doubtful from the naive perturbational point of view. It is the motivation of the present paper to justify Kondo's idea by a more systematic method of many-body theory.

For this purpose, however, it is more convenient to consider a regular lattice of localized spins, e.g. a magnetic rare earth metal in which magnetic electrons are in the well-localized f shells. At present, the analysis of specific heat data of rare earth metals is not conclusive, because one has to single out the electronic part from the observed low temperature specific heat which also contains phonon, nuclear and magnon parts.[6] Although there are some indications of large electronic specific heat of magnetic rare earth metals in comparison with La or Lu, we still have to await further experimental studies. Theoretically, on the basis of his own proposal for the band structure, Kasuya[7] mentioned the value $\lambda_m \sim 0.2$ for Gd, together with the second order perturbational expression $(m_t/m_t^0) = [1-\lambda_p-\lambda_m]^{-1}$. This expression should actually be replaced by $(m_t/m_t^0) = (1+\lambda_p+\lambda_m)$ which holds even for $\lambda_p+\lambda_m \sim 1$. Furthermore, to carry out really a quantitative calculation of λ_m, the following three items should be available: The band structure calculation, which already exists, for instance, for Gd,[8] the *sf* exchange integral J, which has been discussed from the single OPW point of view,[9] and the magnon dispersion curve, which will be supplied from neutron experiment. At present, the information is incomplete and it is not feasible to carry out such quantitative calculations as have been done for λ_p of simple metals.[10]

In view of this circumstance, we shall restrict ourselves to the many-body theoretical formulation of the problem. We start with the usual *sf* Hamiltonian[11]

$$H = \sum_{\boldsymbol{p}} \xi_{\boldsymbol{p}} \phi_{\boldsymbol{p}}^{+} \phi_{\boldsymbol{p}} - \sum_{i=1}^{N} \sum_{\boldsymbol{q}=\boldsymbol{p}-\boldsymbol{p}'} \sum_{\boldsymbol{p}-\boldsymbol{p}'} \left[\frac{J(q)}{V} \right] \exp(-i\boldsymbol{q}\cdot\boldsymbol{R}_i)\, \boldsymbol{S}_i \cdot \phi_{\boldsymbol{p}}^{+} \boldsymbol{\sigma}\, \phi_{\boldsymbol{p}'} . \qquad (1\cdot1)$$

Here $\phi_{\boldsymbol{p}}$ is the two-component spinor which destructs the conduction electron with crystal momentum $\boldsymbol{p}$ (where we take the extended zone scheme and $\hbar=1$) and the energy $\xi_{\boldsymbol{p}}$ measured from the chemical potential μ (which we identify with the usual Fermi energy by measuring μ from the bottom of the conduction band), $\boldsymbol{\sigma}$ is the Pauli vector matrix operating on the spinor, V is the normalization volume, and $\boldsymbol{S}_i$ are spin operators localized at lattice sites $\boldsymbol{R}_i$. Strictly speaking, we should write the *sf* exchange integral as $J(\boldsymbol{p}, \boldsymbol{p}')$ and also allow for Umklapp processes. The electron-phonon coupling ignored in (1·1) will

make additive contribution to the specific heat enhancement. Fermi liquid effect arising from the Coulomb interaction among conduction electrons is also ignored; the *sf* exchange coupling may be enhanced by the Coulomb interaction. We shall not enter into all these details. As we shall see below, they do not qualitatively affect the essential feature of Kondo's idea that $\lambda_m \sim 1$.

This can be seen by casting (1·1) into a form similar to the coupled electron-phonon system. For this purpose, we introduce the magnetic ordering from the outset and describe the system of localized spins by the spin wave approximation. To present the formalism neatly, we assume the ferromagnetic ordering (e.g. Gd). Thus

$$S_{iz} \simeq S - \left(\frac{1}{2S}\right)(S_{ix}^2 + S_{iy}^2), \tag{1·2}$$

$$[S_{ix},\ S_{jy}] \simeq iS\delta_{ij}, \tag{1·3}$$

where the magnitude S of the localized spin should not be too small. The first term on the right of (1·2), when inserted into (1·1), causes a uniform spin polarization of conduction electrons. The second term, when inserted into (1·1) and averaged over the uniformly polarized state of electrons, gives the "unperturbed" magnon Hamiltonian which corresponds to the free phonon Hamiltonian of the electron-phonon system. Throughout the present paper, we assume the weak coupling, i.e. a small value of $nJN(0)$, where n is the atomic density and $N(0)$ is the density of one-electron states at the Fermi surface in the absence of the *sf* coupling. The unperturbed magnon energy is then given by

$$\omega_0 = 2nSJ^2(0)\,N(0). \tag{1·4}$$

According to the APW calculation of Dimmock and Freeman[8] for Gd, $\mu = 0.25$ Ry, $N(0) = 0.9$ electrons per atom per spin per eV, and $nJ(0) = 0.08$ eV, so that (ω_0/μ) is of the order of 10^{-2}.

We then introduce the transverse part, i.e. the part of (1·1) which contains σ_x, σ_y and describes the emission and absorption of the magnon by the conduction electron. We cannot regard this transverse part as perturbation, since it is proportional to $\omega_0^{1/2}$. The situation is quite similar in the problem of electron-phonon coupling, which Migdal[12] has solved by the method of Green functions. In § 3, we shall see that the Migdal approximation can be applied also to our present problem. In the terminology of Green functions, the effects of the transverse part are classified as the electron self-energy Σ, the magnon self-energy Π and the renormalized electron-magnon vertex Γ. The Migdal approximation means the expansion of this Γ into powers of (ω_0/μ). As far as we accept the above estimate of this parameter, we need only to retain the lowest order term, i.e. the bare coupling. Then the Dyson equations can be solved to determine Σ and Π.

In analogy with the electron-phonon case and as we shall indeed see in § 5, the electron self-energy Σ at zero temperature, when regarded as a function of excitation energy ϵ (measured from μ), is appreciable only when $|\varepsilon| \lesssim \omega_m$. Here ω_m is the maximum magnon energy and, at most, of the same order of magnitude as ω_0 which is the self-interaction energy of the localized spin. That $\Sigma(\varepsilon)$ is appreciable only for $|\varepsilon| \lesssim \omega_m$ does not depend on the detail of the model, because it physically means that the magnon system cannot respond to the time variation of the electron system faster than ω_m. It leads to two important consequences.

First, as will be discussed in § 4, the magnon self-energy Π may be calculated with use of the spin polarization function of unperturbed electrons,[13] and thus leads to the same magnon dispersion curve as given by the traditional formalism.[11]

Secondly, since Σ itself is of the order of ω_0, we get the so-called wave function renormalization factor as $Z^{-1} = (1 - \partial\Sigma/\partial\varepsilon) \sim 2$. By the use of the general argument given by Luttinger and Ward,[14] the low temperature entropy of our system can be expressed in terms of Σ and Π at zero temperature (§ 6), and we thus see that $Z^{-1} \sim 2$ implies $\lambda_m \sim 1$. This is the most important conclusion we obtain without going into the detail of the model.

