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ON ELLIPTIC EQUATIONS WITH PIECEWISE CONSTANT COEFFICIENTS. II

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SUMMARY - In this work we prove an existence and uniqueness theorem for solutions in $W^{2,p}(R^n)$ of second order linear elliptic equations, whose coefficients are constant-valued in the half-spaces R^n_+ and R^n_-

1. Introduction and statement of the problem.

In this paper we are interested in solving a second order linear partial differential equation of elliptic type, whose coefficients are constant-valued in the half-spaces $R^n_+ = \{x \in R^n : x_1 > 0\}$ and $R^n_- = \{x \in R^n : x_1 < 0\}$. We carry on our research, begun in [6], where square summable solutions with square summable second derivatives are dealt with: in this work we look for solutions in $W^{2,p}(R^n)(1 < p < +\infty)$. We recall that $W^{2,p}(R^n)$ denotes the Sobolev space (1) of all functions of $L^p(R^n)$, that have derivatives in the sense of distributions of the first two orders belonging to $L^p(R^n)$; $W^{2,p}(R^n)$ is a Banach space with respect to the norm:

$$\| u \|_{W^{2,p}(R^n)} = \left\{ \int_{R^n} \left[| u |^p + \sum_{r,j=1}^{n} \left| \frac{\partial^2 u}{\partial x_r \, \partial x_j} \right|^p \right] dx \right\}^{1/p}$$

Our equation is

$$(1) \qquad \begin{cases} L^+ u \equiv \sum_{r,j=1}^{n} a^+_{rj} \dfrac{\partial^2 u}{\partial x_r \, \partial x_j} + \sum_{j=1}^{n} a^+_j \dfrac{\partial u}{\partial x_j} - h^+ u = f & \text{in } R^n_+ \\ L^- u \equiv \sum_{r,j=1}^{n} a^-_{rj} \dfrac{\partial^2 u}{\partial x_r \, \partial x_j} + \sum_{j=1}^{n} a^-_j \dfrac{\partial u}{\partial x_j} - h^- u = f & \text{in } R^n_- \end{cases}$$

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(1) For the properties of Sobolev spaces see, for instance, [2] or [9].

where a_{rj}^{+}, a_{rj}^{-}, a_{j}^{+}, a_{j}^{-}, h^{+}, h^{-} $(r, j = 1, 2, \dots, n)$ are real constants with the following properties:

i) the $n \times n$ matrices $A^{+} = (a_{rj}^{+})$ and $A^{-} = (a_{rj}^{-})$ are symmetric and positive definite;

ii) $$h^{+} > 0, \; h^{-} > 0;$$

and f is an assigned function in $L^{p}(R^{n})$ $(1 < p < +\infty)$.

In the following we shall be interested in the case $n \geq 3$.

THEOREM. *Equation* (1) *has a unique solution* $u \in W^{2,p}(R^{n})$ *for every* $f \in L^{p}(R^{n})$ $(1 < p < +\infty)$. *There exists a constant* C *independent of* u *such that the following estimate holds*:

$$\| u \|_{W^{2,p}(R^{n})} \leq C \| f \|_{L^{p}(R^{n})} \tag{2}$$

From the theorem it follows that, if p is large enough $(p > n/2)$, the solution is continuous across the interface $x_1 = 0$: if $p > n$, also the first derivatives are continuous across the interface.

