N. TSUCHIYA
KODAI MATH. J.
13 (1990), 289—297

ON ELEMENTARY TRANSVERSELY AFFINE FOLIATIONS I

BY NOBUO TSUCHIYA

§1. Introbuction.

Let $\mathcal{F}$ be a codimension one foliation on a closed C^∞-manifold M. Recall that a *transversely affine structure* on $(M, \mathcal{F})$ is given by an $\mathcal{F}$-atlas whose transition functions are elements of the group $\mathbf{Aff}(\boldsymbol{R})$ of affine automorphisms of the real line. By gluing together the transition functions of a transversely affine structure, we get the *holonomy homomorphism* $h: \pi_1(M) \to \mathbf{Aff}(\boldsymbol{R})$. We say that the transversely affine structure is *elementary* it the image of the holonomy homomorphism is abelian. A foliation which admits an elementrry transversely affine structure is called an *elementary transversely affine foliation.* In this and subsequent papers, we study some properties of elementary transversely affine foliations.

It is known that an elementary transversely affine foliation is *almost without holonomy* (i.e., the holonomy group of each non-compact leaf is trivial) (see [Bo] and [In]). Actually, Inaba's proof ([In]) shows that the above result is true for elementary *topological* transversely affine foliations (see §2 for the definition of topological transversely affine structures). In this paper, we consider the converse problem. That is, we study when an almost without holonomy foliation admits a topological transversely affine structure.

Let $\mathcal{F}$ be a transversely oriented codimension one foliation on a closed C^∞-manifold M. Assume $\mathcal{F}$ is almost without holonomy. Let U be a connected component of $M - \cup\{K | K$ is a compact leaf of $\mathcal{F}\}$. Then there exists a holonomy invariant measure μ_U on U. Since $\mathcal{F}$ is transversely oriented, μ_U defines a singular cochain on U. Let $A_U \in \mathbf{SH}^1(U; \boldsymbol{R})$ denote its spherical cohomology class, where $\mathbf{SH}^1(U; \boldsymbol{R}) = \mathbf{H}^1(U; \boldsymbol{R})/\boldsymbol{R}^+$. It is seen that the class A_U does not depend on the choice of the holonomy invariant measure μ_U. We call A_U the *cohomology direction* of $\mathcal{F}_{|U}$ (see §3).

Now we can state the main theorem of this paper.

MAIN THEOREM. *Let $\mathcal{F}$ be a smooth, transversely oriented codimension one foliation on a closed C^∞-manifold M. Then $\mathcal{F}$ admits an elementary topological transversely affine structure if and only if the following conditions are satisfied*

(1) *$\mathcal{F}$ is almost without holonomy.*

Received October 12, 1989; Revised February 1, 1990.

(2) *Let* $\mathcal{U}$ *be the set of connected components of* $M-\cup\{K \mid K$ *is a compact leaf of* $\mathcal{F}\}$. *Then for each* $U\in\mathcal{U}$, *the transverse orientation is directed simultaneously inward on all the compact leaves in* $\partial\bar{U}$ *or simultaneously outward on them.*

(3) *There is a spherical cohomology class* $A\in\mathbf{SH}^1(M; \boldsymbol{R})$ *which satisfies the following condition; for each* $U\in\mathcal{U}$, *we have* $i_U{}^*(A)=sign(U)\cdot A_U$, *where* i_U *denotes the inclusion map of* U *into* M, A_U *denotes the cohomology direction of* $\mathcal{F}_{|U}$ *and* $sign(U)$ *is* $+1$ (*resp.* -1) *if the transverse orientation is directed inward* (*resp. outward*) *on a compact leaf in* $\partial\bar{U}$.

In § 2, we define topological transversely affine structures. In § 3, we prepare some facts about open saturated sets without holonomy. Finally, in § 4, we prove our main theorem. In this paper, we assume that all the foliations are of codimension one and transversely oriented.

§ 2. Topological transversely affine structures.

In this section we define topological transversely affine structures on a manifold. We start with a notation. An orientation preserving affine automorphism of the real line $\boldsymbol{R}$ is a transformation of the form $t\mapsto a\cdot t+b$ where a is a positive real number and b is a real number. We denote the group of all these transformations by $\mathbf{Aff}^+(\boldsymbol{R})$.

Now, let M be a connected C^∞-manifold. Let $\tilde{M}$ be the universal covering space of M.