§ 2. Reduction of the Hamiltonian

As has been outlined in § 1, we insert the spin wave approximation (1·2) into our Hamiltonian (1·1). The first term on the right of (1·2) then gives the effective field acting on the spin of the conduction electron. Including this field, we write the Hamiltonian of the "free" electron system as

$$H_e = \sum_{\boldsymbol{p}} \psi_{\boldsymbol{p}}^{+} (\xi_{\boldsymbol{p}} - \eta\sigma_z) \psi_{\boldsymbol{p}}, \tag{2·1}$$

$$\eta = nSJ(0), \tag{2·2}$$

where $n = (N/V)$ is the atomic density. In the ground state of this Hamiltonian, the electron system has a uniform spin polarization and we have two Fermi surfaces in momentum space. In our weak coupling limit, $J(0)N(0) \ll 1$, the spin polarization is given by the usual Pauli formula:

$$\frac{1}{V} \sum_{\boldsymbol{p}} \langle \psi_{\boldsymbol{p}}^{+} \sigma_z \psi_{\boldsymbol{p}} \rangle_0 \cong 2\eta N(0). \tag{2·3}$$

Note that the distance between the two Fermi surfaces is of the order of

$$\Delta p = (2\eta/v), \tag{2·4}$$

where v is a certain average electron velocity at the Fermi surface of the unpolarized system.

The second term on the right of (1·2), when inserted into (1·1), gives the longitudinal part of the *sf* exchange coupling.

$$H_{\text{long}}=\sum_{p}\left(\frac{(J(0)}{2V}\right)\psi_p^{+}\sigma_z\psi_p\cdot\sum_{q}(b_q b_q^{+}+b_q^{+}b_q)$$

$$+\sum_{q\neq q'}\sum_{p}\left(\frac{1}{2V}\right)J(\boldsymbol{q}-\boldsymbol{q}')\psi_p^{+}\sigma_z\psi_{p-q+q'}(b_q^{+}b_{q'}+b_{q'}^{+}b_q). \tag{2·5}$$

Here we have introduced the destruction and creation operators of magnons by

$$S_{ix}=\left[\frac{S}{2N}\right]^{1/2}\sum_{q}(b_q+b_{-q}^{+})\exp(i\boldsymbol{q}\cdot\boldsymbol{R}_i),$$

$$S_{iy}=\left[\frac{S}{2N}\right]^{1/2}\sum_{q}i(b_{-q}^{+}-b_q)\exp(i\boldsymbol{q}\cdot\boldsymbol{R}_i), \tag{2·6}$$

$$[b_q,\ b_{q'}^{+}]=\delta_{qq'}. \tag{2·7}$$

The magnon momentum $\boldsymbol{q}$ is, of course, restricted to the reduced zone. For qualitative purposes, we shall often replace the reduced zone by the "Debye" sphere $0\leqq q\leqq q_{\max}$.

On the right of (2·5), we now replace the first term by its average over the uniformly polarized ground state of (2·1). Then we get the unperturbed magnon Hamiltonian

$$H_m=\sum_{q}(\omega_0/2)\,(b_q b_q^{+}+b_q^{+}b_q), \tag{2·8}$$

where

$$\omega_0=\sum_{p}\left(\frac{J(0)}{V}\right)\langle\psi_p^{+}\sigma_z\psi_p\rangle_0. \tag{2·9}$$

Inserting (2·3), we obtain the expression (1·4) for ω_0. In terms of Green functions, we can actually take a more rigorous method of self-consistent subtraction to obtain ω_0, but this gives the same ω_0 as ours in the weak coupling limit. The second term on the right of (2·5) describes the mutual scattering of electron and magnon. The term, when compared with the transverse part given below, is smaller by the spin wave theoretical factor S^{-1}, so that we shall entirely ignore it.

There remains the transverse part of the *sf* exchange coupling

$$H_{\text{trans}}=-\sum_{p}\sum_{q}\frac{\alpha(\boldsymbol{q})}{\sqrt{V}}\,(\psi_{p\downarrow}^{+}\psi_{p-q\uparrow}\,b_q+\psi_{p\uparrow}^{+}\psi_{p-q\downarrow}\,b_{-q}^{+}). \tag{2·10}$$

Here

$$\alpha(\boldsymbol{q})=[2nS]^{1/2}J(\boldsymbol{q})=\left[\frac{\lambda(\boldsymbol{q})\omega_0}{N(0)}\right]^{1/2} \tag{2·11}$$

so that

$$\lambda(\boldsymbol{q})=[J(\boldsymbol{q})/J(0)]^2. \tag{2·12}$$

We thus define our problem by the reduced Hamiltonian

$$H=H_e+H_m+H_{\text{trans}}\,. \tag{2·13}$$

When H_{trans} is regarded as perturbation, the expansion parameter is $\lambda(\boldsymbol{q})$ which is of the order of unity. The simple perturbational expansion makes no sense at any finite order. In terms of Green functions, however, we can take a partial sum of the perturbational series in such a way that the correction arising from residual terms is of the relative order of $(\omega/\mu)^2$. The method has been first applied by Migdal to the coupled electron-phonon system, and will be described briefly in the next section, because our problem slightly differs from the electron-phonon case in some respects.

§ 3. Dyson equations and Migdal approximation

We shall formulate the Migdal approximation in terms of thermal Green functions defined along the imaginary time axis $-it$, where the real variable t runs from $t=0$ to $t=T^{-1}$. One-electron and one-magnon Green functions are defined respectively by

$$G_\sigma(\boldsymbol{p}\,;\,t-t')=-\langle \mathrm{T}_t\,\psi_{\boldsymbol{p}\sigma}(t)\psi^+_{\boldsymbol{p}\sigma}(t')\rangle, \tag{3·1}$$

$$D(\boldsymbol{q}\,;\,t-t')=-\langle \mathrm{T}_t\,b_{\boldsymbol{q}}(t)\,b^+_{\boldsymbol{q}}(t')\rangle. \tag{3·2}$$

Here T_t is Wick's t-ordering operator, $\langle\ \rangle$ means the average over the grand canonical ensemble $\exp[-H/T]$, and

$$\psi_{\boldsymbol{p}\sigma}(t)=\exp(Ht)\ \psi_{\boldsymbol{p}\sigma}\exp(-Ht),\ \ b_{\boldsymbol{q}}(t)=\exp(Ht)\,b_{\boldsymbol{q}}\exp(-Ht). \tag{3·3}$$

As usual, we introduce Fourier transforms by

$$\left.\begin{aligned} G_\sigma(\boldsymbol{p}\,;\,t)&=T\sum_{n=-\infty}^{+\infty}G_\sigma(\boldsymbol{p},\,\epsilon_n)\exp(-\epsilon_n t),\\ D(\boldsymbol{q}\,;\,t)&=T\sum_{m=-\infty}^{+\infty}D(\boldsymbol{q},\,\omega_m)\exp(-\omega_m t),\end{aligned}\right\} \tag{3·4}$$

where

$$\epsilon_n=2\pi i(n+\tfrac{1}{2})T,\ \ \omega_m=2\pi i m T. \tag{3·5}$$

Note that we have ignored Umklapp processes for simplicity, so that Green functions are diagonal in momentum space.