The method used to prove the existence of the solution consists in solving the Neumann problems

$$\begin{cases} L^{+} u^{+} = f \\ u^{+} \in W^{2,p}(R_{+}^{n}) \\ \dfrac{\partial u^{+}}{\partial x_1}(0+, \cdot) = g \end{cases} \tag{3} \qquad \begin{cases} L^{-} u^{-} = f \\ u^{-} \in W^{2,p}(R_{-}^{n}) \\ \dfrac{\partial u^{-}}{\partial x_1}(0-, \cdot) = g \end{cases} \tag{4}$$

where g is some function in $W^{1-\frac{1}{p},p}(R^{n})$, $\dfrac{\partial u^{+}}{\partial x_1}(0+, \cdot)$ and $\dfrac{\partial u^{-}}{\partial x_1}(0-, \cdot)$ denote respectively the traces of $\dfrac{\partial u^{+}}{\partial x_1}$ and $\dfrac{\partial u^{-}}{\partial x_1}$ on $x_1 = 0$. Remember that $W^{s,p}(R^{n})$, for $s > 0$ non integer, is the space of all functions which together with all derivatives of order $< s$ (in the sense of distributions) are in $L^{p}(R^{n})$ and satisfy the inequality

$$\| u \|_{W^{s,p}(R^{n})} = \left\{ \sum_{j=0}^{[s]} \sum_{|\alpha| = j} \left[\int_{Rn} | D^{\alpha} u |^{p} \, dx + \right.\right.$$
$$\left.\left. + \int_{R^{n}} dx \int_{R^{n}} \frac{| D^{\alpha} u(x) - D^{\alpha} u(y) |^{p}}{| x - y |^{n + p(s - [s])}} \, dy \right] \right\}^{1/p} < +\infty \tag{5}$$

where $[s]$ is the largest integer $< s$. $W^{s,p}(R^n)$ is a Banach space with respect to the norm defined by the left side of (5).

We observe that $W^{1-\frac{1}{p},p}(R^n)$ is exactly the space of traces of first derivatives of functions in $W^{2,p}(R^n)$.

We shall prove that problems (3) and (4) have a unique solution. Then we shall choose g so that the function u so defined:

$$u(x) = \begin{cases} u^+(x) & x \in R^n_+ \\ u^-(x) & x \in R^n_- \end{cases}$$

is the wanted solution of (1). To do thus, we shall have to solve an integral equation in g: the solution of such an equation is obtained by interpolation-techniques.

2. Fundamental solution of an operator with constant coefficients.

In this section we shall, for the sake of convenience, denote the variables by $(x_0, x_1, \ldots, x_n)$ $(n \geq 2)$.

Consider the second order linear differential operator with real constant coefficients

$$L = \sum_{r,j=0}^{n} a_{rj} \frac{\partial^2}{\partial x_r \, \partial x_j} + \sum_{j=0}^{n} a_j \frac{\partial}{\partial x_j} - h^2 \tag{6}$$

where $A = (a_{rj})$ is a symmetric, positive definite $(n+1) \times (n+1)$ matrix and $h > 0$.

We shall show some properties of the function

$$E(x) = c^{n-1} (\det A)^{-1/2} \, l_n[cr(x)] \exp\left[-\frac{1}{2}(A^{-1} a, x)\right] \quad x \neq 0 \tag{7}$$

that turns out to be a fundamental solution of L. In formula (7)

$$l_n(r) = -(2\pi)^{-(n+1)/2} \, r^{(1-n)/2} K_{(n-1)/2}(r) \tag{8}$$

$K_{(n-1)/2}$ being the modified Hankel function, [2], a being the vector

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[2] For the properties of Bessel functions see [11].

$(a_1 \dots a_n)$,

$$c = \left[h^2 + \frac{1}{4}(A^{-1}a, a)\right]^{1/2} \quad (9)$$

[3]

$$r(x) = (A^{-1}x, x)^{1/2} \quad (10)$$

For the following it will be useful to recall the integral representation formula

$$l_n(r) = -\frac{1}{2}(2\pi)^{-(n+1)/2} \int_0^{+\infty} s^{(n-3)/2} \exp\left(-\frac{1}{2}r^2 s - \frac{1}{2s}\right) ds \quad (11)$$

that is an immediate consequence of known formulas for Bessel functions.

Then, we should like to call to mind that, if $L = \Delta - h^2$, the fundamental solution (7) becomes the familiar function $h^{n-1} l_n(h|x|)$.