(2.1) DEFINITION. A *transversely affine structure* (resp. a *topological transversely affine structure*) on M is a pair (D, h) where D is a C^∞-submersion (resp. a topological submersion with the property that $D^{-1}(t)$ is a smooth submanifold of $\tilde{M}$ for each $t\in Im(D)$) from $\tilde{M}$ to $\boldsymbol{R}$ and h is a homomorphism from $\pi_1(M)$ to $\mathbf{Aff}^+(\boldsymbol{R})$ which satisfies the following equivariance condition:

$$D(\gamma x)=h(\gamma)(D(x)), \qquad \text{where } \gamma\in\pi_1(M) \text{ and } x\in\tilde{M}.$$

We call D the *developing submersion* and h the *holonomy homomorphism.*

(2.2) *Remark.* Let (D, h) be a (topological) transversely affine structure on M. Let K be a normal subgroup of $\pi_1(M)$ which is contained in the kernel of h. Let $\bar{M}=K\backslash\tilde{M}$. Then the developing submersion D naturally defines a submersion $\bar{D}: \bar{M}\to\boldsymbol{R}$. The map $\bar{D}$ is also called a developing submersion.

Let (D, h) be a (topological) transversely affine structure on M. Let $\tilde{\mathcal{F}}$ be the foliation on $\tilde{M}$ defined by the level surfaces of D. Then, since D is h-equivariant, the foliation $\tilde{\mathcal{F}}$ projects down to a foliation $\mathcal{F}$ on M. We call $\mathcal{F}$ the *underlying foliation* of the (topological) transversely affine structure (D, h).

(2.3) DEFINITION. Let $\mathcal{F}$ be a smooth foliation on M. We say that $\mathcal{F}$ *admits a transversely affine structure* (D, h) if the underlying foliation of (D, h) is diffeomorphic to $\mathcal{F}$ under a diffeomorphism which preserves the transverse orientation. We say that $\mathcal{F}$ *admits a topological transversely affine structure* (D, h) if there is a homeomorphism f of M which sends each leaf of the underlying foliation of (D, h) diffeomorphically onto a leaf of $\mathcal{F}$ and which preservcs the transverse orientation. We alse say that $\mathcal{F}$ is a *(topological) transversely affine foliation* if $\mathcal{F}$ admits a (topological) transversely affine structure.

(2.4) *Remark.* There are many C^∞-foliations which admit topological transversely affine structures but which does not admit transversely affine structures. We give two examples.

A first example is given by foliations of the 2-torus T^2 consisting of two Reeb components. Let $\hat{\mathcal{F}}$ be the foliation of the punctured plane $\boldsymbol{R}^2-\{0\}$ defined by the level surfaces of the submersion $\hat{D}(x, y)=y$. The quotient of $\hat{\mathcal{F}}$ by the $\boldsymbol{Z}$-action generated by $(x, y)\mapsto(2x, 2y)$ is a Reeb foliation $\mathcal{F}$ of T^2, which has an obvious transversely affine structure (see [Bo]). Consider a C^∞-foliation $\mathcal{F}_1$ of T^2 such that

(1) $\mathcal{F}_1$ is homeomorphic to $\mathcal{F}$ and
(2) the holonomy group of each compact leaf of $\mathcal{F}_1$ is infinitely tangent to the identity.

Then the foliation $\mathcal{F}_1$ does not admit a transversely affine structure since each non-trivial holonomy of a transversely affine foliation has non-trivial linear part. But it is obvious that the foliation $\mathcal{F}_1$ does admit a topological transversely affine structure.

A second example is given by the suspension foliation $\mathcal{F}_g$ of a diffeomorphism g of S^1 which is topologically conjugate to a rotation. It is easy to see that the foliation $\mathcal{F}_g$ admits a topological transversely affine structure. It is also easy to see that the foliation $\mathcal{F}_g$ admits a transversely affine structure if and only if g is smoothly conjugate to a rotation.

Let $\mathcal{F}$ be a topological transversely affine foliation on a C^∞-manifold M. As in [S, Theorem 6], one can prove that M admits a possibly new C^∞-structure with respect to which $\mathcal{F}$ is transversely affine. The precise statement is the following.