If we ignore H_{trans} in (2·13), we can easily find the explicit expressions for (3·3) and therefore Green functions (3·2). Green functions of free electrons and free magnons are thus

$$G_\sigma^{(0)}(P)=[\epsilon_n-\xi_{\boldsymbol{p}\sigma}]^{-1}, \tag{3·6}$$

$$D^{(0)}(Q)=[\omega_m-\omega_0]^{-1}, \tag{3·7}$$

where

$$P=(\boldsymbol{p},\ \epsilon_n),\quad Q=(\boldsymbol{q},\ \omega_m)$$

and

$$\xi_{p\sigma}=\xi_p-\sigma\eta. \tag{3·8}$$

We may introduce H_{trans} through the perturbational expansions of G and D into powers of H_{trans}, so that we can apply the usual diagram analysis. Thus, the $G_{\uparrow}^{(0)}$ function will be represented by a full line, the $G_{\downarrow}^{(0)}$ function by a dotted line, the $D^{(0)}$ function by a wavy line, and the bare interaction $\alpha(\boldsymbol{q})$ by a vertex, at which one full line, one dotted line and one wavy line are combined together.

Fig. 1.

In contrast to the electron-phonon case, our wavy line should also be directed. For example, the second order corrections to Green functions are shown in Fig. 1. It should also be noticed that, in our case, the vertex correction appears only from the fourth order, as shown in Fig. 2.

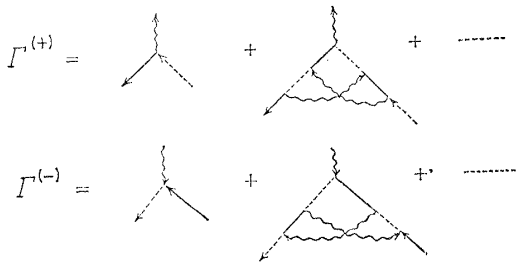

Fig. 2.

Apart from these two points, we can apply a diagram analysis similar to that of the electron-phonon coupling; corrections to Green functions are classified into the electron self-energy $\Sigma_\sigma(P)$, the magnon self-energy $\Pi(Q)$ and vertex functions $\Gamma^{(\pm)}(P', P)$ which represent the renormalization of $\alpha(\boldsymbol{q})$. We thus obtain Dyson equations as exact relations between various functions:

$$G_\sigma(P)=G_\sigma^{(0)}(P)+G_\sigma^{(0)}(P)\Sigma_\sigma(P)G_\sigma(P), \tag{3·9}$$

$$D(Q)=D^{(0)}(Q)+D^{(0)}(Q)\Pi(Q)D(Q), \tag{3·10}$$

$$\left.\begin{aligned}
\Sigma_\uparrow(P)&=-\left(\frac{T}{V}\right)\sum_{P'}\alpha(\boldsymbol{p}-\boldsymbol{p}')D(P'-P)\Gamma^{(-)}(P', P)G_\downarrow(P'),\\
\Sigma_\downarrow(P)&=-\left(\frac{T}{V}\right)\sum_{P'}\alpha(\boldsymbol{p}-\boldsymbol{p}')D(P-P')\Gamma^{(+)}(P', P)G_\uparrow(P'),\\
\Pi(Q)&=\left(\frac{T}{V}\right)\alpha(\boldsymbol{q})\sum_{P}G_\downarrow(P)\Gamma^{(-)}(P, P-Q)G_\uparrow(P-Q).
\end{aligned}\right\} \tag{3·11}$$

Vertex functions may be defined by their perturbational expansions shown in Fig. 2. For our purpose, however, we need only to take the lowest order term, i.e. the bare interaction $\alpha(\boldsymbol{p}'-\boldsymbol{p})$. In fact, we can see

$$\Gamma^{(\pm)}(P+Q, P)=\alpha(\boldsymbol{q})\left\{1+O\left(\frac{\omega_0}{\mu}\right)^2\right\} \tag{3·12}$$

unless the energy-momentum transfer Q is so small that $|\omega_m|\gtrsim vq$. This exceptional case appears nowhere in our following calculation. Therefore we may replace (3·11) by Migdal equations

$$\Sigma_\uparrow(P)=-\left(\frac{T}{V}\right)\sum_{P'}\alpha^2(\boldsymbol{p}-\boldsymbol{p}')D(P'-P)G_\downarrow(P'), \tag{3·13}$$

$$\Sigma_\downarrow(P)=-\left(\frac{T}{V}\right)\sum_{P'}\alpha^2(\boldsymbol{p}-\boldsymbol{p}')D(P-P')G_\downarrow(P'), \tag{3·14}$$

$$\Pi(Q)=\left(\frac{T}{V}\right)\alpha^2(\boldsymbol{q})\sum_{P}G_\downarrow(P)G_\uparrow(P-Q). \tag{3·15}$$

To see (3·12), we take the fourth order term of $\Gamma^{(-)}$ shown in Fig. 2 and ignore the q-dependence of (2·13) for simplicity. Then

$$\Gamma_4^{(-)}(P+Q, P)=\left(\frac{\omega_0}{N(0)}\right)^{5/2}\left(\frac{T}{V}\right)^2\sum_{P''}\sum_{P'}D^{(0)}(P''-P)D^{(0)}(P''-P')$$
$$\times G_\downarrow^{(0)}(P'')G_\uparrow^{(0)}(P+P'+Q-P'')G_\uparrow^{(0)}(P')G_\downarrow^{(0)}(P'+Q). \tag{3·16}$$