Moreover, we observe that

$$E(x) = -\frac{\Gamma\left(\frac{n+1}{2}\right)}{2\pi^{(n+1)/2}(\det A)^{1/2}(n-1)}[r(x)]^{1-n}[1 + 0(|x|)] \text{ as } x \to 0, \quad (12)$$

where the function $-\dfrac{\Gamma\left(\frac{n+1}{2}\right)}{2\pi^{(n+1)/2}(\det A)^{1/2}(n-1)}[r(x)]^{1-n}$ is a fundamental solution of the operator $\sum_{r,j=0}^{n} a_{rj}\dfrac{\partial^2}{\partial x_r \partial x_j}$. (12) is easily proved by using the formula

$$K_m(r) = 2^{m-1}\Gamma(m) r^{-m} e^{-r}(1 + r\,0(r)) \text{ as } r \to 0 \ (m > 1/2) \quad (13)$$

(see, for instance, [10], appendix) and the inequalities

$$|(A^{-1}a, x)| \le (A^{-1}x, x)^{1/2}(A^{-1}a, a)^{1/2} \quad (14)$$

(that is valid for all symmetric, positive definite $(n+1)\times(n+1)$ matrices and for all vectors $a, x \in R^{n+1}$)

$$|r(x)| \le \nu^{-1/2}|x| \quad (15)$$

ν being the largest eigenvalue of A.

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[3] $(\cdot,\cdot)$ denotes the scalar product in R^n.

Finally, we observe the following estimates for E and the gradient DE of E, that, in particular, imply $E \in W^{1,1}(R^{n+1})$:

$$(16) \qquad |E(x)| \leq \frac{\nu^{1/2}}{4\pi (\det A)^{1/2}} \cdot \frac{\exp(-\alpha |x|)}{|x|} \qquad n = 2$$

$$(16) \qquad |E(x)| \leq \frac{\nu}{4\pi^2 (\det A)^{1/2}} \cdot \frac{\exp(-\alpha |x|)}{|x|^2} \left[1 + \left(\frac{c\pi}{2\nu^{1/2}} |x|\right)^{1/2}\right] \qquad n = 3$$

$$(16) \qquad |E(x)| \leq \frac{\Gamma\left(\frac{n+1}{2}\right) \nu^{\frac{n-1}{2}}}{2\pi^{\frac{n+1}{2}} (\det A)^{1/2} (n-1)} \cdot \frac{\exp(-\alpha |x|)}{|x|^{n-1}} \left[1 + \frac{c}{C_n \nu^{1/2}} |x|\right]^{\frac{n-2}{2}} \qquad n \geq 4$$

$$(17) \qquad |DE(x)| \leq \frac{\nu^{3/2}}{2\pi \mu (\det A)^{1/2}} \cdot \frac{\exp(-\alpha |x|)}{|x|^2} \left[1 + \frac{2c\nu^{1/2} + |a|}{4\nu} |x|\right] \qquad n = 2$$

$$(17) \qquad |DE(x)| \leq \frac{\nu^2}{2\pi^2 \mu (\det A)^{1/2}} \left(1 + \frac{|a|}{c\nu^{1/2}}\right) \frac{\exp(-\alpha |x|)}{|x|^3} \left[1 + \frac{c\pi^{1/3}}{2\nu^{1/2}} |x|\right]^{3/2} \qquad n = 3$$

$$(17) \qquad |DE(x)| \leq \frac{\Gamma\left(\frac{n+1}{2}\right) \nu^{\frac{n+1}{2}}}{2\pi^{\frac{n+1}{2}} \mu (\det A)^{1/2}} \cdot$$

$$\cdot \frac{\exp(-\alpha |x|)}{|x|^n} \left(1 + \frac{|a|}{2c\nu^{1/2}}\right) \left(1 + \frac{c}{C_{n+2} \nu^{1/2}} |x|\right)^{n/2} \qquad n \geq 4$$