(2.5.) PROPOSITION. *Let (D, h) be a topological transversely affine structure on a connected C^∞-manifold M. Let $\mathcal{F}$ be the underlying foliation. Then there is a C^∞-manifold M' and a transversely affine foliation $\mathcal{F}'$ on M' which satisfies the following conditions.*

(1) *M' is identical to M as a topological manifold.*
(2) *The identity map ι from M to M' sends each leaf of $\mathcal{F}$ diffeomorphically onto a leaf of $\mathcal{F}'$.*

Proof. Let $\tilde{M}$ be the universal covering space of M and $\tilde{\mathcal{F}}$ the lift of $\mathcal{F}$.

Since $\tilde{\mathcal{F}}=\{D^{-1}(t)\}_{t\in Im(D)}$, there exists uniquely a new differentiable structure on $\tilde{M}$ in which each leaf of $\tilde{\mathcal{F}}$ is a differentiable submanifold and in which the developing submersion D is a smooth map. Let $\tilde{M}'$ be this differentable manifold and $\tilde{i}: \tilde{M}\to\tilde{M}'$ the identity map. From the equivariance condition it follows that the action of $\pi_1(M)$ on $\tilde{M}'$ given by $\gamma(x')=\tilde{i}\circ\gamma\circ\tilde{i}^{-1}(x')$ (where $\gamma\in\pi_1(M)$ and $x'\in\tilde{M}'$) is differentiable. Thus the differentiable manifold $M'=\pi_1(M)\diagdown\tilde{M}'$ satisfies the desired properties. We have proved the proposition.

Let (D, h) be a (topological) transversely affine structure. Let Γ denote the image of the holonomy homomorphism h. We call Γ the *global holonomy group* of (D, h).

(2.6.) DEFINITION. A (topological) transversely affine structure is said to be *elementary* if the global holonomy group Γ is abelian. A foliation which admits an elementary (topological) transversely affine structure will be called an *elementary (topological) transversely affine foliation.*

See [Bo] and [G, Chapter 3] for fundamental properties on transversely affine foliations. Some of the results there are valid for topological transversely affine foliations. See [E, § 1] for general backgrounds on transversely (G, X)-foliations.

§ 3. Open saturated sets without holonomy.

Let $\mathcal{F}$ be a C^∞-foliation on a closed manifold M. Let U be a connected component of $M-\cup\{K|K$ is a compact leaf of $\mathcal{F}\}$. Assume that each leaf in U has trivial holonomy group. Such a set is a building block of an almost without holonomy foliation, and the structure of such a set is fairly well-understood (see e.g., [He], [Im], [C-C, § 4] and [T, § 6]). There are several equivalent ways to describe the situation. One can use the notion of:

(1) holonomy invariant measures ([P]),
(2) leaf preserving flows ([C-C], [T]),
(3) C^0-closed one forms defining $\mathcal{F}$ ([Im])
or (4) foliated J-bundles (M-M-T]).

In this paper we adopt the notion of holonomy invariant measures, and show that it is also equivalent to

(5) topological transversely affine structures whose global holonomy groups are contained in the group of translations.

Let U be as above. Let X be the disjoint union of all immersed transverse arcs in U. A *holonomy invariant measure* on $(U, \mathcal{F}_{|U})$ is a non-trivial Borel measure on X which is invariant under the action of the holonomy pseudogroup of $\mathcal{F}_{|U}$ (see [H-H, p. 260]).

(3.1) LEMMA (see [C-C, p. 103]). *There exists a holonomy invariant measure* μ_U *on* $(U, \mathcal{F}_{|U})$.

(3.2) LEMMA. *Let* μ_U *and* μ'_U *be holonomy invariant measures on* $(U, \mathcal{F}_{|U})$. *Then there exists a positive constant* λ *such that* $\mu'_U(\gamma)=\lambda\mu_v(\gamma)$ *for every closed curve* γ *transverse to* $\mathcal{F}_{|U}$.