Suppose that we first take the sum over ϵ_n''. When $|\epsilon_n''|$ is much larger than $|\varepsilon_n|$ and $|\varepsilon_n'|$, the product of $D^{(0)}$ functions is asymptotically equal to $(\varepsilon_n'')^{-2}$. Hence the main contribution to the sum comes from the regions where either $|\varepsilon_n''-\varepsilon_n'|\lesssim\omega_0$ or $|\varepsilon_n''-\varepsilon_n|\lesssim\omega_0$. Then the product of $D^{(0)}$ functions may be replaced either by $[\omega_0(\varepsilon_n'-\varepsilon_n-\omega_0)]^{-1}$ or by $[\omega_0(\varepsilon_n-\varepsilon_n'-\omega_0)]^{-1}$. Similarly, in taking the sum over ε_n', we receive the factor $(\varepsilon_n'-\varepsilon_n)^{-2}$ when $|\varepsilon_n'-\varepsilon_n|\gg\omega_0$, and the factor ω_0^{-2} when $|\varepsilon_n'-\varepsilon_n|\ll\omega_0$. When we take the sums over $\boldsymbol{p}'$, $\boldsymbol{p}''$, we should remember that $D^{(0)}(Q)$ vanishes for $\boldsymbol{q}$ lying outside the reduced zone. The radius of the reduced zone is of the same order of magnitude as the average radius p_0 of the Fermi surface. Thus, we may estimate (3·16) as

$$\Gamma_4^{(-)}\sim\left[\frac{\omega_0}{N(0)}\right]^{5/2}(TN(0))^2\left(\frac{\omega_0}{T}\cdot\frac{1}{\omega_0}\cdot\frac{1}{\mu}\right)^2=\left[\frac{\omega_0}{N(0)}\right]^{1/2}\left(\frac{\omega_0}{\mu}\right)^2.$$

§ 4. The renormalization of the magnon energy

We now calculate the magnon self-energy (3·15). As we can anticipate from (3·13), (3·14), the electron self-energy has an appreciable value of the order of ω_0 only when $|\varepsilon_n|\lesssim\omega_0$. In (3·15), there is no restriction about the sum over ϵ_n. To the lowest order of (ω_0/μ), therefore, we may replace G functions in (3·15) by $G^{(0)}$ functions. As usual, the sum over ϵ_n is then trans-

formed into the contour integral with the use of the Fermi function $f(z)=[\exp(z/T)+1]^{-1}$. We thus obtain the familiar expression for the polarization function

$$\Pi(Q)=\left[\frac{\alpha^2(\boldsymbol{q})}{V}\right]\sum_{\boldsymbol{p}}{}'\left[\frac{f(\xi_{\boldsymbol{p}-\boldsymbol{q}\uparrow})-f(\xi_{\boldsymbol{p}\downarrow})}{\xi_{\boldsymbol{p}-\boldsymbol{q}\uparrow}-\xi_{\boldsymbol{p}\downarrow}+\omega_m}\right]. \tag{4·1}$$

The analytic continuation can be made by replacing ω_m by any complex energy. In particular, when we approach the real energy ω from above, we get

$$\Pi(\boldsymbol{q},\ \omega+i0^+)=-\alpha^2(\boldsymbol{q})\chi(\boldsymbol{q},\ \omega), \tag{4·2}$$

$$\chi(\boldsymbol{q},\ \omega)=-\frac{1}{V}\sum_{\boldsymbol{p}}{}'\left\{\frac{f(\xi_{\boldsymbol{p}-\boldsymbol{q}\uparrow})-f(\xi_{\boldsymbol{p}\downarrow})}{\xi_{\boldsymbol{p}-\boldsymbol{q}\uparrow}-\xi_{\boldsymbol{p}\downarrow}+\omega+i0^+}\right\}. \tag{4·3}$$

As we shall see below, the damping of the magnon is small. From (3·10), therefore, the renormalized magnon energy $\omega(\boldsymbol{q})$ may be determined as the solution of

$$\omega-\omega_0+\alpha^2\chi'(\boldsymbol{q},\ \omega)=0, \tag{4·4}$$

where χ' is the real part of (4·3). The imaginary part χ'', on the other hand, determines the damping constant by

$$\gamma(\boldsymbol{q})=\alpha^2(\boldsymbol{q})\chi''(\boldsymbol{q},\ \omega(\boldsymbol{q})). \tag{4·5}$$

Now, in accordance with our approximation (2·3), we have

$$\chi'(0,\ 0)\cong-\frac{1}{V}\sum_{\boldsymbol{p}}{}'f'(\xi_{\boldsymbol{p}})=N(0), \tag{4·6}$$

where the prime indicates the derivative of the Fermi function. From (1·4), (2·12) and (4·6), we see that Eq. (4·4) has the solution $\omega=0$ for $\boldsymbol{q}=0$ as it should. This checks the consistency of our approximations made in § 2 and in the present section. For small q, we may thus expect the usual quadratic dispersion

$$\omega(\boldsymbol{q})=\frac{q^2}{2M}. \tag{4·7}$$

The effective mass M of the magnon is of the order of $(q_{\max}^2/\omega_0)$ and, of course, a tensor of second rank in general. We cannot entirely ignore the possibility of obtaining a q-linear term in the expansion of $J(\boldsymbol{q})$ or $\chi'(\boldsymbol{q},\ \omega)$. Then we get

$$\omega(\boldsymbol{q})=v_s q, \tag{4·8}$$

where the magnon velocity v_s is of the order of $(\omega_0/q_{\max})$ and much smaller than the Fermi velocity v. In all cases, we have $vq\gg\omega(\boldsymbol{q})$. This is also true in the short wave length region, $q\sim q_{\max}$, where $\omega(\boldsymbol{q})$ must be of the order of

ω_0. Hence we can put $\omega=0$ in χ' of (4·4), so that the magnon energy is given by

$$\omega(q)=2nS[J^2(0)\chi'(0,0)-J^2(q)\chi'(q,0)]. \tag{4·9}$$

When $vq\ll\mu$, we can introduce the expansion

$$\chi'(q,0)=-\frac{1}{V}\sum_{p}\sum_{n=1}^{\infty}\frac{1}{n!}f^{(n)}(\xi_p+\eta)(\xi_{p-q\uparrow}-\xi_{p\downarrow})^{n-1}$$

$$=-\frac{1}{V}\sum_{n=1}^{\infty}\sum_{p}\frac{1}{n!}(\xi_{p-q}-\xi_p)^{n-1}\left\{f^{(n)}(\xi_p)-\binom{2n-1}{n+1}\eta f^{(n+1)}(\xi_p)+\cdots\right\}.$$

The terms containing η in the curly bracket give rise to corrections of the relative order of (η/μ) or higher and may therefore be ignored. The same is also true when $q\sim q_{\max}$, so that $\eta\ll vq\sim\mu$. In our weak coupling limit, the magnon energy (4·9) is practically the same as given by the traditional formalism.[11]

As for the damping, it is physically obvious that the electron cannot absorb the low energy magnon because of the exchange splitting. The process is prohibited by the energy-momentum conservation law when $vq\ll 2\eta$. The threshold of momentum is of the order of Δp defined by (2·4):

$$\Delta p\sim(J(0)N(0))q_{\max}.$$

In fact, when $q\ll\Delta p$, the expression

$$\chi''(q,\omega)=\frac{1}{V}\sum_{p}[f(\xi_{p-q}-\eta)-f(\xi_p+\eta)]\delta(\xi_{p-q}-\xi_p-2\eta+\omega)$$

vanishes unless $\omega\sim 2\eta(\sim(\omega_0/J(0)N(0)))$. When $q\gg\Delta p$, we can estimate (4·5) as

$$\gamma(q)\cong\lambda(q)\left(\frac{\omega_0}{vq}\right)\omega(q). \tag{4·11}$$

For $q\sim q_{\max}$, the relative damping $[\gamma(q)/\omega(q)]$ is of the order of $(\omega_0/\mu)\sim(J(0)N(0))^2$. Even for $q\sim\Delta p$, it is of the order of $(\omega_0/2\eta)\sim J(0)N(0)$. Therefore, in our weak coupling limit, the damping of the magnon is very small over the whole range of q.