where μ and ν are the smallest and largest eigenvalues of A and α, $|a|$, C_n are given respectively by

$$(18) \qquad \alpha = \frac{c - \frac{1}{2} (A^{-1} a, a)^{1/2}}{\nu^{1/2}}$$

$$|a| = \left(\sum_{j=0}^{n} a_j^2\right)^{1/2}$$

$$(19) \qquad C_n = \frac{1}{2} \left[\pi^{-1/2} \Gamma\left(\frac{n-1}{2}\right)\right]^{\frac{2}{n-2}}$$

Estimates (16) follow immediately from (14), (15) and known inequalities for Bessel functions (see, for istance, [10], appendix). The same arguments and the formulas

$$DE(x) = \frac{c^n}{(\det A)^{1/2}} \cdot \frac{A^{-1} x}{r(x)} l_n'(cr(x)) \exp\left[-\frac{1}{2}(A^{-1} a, x)\right] - \frac{1}{2} A^{-1} a E(x)$$

$$\frac{d}{dr}(r^{-m} K_m(r)) = -r^{-m} K_{m+1}(r) \tag{20}$$

give (17).

Now we can prove two lemmas:

LEMMA 1. *E is a fundamental solution of L, that is*

$$\mathcal{F} E(\xi) = -\frac{1}{(A\xi, \xi) - i(a, \xi) + h^2} \qquad \xi \in R^n$$

where $\mathcal{F} E(\xi) = \int_{R^n} \exp[-i(x, \xi)] E(x)\, dx$ *is the Fourier transform of E.*

LEMMA 2. *If $f \in L^p(R^{n+1})$ $(1 < p < +\infty)$, the following properties hold:*

i) $E * f \in W^{2,p}(R^{n+1})$, *where $*$ denotes convolution;*

ii) $L(E * f) = f$;

iii) $\| E * f \|_{W^{2,p}(R^{n+1})} \leq C \| f \|_{L^p(R^{n+1})}$, *$C$ being a constant independent of f.*

PROOF OF LEMMA 1 We observe that, for (11), the following formula is easily seen to hold:

$$c^{n-1} \int_{R^{n+1}} \exp[-i(x, \xi)] l_n(c|x|)\, dx = \frac{-1}{|\xi|^2 + c^2}$$

With the change of variables $x = M_\tau^{-1} z$, where M is a $(n+1) \times (n+1)$ matrix such that $M_\tau M = A$ [4], we get easily

$$c^{n-1} \int_{R^{n+1}} \exp[-i(x, \xi)] l_n[cr(x)]\, dx = -\frac{(\det A)^{1/2}}{(A\xi, \xi) + c^2}$$

[4] M_τ is the transposed matrix of M.

To conclude, we remember that, if f is a well-behaved function, the Fourier transform of $f(x)\exp(a,x)$ is $\mathcal{F}f(\xi - ia)$: thus the lemma is proved.

PROOF OF LEMMA 2. To prove the lemma we can suppose $f \in C_0^\infty(R^{n+1})$. From Young's inequality it follows that

$$\| E * f \|_{L^p(R^{n+1})} \leq \| f \|_{L^p(R^{n+1})} \int_{R^{n+1}} | E(x) | \, dx, \tag{21}$$

the integral of E being finite on account of estimates (16). Incidentally, we notice that Lemma 1 and the obvious fact that E is negative imply the equation

$$\int_{R^{n+1}} | E(x) | \, dx = 1/h^2$$

The formula

$$\frac{\partial^2 (E * f)}{\partial x_r \, \partial x_j} = - \mathcal{F}^{-1} [\xi_r \, \xi_j \, \mathcal{F} E \, \mathcal{F} f] = (\mathcal{F}^{-1} \varphi_{rj}(\xi) \, \mathcal{F}) f,$$

where

$$\varphi_{rj}(\xi) = \frac{\xi_r \, \xi_j}{(A\xi, \xi) - i(a, \xi) + h^2} \quad (r, j = 0, \dots, n),$$

and a theorem of Hörmander-Mihlin show that the second derivatives of $E * f$ are in $L^p(R^{n+1})$ and that, taking into account (21), iii) is fulfilled. For, it is easy to see that the functions φ_{rj} are, following Hörmander's terminology, multipliers of type (p, p) for every $1 < p < +\infty$. For more details, see the appendix of this paper.