Proof. Either all leaves of $\mathcal{F}_{|v}$ are dense in U or all leaves of $\mathcal{F}_{|U}$ are closed in U (see e.g., [Im]). If all leaves of $\mathcal{F}_{|U}$ are dense, it is easy to see that the holonomy invariant measure μ_U is unique up to multiplicative constant. Assume all leaves of $\mathcal{F}_{|U}$ are closed in U. Then, by standard arguments in foliation theory, one can find a transverse simple closed curve C in U which intersects with every leaf of $\mathcal{F}_{|U}$ exactly once. And furthermore, one can see that every closed curve C' transverse to $\mathcal{F}_{|U}$ is freely homotopic to a multiple C^m of C, $m\in \boldsymbol{Z}$ (see e.g., [T, §5]). Hence, it we put $\lambda=\mu'(C)/\mu(C)$, we have $\mu'(C')=\mu'(C^m)=m\cdot\mu'(C)=m\cdot\lambda\cdot\mu(C)=\lambda\cdot\mu(C^m)=\lambda\cdot\mu(C')$. This proves the lemma.

Let μ_U be a holonomy invariant measure on $(U, \mathcal{F}_{|U})$. Then μ_U defines a real valued singular cocycle Φ_{μ_U} which is characterized by the following conditions (see [P, p. 345] and [H-H, p. 278]):

(1) Let $\gamma: [0, 1]\to U$ be an arc transverse to $\mathcal{F}$ whose orientation is compatible (resp. incompatible) with the transverse orientation of $\mathcal{F}$. Then we have $\Phi_{\mu_U}(\gamma)=\mu_U(\gamma)$ (resp. $-\mu_U(\gamma)$).

(2) Let $\gamma: [0, 1]\to U$ be an arc which is contained in a leaf of $\mathcal{F}$. Then we have $\Phi_{\mu_U}(\gamma)=0$.

(3) Let $\gamma_0, \gamma_1: [0, 1]\to U$ be arcs in U which are homotopic relative to endpoints. Then we have $\Phi_{\mu_U}(\gamma_0)=\Phi_U(\gamma_1)$.

(3.3) DEFINITON. Let $q_{\mu_U}\in \boldsymbol{H}^1(U; \boldsymbol{R})\cong\mathrm{Hom}(\pi_1(U), \boldsymbol{R})$ be the cohomology class of the cocycle Φ_{μ_U}. We consider q_{μ_U} as a homomorphism from $\pi_1(U)$ to $\boldsymbol{R}$ and call it the *period homomorphism* associated with the holonomy invariant measure μ_U.

Let μ_U and q_{μ_U} be as above. Let $A_{\mu_U}=\{q_{\mu_U}\}\in\mathbf{SH}^1(U; \boldsymbol{R})=\mathbf{H}^1(U; \boldsymbol{R})/\boldsymbol{R}^+$ be the spherical cohomology class of q_{μ_U}. Since it is known (see e.g., [Im]) that every closed curve in U is homotopic either to a closed curve transverse to $\mathcal{F}$ or to a closed curve contained in a leaf of $\mathcal{F}$, it follows from (3.2) and the definition of Φ_{μ_U} that the class A_{μ_U} does not depend on the choice of the holonomy invariant measure μ_U. So we denote the class A_{μ_U} simply by A_μ. We summarize this in the following definition.

(3.4) DEFINITION (see [H-H, p. 194]). The spherical cohomology class of the period homomorphism q_{μ_U} is called the *cohomology direction* of $\mathcal{F}_{|U}$, and is denoted by A_U.

We consider a special class of holonomy invariant measures.

(3.5) DEFINITION. A holonomy invariant measure μ_U on $(U, \mathcal{F}_{|U})$ is said to to be *non-singular* if the following condition is satisfied:

Let $\gamma: [0, 1]\to U$ be an immersion transverse to $\mathcal{F}$. Then there is a positive continuous function $f_\gamma: [0, 1]\to \boldsymbol{R}^+$ such that $\gamma^*(\mu)=f_\gamma(t)\cdot dt$.

(3.6) LEMMA. *There exists a non-singular holonomy invariant measure μ_U on $(U, \mathcal{F}_{|U})$.*

Proof. By a theorem of Imanishi [Im, Theorem S′], there exist a smooth non-singular closed one form ω on U and a homeomorphism h of U such that h sends each leaf of $\mathcal{F}_{|U}$ diffeomorphically onto a leaf of the foliation $\mathcal{F}_\omega$ defined by ω. Let $\mu_U=h^*\mu_\omega$ where μ_ω denotes the holonomy invariant measure on $(U, \mathcal{F}_\omega)$ defined by ω. It is obvious that μ_U is a non-singular holonomy invariant measure. This proves the lemma.