§ 5. The electron self-energy

We are now going to calculate the electron self-energy by (3·13) and (3·14) at $T=0$. We neglect the small damping of the magnon, so that

$$D(Q)=[\omega_m-\omega(q)]^{-1}. \tag{5·1}$$

It is convenient to introduce the spectral representation

$$G_\sigma(P)=\int_{-\infty}^{\infty} d\epsilon \frac{A_\sigma(\boldsymbol{p},\epsilon)}{\epsilon_n-\epsilon}\,. \tag{5·2}$$

As is well known,

$$A_\sigma(\boldsymbol{p},\epsilon)=\left(\frac{1}{\pi}\right)\left(\frac{b_\sigma(\boldsymbol{p},\epsilon)}{[\epsilon-\xi_{p\sigma}-a_\sigma(\boldsymbol{p},\epsilon)]^2+b_\sigma^2(\boldsymbol{p},\epsilon)}\right), \tag{5·3}$$

where

$$\Sigma_\sigma(\boldsymbol{p},\epsilon\pm i0^+)\equiv a_\sigma(\boldsymbol{p},\epsilon)\mp ib_\sigma(\boldsymbol{p},\epsilon). \tag{5·4}$$

Then the sum over ϵ_n' in (3·13) and (3·14) is transformed into the contour integral with the use of the Fermi function. The analytic continuation of $\Sigma(\boldsymbol{p},\epsilon_n)$ can be made by replacing ϵ_n by any complex energy ζ. In particular, at $T=0$, we get

$$\Sigma_\uparrow(\boldsymbol{p},\zeta)=-\frac{1}{V}\sum_{\boldsymbol{p}'}\int_{-\infty}^{\infty} d\epsilon\alpha^2(\boldsymbol{p}-\boldsymbol{p}')A_\downarrow(\boldsymbol{p}',\epsilon)\times\left[\frac{f(\epsilon)}{\epsilon-\zeta-\omega(\boldsymbol{p}'-\boldsymbol{p})}\right], \tag{5·5}$$

$$\Sigma_\downarrow(\boldsymbol{p},\zeta)=-\frac{1}{V}\sum_{\boldsymbol{p}'}\int_{-\infty}^{\infty} d\epsilon\alpha^2(\boldsymbol{p}-\boldsymbol{p}')A_\uparrow(\boldsymbol{p}',\epsilon)\times\left[\frac{1-f(\epsilon)}{\epsilon-\zeta+\omega(\boldsymbol{p}-\boldsymbol{p}')}\right]. \tag{5·6}$$

Now, the self-energy itself is of the order of $\lambda\omega_0$ and therefore not important, as far as we are interested in electronic states near the Fermi surfaces. Its momentum dependence is also small. As we can see from the following calculation, the change of the self-energy will be of the order of $\omega(\Delta p)$, when the momentum changes from one Fermi surface to the other, the distance between them being of the order of Δp. Thus

$$\left(\frac{\partial p}{\partial\xi_p}\right)\left(\frac{\partial\Sigma}{\partial p}\right)\sim\left(\frac{\omega(\Delta p)}{2\eta}\right)\sim J(0)N(0).$$

We need only to take into account the dependence on the direction of momentum in the case of anisotropic Fermi surfaces. Thus, in calculating $\Sigma_\uparrow(\boldsymbol{p},\zeta)$, we take $\boldsymbol{p}=\boldsymbol{p}_\uparrow$, where $\boldsymbol{p}_\uparrow$ means a point on the up spin Fermi surface. Similarly, we calculate $\Sigma_\downarrow(\boldsymbol{p}_\downarrow,\zeta)$ where $\boldsymbol{p}_\downarrow$ means a point on the down spin Fermi surface.

In the following calculation, we restrict ourselves to the upper half-plane of the complex energy, $\mathrm{Im}\,\zeta>0$. Since $\Sigma_\sigma(\boldsymbol{p},i0^+)$ give rise only to small shifts of the chemical potential from the free electron value and also of the exchange splitting, we calculate

$$\Delta\Sigma_\sigma(\boldsymbol{p},\zeta)=\Sigma_\sigma(\boldsymbol{p},\zeta)-\Sigma_\sigma(\boldsymbol{p},i0^+).$$

In calculating $\Delta\Sigma_\uparrow(\boldsymbol{p},\zeta)$ for $|\zeta|\lesssim\omega_0$, the main contribution to the integral over ϵ in (5·5) comes from the region, where $|\epsilon|\lesssim\omega_0$, and the spectral function $A_\downarrow(\boldsymbol{p}',\epsilon)$ is large only when $|\xi_{p'\downarrow}|\lesssim\omega_0$, since a, b are of the order of ω_0. In

taking the sum over $\boldsymbol{p}'$, we may thus replace $\boldsymbol{p}'$ by $\boldsymbol{p}_{\downarrow}'$ everywhere except for $\xi_{\boldsymbol{p}'\downarrow}$; we then transform the sum over $\boldsymbol{p}'$ into integrals over $\xi_{\boldsymbol{p}'\downarrow}$ and over the down spin Fermi surface. Remembering that $A_{\downarrow}(\boldsymbol{p}', \epsilon)$ is Lorentzian in integrating over $\xi_{\boldsymbol{p}'\downarrow}$, we get

$$\Delta\Sigma_{\uparrow}(\boldsymbol{p}_{\uparrow}, \zeta) = -\frac{1}{(2\pi)^3}\int \frac{dF'}{v(\boldsymbol{p}_{\downarrow}')}\int_{-\infty}^{\infty} d\epsilon\alpha^2(\boldsymbol{p}_{\downarrow}-\boldsymbol{p}_{\downarrow}')$$

$$\times f(\epsilon)\left[\frac{1}{\epsilon-\zeta-\omega(\boldsymbol{p}_{\downarrow}'-\boldsymbol{p}_{\uparrow})}-\frac{1}{\epsilon-\omega(\boldsymbol{p}_{\downarrow}'-\boldsymbol{p}_{\uparrow})-i0^{+}}\right]. \tag{5·7}$$

Here $v(\boldsymbol{p})=|\partial\xi_{\boldsymbol{p}}/\partial\boldsymbol{p}|$ is the electron velocity and the surface integral is to be taken over the down spin Fermi surface.