Then, from lemma 1 it follows that

$$\mathcal{F}[L(E * f)](\xi) = - [(A\xi, \xi) - i(a, \xi) + h^2] \, \mathcal{F} E(\xi) \, \mathcal{F} f(\xi) = \mathcal{F} f(\xi),$$

that is ii). The lemma is proved.

For the following we need to have the expression of the Fourier transform of $E(t, \cdot)$, where we have put, for the sake of convenience, $t = x_0$, $x = (x_1, \dots, x_n)$: moreover, ξ will denote the dual variable of x and, occasionally, $\mathcal{F}_x$ will denote the Fourier transform with respect to x.

The following lemma holds:

LEMMA 3. *We have*

$$- \int_{R^n} \exp[-i(x, \xi)] \, E(t, x) \, dx = \psi(t, \xi) = \begin{cases} \dfrac{\exp[t z_1(\xi)]}{\sqrt{H(\xi)}} & t \geq 0 \\[2ex] \dfrac{\exp[t z_2(\xi)]}{\sqrt{H(\xi)}} & t < 0 \end{cases} \tag{22}$$

where $z_1(\xi)$ and $z_2(\xi)$ are the roots of the equation

$$a_{00} z^2 + \alpha(\xi) z - \beta(\xi) = 0 \tag{23}$$

respectively with negative and positive real part and

$$H(\xi) = \alpha^2(\xi) + 4a_{00}\,\beta(\xi) \tag{24}$$

is the discriminant [5], $\alpha(\xi)$ and $\beta(\xi)$ being defined as follows:

$$\alpha(\xi) = a_0 + 2i \sum_{j=1}^{n} a_{0j}\,\xi_j \tag{25}$$

$$\beta(\xi) = h^2 + \sum_{r,j=1}^{n} a_{rj}\,\xi_r\,\xi_j - i \sum_{j=1}^{n} a_j\,\xi_j \tag{26}$$

REMARK 1. Observe that the discriminant $H(\xi)$ has the following properties:

i) $\operatorname{Re} H(\xi) \geq a_0^2 + 4a_{00} h^2$

ii) $\operatorname{Re} \sqrt{H(\xi)} \geq \sqrt{\operatorname{Re} H(\xi)}$

iii) $|H(\xi)| \geq C(1 + |\xi|^2)$, $\left[C \text{ being a strictly positive constant, } |\xi| = \left(\sum_{j=1}^{n} \xi_j^2\right)^{1/2}\right]$.

Moreover, the roots $z_1(\xi)$ and $z_2(\xi)$ have the properties:

$$\operatorname{Re} z_1(\xi) \leq \frac{-a_0 - \sqrt{a_0^2 + 4a_{00} h^2}}{2a_{00}} < 0 \tag{27}$$

$$\operatorname{Re} z_2(\xi) \geq \frac{-a_0 + \sqrt{a_0^2 + 4a_{00} h^2}}{2a_{00}} > 0 \tag{28}$$

$$C'(1 + |\xi|^2)^{1/2} \leq |\operatorname{Re} z_j(\xi)| \leq |z_j(\xi)| \leq C''(1 + |\xi|^2)^{1/2} \quad (j = 1, 2) \tag{29}$$

C' and C'' being positive constants.

For the proof, see [6], lemmas 1 and 3.

(5) In this paper the square root of a complex number is the one with non-negative real part.