(3.7) LEMMA. *Let μ_U be a non-singular holonomy invariant measure and q_{μ_U} the period homomorphism associated with μ_U. Let K be a subgroup of $\pi_1(U)$ which is contained in the kernel of q_{μ_U} and $\hat{U}$ the covering space of U defined by the group K.*

Then there exists a topological submersion $F_U: \hat{U}\to \boldsymbol{R}$ which satisfies the following conditions:

(1) *Let $\hat{\mathcal{F}}$ be the foliation on $\hat{U}$ induced from $\mathcal{F}_{|U}$. Then $\hat{\mathcal{F}}$ is the pullback of the point foliation of $\boldsymbol{R}$ by F_U.*

(2) *For each $\gamma\in\pi_1(U)$ and $x\in\hat{U}$, we have $F_U(\gamma x)=q_{\mu_U}(\gamma)+F_{\hat{U}}(x)$.*

(3) *The lift of the non-singular holonomy invariant measure μ_U to $\hat{U}$ coincides with the pullback of the Lebesgue measure of $\boldsymbol{R}$ up to multiplication by a positive continuous function.*

(4) *Let $C(\hat{U})$ be the completion of $\hat{U}$ with respect to a metric induced from a Riemannian metric of M. Let $c: [0, 1]\to C(\hat{U})$ be a continuous path such that $c([0, 1])\cap\hat{U}=c((0, 1])$ and $c_{|(0,1]}$ is transverse to $\hat{\mathcal{F}}$. If the orientation of c is compatible (resp. incompatible) with the transverse orientation of $\hat{\mathcal{F}}$, then we have $\lim_{t\to 0} F_U(c(t))=-\infty$ (resp. $+\infty$).*

Proof. Choose a base point $x_U\in U$, and consider the space $P(U, x_U)$ of all continuous paths in U based at x_U: $P(U, x_U)=\{a: [0, 1]\to U, a(0)=x_U\}$. Then the cochain Φ_{μ_U} restricts to a map $F: P(U, x_U)\to\boldsymbol{R}$. Define an equivalence relation $\sim$ on $P(U, x_U)$ by $a\sim b$ if and only if $a(1)=b(1)$ and $a*b^{-1}\in K$. Then the space $P(U, x_U)/\sim$ is homeomorphic to the space $\hat{U}$. Since $K\subset \mathrm{Ker}(q_{\mu_U})$, we have $F(a)=F(b)$ if $a\sim b$. So the map F induces a map $F_{\hat{U}}: \hat{U}\to\boldsymbol{R}$. It is easy to see that the map $F_{\hat{U}}$ is a topological submersion and satisfies the conditions (1), (2) and (3).

We prove the condition (4). Let $c: [0, 1] \to C(\hat{U})$ be a path as in (4) whose orientation is compatible with the transverse orientation of $\hat{\mathcal{F}}$. Then there exists an element g of the holonomy pseudogroup of $\hat{\mathcal{F}}$ which sends $c(t)$ $(0<t\leqq 1)$ to $g(c(t))=c(\alpha(t))$ where $\alpha: (0, 1] \to (0, 1]$ is a contraction to 0 (see e.g., [C-C, Lemma 2]). Then we have, by the holonomy invariance of μ_U,

$$\begin{aligned}\lim_{t\to 0} F_U(c(t)) &= \lim_{n\to\infty} F_U(c(\alpha^n(1)))\\ &= \lim_{n\to\infty} [F_U(c(1)) + \{-F_{\hat{U}}(c(1)) + F_{\hat{U}}(c(\alpha(1)))\}\\ &\qquad + \{-F_{\hat{U}}(c(\alpha(1))) + F_{\hat{U}}(c(\alpha^2(1)))\} + \cdots\\ &\qquad + \{-F_{\hat{U}}(c(\alpha^{n-1}(1))) + F_{\hat{U}}(c(\alpha^n(1)))\}]\\ &= \lim_{n\to\infty} [F_U(c(1)) - n\cdot\{F_U(c(1)) - F_U(c(\alpha(1)))\}]\\ &= -\infty\,.\end{aligned}$$

This completes the proof of the lemma.

The lemma (3.7) says that the pair (F_U, q_{μ_U}) defines a topological transversely affine structure on $\mathcal{F}_{|U}$ whose global holonomy group is contained in the group of translations $\boldsymbol{R} \subset \mathbf{Aff}^+(\boldsymbol{R})$. We remark that the converse statement holds.