It is convenient to define spectral functions of the magnon by

$$\omega_0 I_{\sigma}(\omega\,;\,\boldsymbol{p}_{\sigma}) = \frac{1}{(2\pi)^3}\int \frac{dF'}{v(\boldsymbol{p}_{-\sigma}')}\alpha^2(\boldsymbol{p}_{\sigma}-\boldsymbol{p}_{-\sigma}')\delta(\omega-\omega(\boldsymbol{p}_{\sigma}-\boldsymbol{p}_{-\sigma}')). \tag{5·8}$$

Remember that $I_{\sigma}(\omega)$ have not only the "Debye cut" at high ω, but also a cutoff at low ω because of the exchange splitting of Fermi surfaces. Unless ω is near to this lower cutoff energy, $I_{\sigma}(\omega)$ do not actually depend on the spin direction, since the separation between two Fermi surfaces is small, $\Delta p \ll q_{\max}$. Furthermore, $I_{\sigma}(\omega)$ are not quite the same as the density of magnon states observed by neutron scattering experiment, because we have the two-dimensional integral in (5·8) and also the factor α^2. As a simple example of (5·8), let us take the case where $\omega(q)$, $\lambda(q)$, and Fermi surfaces are all isotropic (see Fig. 3). Ignoring the exchange splitting wherever possible, we then obtain the spin independent I function

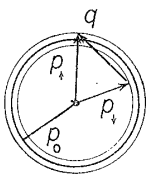

Fig. 3.

$$I(\omega) = p_0^{-2}\int_{q_1}^{q_2} qdq\lambda(q)\delta(\omega-\omega(q)). \tag{5·9}$$

Here $q_1=p_{\uparrow}-p_{\downarrow}\cong(2\eta/v)$ and $q_2\cong\min\{q_{\max}, 2p_0\}$. In particular, if $\lambda(q)$ is independent of q and $\omega(q)$ is quadratic, $I(\omega)$ is constant;

$$I(\omega) = \begin{cases} I_0, & \omega_1<\omega<\omega_2 \\ 0, & \text{otherwise} \end{cases} \tag{5·10}$$

where $\omega_i=\omega(q_i)$ and

$$I_0 = \frac{1}{2}\left(\frac{q_{\max}}{q_0}\right)^2\left(\frac{1}{\omega(q_{\max})}\right). \tag{5·11}$$

On the other hand, if $\omega(q)\propto q$, $\alpha^2(q)\propto q$, we get $I(\omega)\propto\omega^2$; this case corresponds to Fröhlich's model of the coupled electron-phonon system. These models are

not realistic particularly in the short wave length region $q\sim q_{\max}$, where $\lambda(q)$ will decrease and $\omega(q)$ will be more or less constant. In actual crystals, $I_\sigma(\omega)$ may have a number of sharp peaks corresponding to the peaks of the density of magnon states.

Now by use of (5·8), we can write (5·7) as

$$\Delta\Sigma_\uparrow(\boldsymbol{p}_\uparrow,\zeta)=-\omega_0\int_0^\infty d\omega\int_{-\infty}^0 d\epsilon I_\uparrow(\omega;\boldsymbol{p}_\uparrow)\times\left[\frac{1}{\epsilon-\zeta-\omega}-\frac{1}{\epsilon-\omega-i0^+}\right]$$

$$=-\omega_0\int_0^\infty d\omega I_\uparrow(\omega;\boldsymbol{p}_\uparrow)\log\left(\frac{\omega+\zeta}{\omega+i0^+}\right). \tag{5·12}$$

Similarly

$$\Delta\Sigma_\downarrow(\boldsymbol{p}_\downarrow:\zeta)=\omega_0\int_0^\infty d\omega I_\downarrow(\omega;\boldsymbol{p}_\downarrow)\log\left(\frac{\omega-\zeta}{\omega-i0^+}\right). \tag{5·13}$$

When we calculate the low temperature entropy in the next section, we need real parts of $\Delta\Sigma$ for $\zeta=\epsilon+i0^+$, where ϵ is a real energy.

$$\Delta a_\uparrow(\boldsymbol{p}_\uparrow,\epsilon)=\mathrm{Re}\,\Delta\Sigma_\uparrow(\boldsymbol{p}_\uparrow,\epsilon+i0^+)$$

$$=-\omega_0\int_0^\infty d\omega I_\uparrow(\omega;\boldsymbol{p}_\uparrow)\log\left|\frac{\omega+\epsilon}{\omega}\right|, \tag{5·14}$$

$$\Delta a_\downarrow(\boldsymbol{p}_\downarrow,\epsilon)=\omega_0\int_0^\infty d\omega I_\downarrow(\omega;\boldsymbol{p}_\downarrow)\log\left|\frac{\omega-\epsilon}{\omega}\right|. \tag{5·15}$$

§ 6. The low temperature entropy

We now calculate the entropy of our system at low temperatures. According to Luttinger and Ward,[14] we can write the thermodynamic potential of our system in the form

$$\Omega(T)=-T\sum_P\sum_\sigma\exp(\epsilon_n 0^+)\left[\log(\xi_{p\sigma}+\Sigma_\sigma(P)-\epsilon_n)+\Sigma_\sigma(P)G_\sigma(P)\right]$$

$$+T\sum_Q\exp(\omega_m 0^+)\left[\log(\omega_0+\Pi(Q)-\omega_m)+\Pi(Q)D(Q)\right]+\Omega'. \tag{6·1}$$

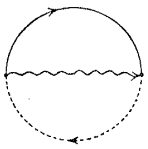

Fig. 4.

Here log means its principal branch and Ω' is the contribution from the so-called skeleton diagrams. The Migdal approximation means that we are to take into account only the lowest order skeleton diagram shown by Fig. 4, in which lines now represent full Green functions. Thus

$$\Omega'(T)=-T^2\sum_{P}{}'\sum_{P'}{}'\alpha^2(\boldsymbol{p}-\boldsymbol{p}')G_{\downarrow}(P)D(P-P')G_{\uparrow}(P'). \tag{6·2}$$

The thermodynamic potential can be regarded as a functional of Σ_σ and Π. By use of Migdal equations (3·13), (3·14) and (3·15), we can show that the first order variation of Ω vanishes;

$$\begin{aligned}\delta\Omega=-T\sum_{P}{}'[&\delta\Sigma_{\uparrow}(P)G_{\uparrow}{}^2(P)\{\Sigma_{\uparrow}(P)+T\sum_{P'}{}'\alpha^2(\boldsymbol{p}-\boldsymbol{p}')D(P'-P)G_{\downarrow}(P')\}\\&+\delta\Sigma_{\downarrow}(P)G_{\downarrow}{}^2(P)\{\Sigma_{\downarrow}(P)+\sum_{P'}{}'\alpha^2(\boldsymbol{p}-\boldsymbol{p}')D(P-P')G_{\uparrow}(P')\}]\\&+T\sum_{Q}{}'\delta\Pi(Q)D^2(Q)\{\Pi(Q)-T\alpha^2(\boldsymbol{q})\sum_{P}{}'G_{\downarrow}(P)G_{\uparrow}(P-Q)\}=0.\end{aligned} \tag{6·3}$$