REMARK 2. The function $-\psi$ is a fundamental solution of the operator

$$a_{00}\frac{\partial^2}{\partial t^2}+\alpha(\xi)\frac{\partial}{\partial t}-\beta(\xi)$$

that admits (23) as its characteristic equation.

PROOF OF LEMMA 3. From the properties of integrability of E (see inequalities (16)) it follows that

$$\int_{R^n}\exp[-i(x,\xi)]\,E(t,x)\,dx=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp(it\tau)\,\mathcal{F}E(\tau,\xi)\,d\tau \tag{30}$$

For lemma 1 and the notations introduced before the statement of lemma 3, we can write

$$\mathcal{F}E(\tau,\xi)=-\frac{1}{a_{00}\tau^2-i\alpha(\xi)\tau+\beta(\xi)} \tag{31}$$

where $\alpha(\xi)$ and $\beta(\xi)$ are defined by (25), (26).

Applying Cauchy's theorem on residues, from (30) and (31) we get

$$\int_{R^n}\exp[-i(x,\xi)]\,E(t,x)\,dx=\begin{cases}-\dfrac{1}{a_{00}}\cdot\dfrac{\exp[tz_1(\xi)]}{z_2(\xi)-z_1(\xi)} & t\geq 0\\[2ex] -\dfrac{1}{a_{00}}\cdot\dfrac{\exp[tz_2(\xi)]}{z_2(\xi)-z_1(\xi)} & t<0\end{cases}$$

i. e. the assertion.

For the following, it will be useful to estimate all the derivatives of E: such estimates imply, in particular that E is analytic in $R^n-\{0\}$.

LEMMA 4. *Let γ be any multi-index: then*

$$|D^\gamma E(x)|\leq(|\gamma|!)\,C^{|\gamma|+1}|x|^{1-n-|\gamma|}\exp(-\delta|x|)\quad x\neq 0 \tag{32}$$

where $D^\gamma=\dfrac{\partial^{\gamma_0}}{\partial x_0^{\gamma_0}}\dfrac{\partial^{\gamma_1}}{\partial x_1^{\gamma_1}}\cdots\dfrac{\partial^{\gamma_n}}{\partial x_n^{\gamma_n}}$, $|\gamma|=\sum_{j=0}^{n}\gamma_j$, c is defined by (9),

$$\delta=\frac{h^2}{2\nu^{1/2}[(A^{-1}a,a)+2h^2]^{1/2}} \tag{33}$$

and C is a constant independent of γ.

PROOF. Leibniz's rule for the derivative of a product, inequalities (14), (15), formulas (9) and (33) imply that it is enough to prove the following inequality

$$(34) \qquad |D^\gamma l_n[cr(x)]| \leq (|\gamma|!)\, C^{|\gamma|+1}[r(x)]^{1-n-|\gamma|} \exp[-\varepsilon cr(x)]$$

where

$$(35) \qquad \varepsilon = \frac{[(A^{-1}a, a) + 2h^2]^{1/2}}{[(A^{-1}a, a) + 4h^2]^{1/2}}$$

For the derivation of (32) from (34) it is useful to bear in mind the inequality

$$(36) \qquad \varepsilon c - \frac{1}{2}(A^{-1}a, a)^{1/2} = \frac{h^2}{[(A^{-1}a, a) + 2h^2]^{1/2} + (A^{-1}a, a)^{1/2}} > \nu^{1/2}\delta$$

Since there exists an orthogonal $(n+1)\times(n+1)$ matrix that maps the quadratic form $r^2(x)$ into the canonical form $\varrho^2(x) = \sum_{j=0}^{n} \lambda_j x_j^2$, it suffices to prove (34) with $l_n[cr(x)]$ substituted by $l_n[c\varrho(x)]$. For the integral representation formula (11) we get

$$(37) \qquad l_n[c\varrho(x)] = -\frac{1}{2(2\pi)^{\frac{n+1}{2}}} \int_0^{+\infty} s^{\frac{n-3}{2}} \exp\left[-\frac{1}{2} sc^2 \sum_{j=0}^{n} \lambda_j x_j^2 - \frac{1}{2s}\right] ds$$