(3.8) LEMMA. *Let U be a connected, open saturated set without holonomy. Let (D, q) be an elementary topological transversely affine structure on $\mathcal{F}_{|U}$ whose global holonomy group is contained in the group of translations. Then the pull-back of the Lebesgue measure of $\boldsymbol{R}$ by the map D defines a non-singular holonomy invariant measure μ_U, and the period homomorphism q_{μ_U} associated with μ_U coincides with the homomorphism q.*

The proof is easy and is omitted.

§ 4. Proof of the main theorem.

In this section we prove our main theorem.

First, we prove the "only if" part. The assertions (1) and (2) were proved by Inaba (see [In, Theorem 1.2 and Lemma 3.4]). (Although Inaba treats transversely affine foliations, his proof is valid for topological transversely affine foliations.)

We prove the assertion (3). Let $\mathcal{F}$ be a foliation on a closed manifold M which admits an elementary topological transversely affine structure (D, h). Since the global holonomy group $\Gamma = Im(q)$ is abelian, either Γ is contained in the group of translations or Γ is a subgroup of $\mathbf{Aff}^+(\boldsymbol{R})$ which fixes a point $t_0 \in \boldsymbol{R}$.

Assume Γ is contained in the group of translations. Then the foliation $\mathcal{F}$ is without holonomy, and the condition (3) is automatically satisfied.

Assume Γ fixes a point $t_0 \in \boldsymbol{R}$. By passing to an equivalent structure, we may assume that $t_0=0$ and $\Gamma \subset \mathbf{GL}^+(1, \boldsymbol{R}) \cong \boldsymbol{R}^+$. Define a homomorphism $q: \pi_1(M) \to \boldsymbol{R}$ by $q=log \circ h$, and let A denote the spherical cohomology class of q. We show that $i_U^*(A)=sign(U) \cdot A_U$ for each $U \in \mathcal{U}$.

Let $\pi: \tilde{M} \to M$ be the universal covering of M. Note that the union of all compact leaves of $\mathcal{F}$ coincides with $\pi \circ D^{-1}(0)$ (see [In]). So, if U is an element of $\mathcal{U}$ with $sign(U)=+1$ (resp. -1), then we have $D(\pi^{-1}(U)) \subset (0, \infty)$ (resp. $(-\infty, 0)$). For each $U \in \mathcal{U}$, define a homomorphism $q_U: \pi_1(U) \to \boldsymbol{R}$ by $q_v= sign(U) \cdot q \circ i_{U*}$ and a continuous map $D_U: \pi^{-1}(U) \to \boldsymbol{R}$ by $D_U=log(sign(U) \cdot D_{|\pi^{-1}(U)})$. Then the pair (D_U, q_U) defines a topological transversely affine structure on $\mathcal{F}_{|U}$ whose global holonomy group $Im(q_U)$ is contained in the group of translations. By (3.8), the pullback of the Lebesgue measure of $\boldsymbol{R}$ by D_U defines a holonomy invariant measure μ_U and the period homomorphism associated with μ_U coincides with q_U. So the cohomology direction A_U of U is the class of q_U. Thus we have $i_U^*(A)=sign(U) \cdot A_U$.

Now we prove the "if" part of the main theorem. Assume that the foliation $\mathcal{F}$ on a closed manifold M satisfies the conditions (1), (2) and (3) of the main theorem. Note that the cardinality of $\mathcal{U}$ is finite by (2). If $\mathcal{U}=\emptyset$, then all leaves of $\mathcal{F}$ are compact and the bundle foliation $\mathcal{F}$ has an obvious elementary transversely affine structure. If $\mathcal{U}=\{M\}$, then the whole mainfold is a component of the type considered in § 3. By (3.7), the foliation $\mathcal{F}$ admits an elementary topological transversely affine structure.

Assume $\mathcal{U}$ has at least two elements. Note then that, for each $U \in \mathcal{U}$, the closure $\bar{U}$ of U in M is naturally homeomorphic to the metric completion $C(U)$ of U by the assumption (2). By the assumptions (1) and (3), there exist a homomorphism $q: \pi_1(M) \to \boldsymbol{R}$ and a holonomy invariant measure μ_U for each $U \in \mathcal{U}$ such that $q \circ i_{U*}=sign(U) \cdot q_{\mu_U}$ where q_{μ_U} is the period homomorphism associated with μ_U. By (3.2) and (3.6), we may assume that each μ_U is non-singular.