In calculating the entropy by

$$S=-\left(\frac{\partial\Omega(T)}{\partial T}\right)_{\mu,V}, \tag{6·4}$$

we need only to know the difference $\Delta\Omega(T)=\Omega(T)-\Omega(0)$. The self-energy determined by Migdal equations depends on T even for a fixed value of the energy variable. Because of the stationary property of (6·3), however, we can ignore this implicit dependence on T, as far as we are interested in the first order $\Delta\Omega$. In the following calculation, therefore, we shall take $\Sigma_\sigma(\boldsymbol{p},\zeta)$ and $\Pi(\boldsymbol{q},\zeta)$ calculated for $T=0$. The T-dependence in (6·1) then appears only from taking the sums over ϵ_n and ω_m given by (3·5). At $T=0$, the sums are transformed into integrals along the imaginary axis. In calculating $\Delta\Omega'$ to first order, we need to replace the integral by the sum only along each one of three lines in Fig. 4, the remaining two lines being left in the integral form at $T=0$. Making use of the Migdal equations, we thus obtain

$$\begin{aligned}\Delta\Omega'(T)=&\left(T\sum_{\epsilon_n}{}'-\int_{-i\infty}^{+i\infty}\frac{d\epsilon_n}{2\pi i}\right)\sum_{\boldsymbol{p}}{}'\sum_{\sigma}{}'G_\sigma(\boldsymbol{p},\epsilon_n)\Sigma_\sigma(\boldsymbol{p},\epsilon_n)\\&+\left(T\sum_{\omega_n}{}'-\int_{-i\infty}^{i\infty}\frac{d\omega_m}{2\pi i}\right)\sum_{\boldsymbol{q}}{}'D(\boldsymbol{q},\omega_m)\Pi(\boldsymbol{q},\omega_m).\end{aligned} \tag{6·5}$$

Therefore the entropy takes the additive form

$$S=S_e+S_m, \tag{6·6}$$

$$S_e=-\frac{\partial}{\partial T}\Omega_e(T), \tag{6·7}$$

$$S_m=-\frac{\partial}{\partial T}\Omega_m(T). \tag{6·8}$$

The effective thermodynamic potentials are defined by

$$\Omega_e(T)=T\sum_{P}\sum_{\sigma}\exp(\epsilon_n\cdot 0^+)\log(\tilde{\xi}_{p\sigma}+\Sigma_\sigma(\boldsymbol{p},\epsilon_n)-\epsilon_n), \tag{6·9}$$

$$\Omega_m(T)=-T\sum_{Q}\exp(\omega_m\cdot 0^+)\log(\omega_0+\Pi(\boldsymbol{q},\omega_m)-\omega_m). \tag{6·10}$$

If we insert our approximate expression for $\Pi(Q)$ obtained in § 4, we get the usual magnon entropy from (6·8). Hence we shall confine ourselves to the electronic entropy (6·7).

We first transform the sum over ϵ_n in (6·9) into the contour integral by use of the Fermi function. Integrating by part, we thus obtain

$$\begin{aligned}\Delta\Omega_e=&\text{const}+T\sum_{P}\sum_{\sigma}\int_0^\infty\frac{d\epsilon}{\pi}\log(1+\exp(-\epsilon/T))\\
&\times\Big[\Big\{\Big(1-\frac{\partial a_\sigma(\boldsymbol{p},\epsilon)}{\partial\epsilon}\Big)b_\sigma(\boldsymbol{p},\epsilon)-\frac{\partial b_\sigma(\boldsymbol{p},\epsilon)}{\partial\epsilon}(\epsilon-\tilde{\xi}_{p\sigma}-a_\sigma(\boldsymbol{p},\epsilon))\Big\}\\
&\times\{(\epsilon-\tilde{\xi}_{p\sigma}-a_\sigma(\boldsymbol{p},\epsilon))^2+b_\sigma^2(\boldsymbol{p},\epsilon)\}^{-1}\\
&+\Big\{\Big(1+\frac{\partial a_\sigma(\boldsymbol{p},\epsilon)}{\partial\epsilon}\Big)b_\sigma(\boldsymbol{p},-\epsilon)-\frac{\partial b_\sigma(\boldsymbol{p},-\epsilon)}{\partial\epsilon}\Big(\epsilon+\tilde{\xi}_{p\sigma}+a_\sigma(\boldsymbol{p},-\epsilon)\Big)\Big\}\\
&\times\{(\epsilon+\tilde{\xi}_{p\sigma}+a_\sigma(\boldsymbol{p},-\epsilon))^2+b_\sigma^2(\boldsymbol{p},-\epsilon)\}^{-1}\Big].\end{aligned}$$

Here const means a T-independent term obtained by changing the variable from ϵ to $-\epsilon$ along the negative axis, and a, b are defined by (5·4). The main contribution to the integral over ϵ comes from the region where $|\epsilon|\lesssim T$. Hence, for $T\lesssim\omega_0$, we can again take $\boldsymbol{p}$ on Fermi surfaces everywhere except for $\tilde{\xi}_{p\sigma}$. As before, we thus obtain

$$\begin{aligned}\Delta\Omega_e=&\text{const}+2T\sum_{\sigma}\int\frac{dF_\sigma}{v(\boldsymbol{p}_\sigma)}\int_0^\infty d\epsilon\log(1+\exp(-\epsilon/T))\frac{\partial}{\partial\epsilon}\{\epsilon+c_\sigma(\boldsymbol{p}_\sigma,\epsilon)\}\\
=&\text{const}+2\sum_{\sigma}\int\frac{dF_\sigma}{v(\boldsymbol{p}_\sigma)}\int_0^\infty d\epsilon f(\epsilon)\{\epsilon+c_\sigma(\boldsymbol{p}_\sigma,\epsilon)\}.\end{aligned} \tag{6·11}$$

Here

$$c_\sigma(\boldsymbol{p}_\sigma,\epsilon)=\tfrac{1}{2}(a_\sigma(\boldsymbol{p}_\sigma,-\epsilon)-a_\sigma(\boldsymbol{p}_\sigma,\epsilon)). \tag{6·12}$$

Inserting (6·11) into (6·9), we get

$$S_e=\frac{2}{T}\sum_{\sigma}\int\frac{dF_\sigma}{v(\boldsymbol{p}_\sigma)}\int_0^\infty d\epsilon\Big(-\frac{\partial f(\epsilon)}{\partial\epsilon}\Big)\epsilon\{\epsilon+c_\sigma(\boldsymbol{p}_\sigma,\epsilon)\}. \tag{6·13}$$

The ϵ^2 term in the integrand gives the band theoretical entropy

$$S_e^{(0)}=\frac{2}{T}\sum_\sigma\int\frac{dF_\sigma}{v(\boldsymbol{p}_\sigma)}\int_0^\infty d\epsilon\left(-\frac{\partial f}{\partial\epsilon}\right)\epsilon^2\cong\left(\frac{\pi^2}{3}\right)T\sum_\sigma\int\frac{dF_\sigma}{v(\boldsymbol{p}_\sigma)}$$

$$\cong\left(\frac{2\pi^2}{3}\right)N(0)\,T. \tag{6·14}$$

The additional entropy arises from the coupling with magnons:

$$\Delta S_e=\frac{2}{T}\sum_\sigma\int\frac{dF_\sigma}{v(\boldsymbol{p}_\sigma)}\int_0^\infty d\epsilon\left(-\frac{\partial f}{\partial\epsilon}\right)\epsilon c_\sigma(\boldsymbol{p}_\sigma,\,\epsilon). \tag{6·15}$$

From (5·14) and (5·15), we find

$$c_\sigma(\boldsymbol{p}_\sigma,\,\epsilon)=\frac{1}{2}\int_0^{\omega_0}d\omega\,I_\sigma(\omega\,;\,\boldsymbol{p}_\sigma)\log\left|\frac{\omega+\epsilon}{\omega-\epsilon}\right|. \tag{6·16}$$

We define the average spectral function of the magnon by

$$2N(0)\,I(\omega)\equiv\sum_\sigma\int\frac{dF_\sigma}{v(\boldsymbol{p}_\sigma)}\,I_\sigma(\omega\,;\,\boldsymbol{p}_\sigma). \tag{6·17}$$

Then

$$\Delta S_e=\left(\frac{2\pi^2}{3}\right)A(T)\,N(0)\,T. \tag{6·18}$$

The enhancement factor is given by

$$A(T)=\left(\frac{3\omega_0}{\pi^2T}\right)\int_0^\infty d\omega\int_0^\infty dx\,I(\omega)\,(-\phi'(x))\,x\log\left|\frac{x+(\omega/T)}{x-(\omega/T)}\right|, \tag{6·19}$$

where $\phi'(x)$ is the derivative of $\phi(x)=[\exp(x)+1]^{-1}$. Obviously $A(T)\geqq 0$, so that A certainly means an enhancement of the electronic specific heat. Note that the integral over x in (6·19) has the following limits:

$$J(\nu)\equiv\int_0^\infty dx(-\phi'(x))\,x\log\left|\frac{x+\nu}{x-\nu}\right|\sim\begin{cases}(\pi^2/3\nu), & \nu\gg 1\\ \nu, & \nu\ll 1\end{cases} \tag{6·20}$$

§ 7. Discussion

We remember that $I(\omega)$ has a lower cutoff energy ω_1 as well as an upper cutoff energy ω_2. By use of (6·20), we thus get

$$A(T)\sim\begin{cases}(3\omega_0\langle\omega\rangle/2\pi^2T^2), & T\gg\omega_2,\\ [\omega_0\langle\omega^{-1}\rangle], & T\ll\omega_1.\end{cases} \tag{7·1}$$

Here $\langle\ \rangle$ means the average over the distribution function $I(\omega)$, the normal-

ization integral of which is of the order of unity. The expression (7·1) is of practical use when $I(\omega)$ is of an "Einstein" type in the sense that it has a sharp and prominent peak at a certain energy $\omega=\omega_s$. Then the enhancement factor is of the order of unity, when $T\ll\omega_s$.

In general, the precise T-dependence of $\Lambda(T)$ depends on the form of $I(\omega)$, which in turn is determined by the q-dependence of $\omega(q)$ and $J(q)$. Here we shall not go into the detail of this problem. Instead, we shall take the simplest model (5·10). As has been mentioned already, this model is not very realistic at higher ω. Nevertheless it gives some interesting results. Inserting (5·10) into (6·19), we have

$$\Lambda(T)=\Lambda_0\left(\frac{3}{\pi^2}\right)[\Phi(\nu_2)-\Phi(\nu_1)]. \tag{7·2}$$

Here

$$\Lambda_0=\frac{1}{2}\left(\frac{q_{\max}}{p_0}\right)^2\left[\frac{\omega_0}{\omega(q_{\max})}\right], \tag{7·3}$$

$$\nu_1=(\omega_1/T),\qquad \nu_2=(\omega_2/T)\ , \tag{7·4}$$

$$\Phi(\nu)=\int_0^\infty dx(-\phi'(x))\left[\nu x\log\left|\frac{x+\nu}{x-\nu}\right|+x^2\log(x^2-\nu^2)\right]. \tag{7·5}$$

Note that[11]

$$\Phi(\nu)\sim\begin{cases}(\pi^2/3)\,(1+\log\nu), & \nu\gg1\\ -\beta+(1/2)\nu^2+\cdots, & \nu\ll1,\end{cases} \tag{7·6}$$

where

$$\beta\equiv-2\int_0^\infty dx(-\phi'(x))\,x^2\log x\cong0.6438. \tag{7·7}$$

Now, the upper cutoff energy ω_2 should, of course, be of the order of the Curie temperature, whereas the lower cutoff energy ω_1 is of the order of $\omega_2\times(2\eta/vq_{\max})^2\sim\omega_2(J(0)N(0))^2$. Therefore, in our weak coupling limit, we have the interesting temperature range $\omega_1\ll T\ll\omega_2$. By use of (7·6), then, we obtain

$$\Lambda(T)\cong\Lambda_0\left[\left(1+\left(\frac{3}{\pi^2}\right)\beta\right)+\log\left(\frac{\omega_2}{T}\right)\right]. \tag{7·8}$$

In this model, therefore, we have the specific heat proportional to $T\log(\omega_2/T)$. Such a logarithmic term appears also in the electron-phonon case,[15] but there we get a term proportional to $T^3\log(\theta_D/T)$, since $I(\omega)\propto\omega^2$.

In actual metals, $I(\omega)$ would probably consist of a constant part I_0 and a number of Einstein peaks $I_i(\omega-\omega_{si})$. Then, when T is lower than all the ω_{si},

the specific heat would be the sum of a T-linear term and a logarithmic term. Whether we can detect the logarithmic term depends on the intensity I_0 relative to Einstein peaks. In any case, the enhancement factor $A(T)$ must be of the order of unity.

We expect a similar enhancement also in metals of spiral spin ordering, though the problem is more complicated because the spiral ordering produces new magnetic zone boundaries. It is also possible to extend the theory to magnetic dilute alloys in which there exists some sort of spin ordering. These problems will be discussed in separate papers.

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