Let $\pi: \tilde{M} \to M$ be the universal covering of M, and let $G=\pi_1(M)$. For each $U \in \mathcal{U}$, let $H_U=Im(i_{U*}: \pi_1(U) \to \pi_1(M)) \subset \pi_1(M)$, $K_U=Ker(i_{U*}: \pi_1(U) \to \pi_1(M)) \subset \pi_1(U)$ and let $\hat{U}$ be the covering space of U defined by K_U. Then the group $H_U \cong G/K_U$ acts naturally on the space $\hat{U}$. By general theory on covering spaces, the G-spaces $\pi^{-1}(U)$ is G-equivariantly diffeomorphic to the twisted product $G \times_{H_U} \hat{U}$ (see [Br, p 46] for the notation). Define $F_{\hat{U}}: \hat{U} \to \boldsymbol{R}$ as in (3.7), and define $D_U: \pi^{-1}(U) \to \boldsymbol{R}$ by $D_U(g, x)=sign(U) \cdot exp\{q(g)+sign(U) \cdot F_U(x)\}$, where we have identified $\pi^{-1}(U)$ with $G \times_{H_U} \hat{U}$.

By (4) of (3.7), $D_U(x)$ tends to zero as x tends to $\partial(C(\pi^{-1}(U)))=\pi^{-1}(\partial \bar{U})$. So we can define a topological submersion $D: \tilde{M} \to \boldsymbol{R}$ by

$$D(x)=D_U(x) \qquad \text{if } x\in\pi^{-1}(U)$$

and

$$D(x)=0 \qquad \text{if } x\in\cup\{\pi^{-1}(\partial\bar{U}) \mid \mathcal{V}\in\mathcal{U}\}.$$

Then, from the H_U-equivariance of $F_{\hat{U}}$ for each $U\in\mathcal{U}$, one can easily see that the map D is G-equivariant with respect to the homomorphism $h=exp\circ q$: $\pi_1(M)\to \boldsymbol{R}^+\cong \mathbf{GL}^+(1, \boldsymbol{R})\subset \mathbf{Aff}^+(\boldsymbol{R})$. Thus the pair (D, h) defines an elementary topological transversely affine structure on $\mathcal{F}$. This completes the proof of the main theorem.

References

[Bo] Bobo Seke, Sur les structures transversalment affines des feuilletages des codimension un, Ann. Inst. Fourier, Grenoble **30** (1980), 1-29.

[Br] G. Bredon, Introduction to compact transformation groups, Academic Press, 1972.

[C-C] J. Cantwell and L. Conlon, Growth of leaves, Comment. Math. Helvetici **53** (1978), 91-111.

[E] D. B. A. Epstein, Transversely hyperbolic 1-dimensional foliations, Astérisque **116** (1984), 53-69.

[G] C. Godbillon, Feuilletages, Études géométriques I, Publ. Inst. de Recherche Math. Avancée, Univ. Louis Pasteur, Strasbourg, 1986.

[He] G. Hector, Sur les feuilletages presque sans holonomie, C. R. Acad. Sci. Paris t **274** (1972), 1703-1706.

[H-H] G. Hector and U. Hirsch, Introduction to the geometry of foliations, Part B, Aspects of Mathematics E3, Vieweg, 1983.

[Im] H. Imanishi, Structures of codimension one foliations which are almost without holonomy, J. Math. Kyoto Univ. **16** (1976), 93-99.

[In] T. Inaba, Resilient leaves in transversely affine foliations, Tôhoku M. J. **41** (1989), 625-631.

[M-M-T] T. Mizutani, S. Morita and T. Tsuboi, The Godbillon-Vey class of codimension one foliations which are almost without holonomy, Ann. of Math. **113** (1981), 515-527.

[P] J. F. Plante, Foliations with measure preserving holonomy, Ann. of Math. **102** (1975), 327-361.

[S] R. Sackstedeer, Foliations and pseudogroups, Amer. J. Math. **87** (1965), 79-102.

[T] N. Tsuchiya, Growth and depth of leaves, J. Fac. Sci. Univ. Tokyo **26** (1979), 473-500.

Department of Mathematics
Faculty of Science
Tokyo Institute of Technology