First we prove the following estimate:

$$(38) \qquad \left|\frac{d^k}{dx^k}\exp(-ax^2)\right| \leq k!\left(\frac{k}{2}\right)^{-k/2} e^{k/2} a^{k/2}(1-\eta^2)^{-k/2}\exp(-\eta^2 ax^2)$$

for every $a > 0$, $0 \leq \eta < 1$, $k = 0, 1, 2, \ldots$.

In fact, since $\exp(-ax^2)$ is an entire function, we can write the inequality

$$\left|\frac{d^k}{dx^k}\exp(-ax^2)\right| \leq \frac{k!\, r^{-k}}{2\pi}\int_0^{2\pi} \exp[-a\,\mathrm{Re}\,(x + re^{i\varphi})^2]\, d\varphi \qquad r \in (0, +\infty)$$

where

$$\mathrm{Re}\,(x + re^{i\varphi})^2 = x^2 + 2rx\cos\varphi + r^2\cos^2\varphi - r^2\sin^2\varphi$$

From the inequality

$$|2rx\cos\varphi| \leq (1-\eta^2)x^2 + \frac{r^2\cos^2\varphi}{1-\eta^2}$$

valid for every $\eta \in [0, 1)$, we infer that

$$\operatorname{Re}(x + re^{i\varphi})^2 \geq \eta^2 x^2 - r^2 \left[\frac{\eta^2}{1-\eta^2} \cos^2\varphi + \sin^2\varphi\right] \geq \eta^2 x^2 - \frac{r^2}{1-\eta^2}$$

Hence

$$\left|\frac{d^k}{dx^k} \exp(-ax^2)\right| \leq k!\, r^{-k} \exp \frac{ar^2}{1-\eta^2} \exp(-\eta^2 ax^2) \tag{39}$$

(38) follows from (39) minimizing with respect to $r \in (0, +\infty)$.

Then, (37) and (38) imply that, for $\eta \in (0, 1)$, denoting $|\gamma|$ by σ

$$|D^\gamma l_n[c\varrho(x)]| \leq \frac{C^\sigma \left(\frac{\sigma}{2}\right)^{-\sigma/2} e^{\sigma/2}\, \sigma!}{2(2\pi)^{\frac{n+1}{2}}} \int_0^{+\infty} s^{\frac{\sigma+n-3}{2}} \exp\left[-\frac{s\eta^2 c^2 \varrho^2(x)}{2} - \frac{1}{2s}\right] ds = \tag{40}$$

$$= \left(\frac{\sigma}{2}\right)^{-\sigma/2} e^{\sigma/2} (\sigma!)\, |\, l_{n+\sigma}[\eta c \varrho(x)]\,|$$

Use has been made of the inequalities $\lambda_j \leq \frac{1}{\mu}$ $(j = 0, \ldots . n)$, μ being the smallest eigenvalue of A, of formula (11) and the estimates:

$$\prod_{j=0}^{n} \gamma_j! \leq \sigma! \qquad \prod_{j=0}^{n} \left(\frac{\gamma_j}{2}\right)^{-\gamma_j/2} \leq (n+1)^{\sigma/2} \left(\frac{\sigma}{2}\right)^{-\sigma/2}.$$

Then, the estimate

$$|K_m(r)| \leq 2^{m-1} r^{-m} \Gamma(m) (1-\eta^2)^{-m} \exp(-\eta r) \quad r > 0,\ 0 < \eta < 1$$

(see, for instance, [10], appendix), Stirling's formula

$$\lim_{m \to +\infty} \frac{\Gamma(m+\varrho)}{m^\varrho \Gamma(m)} = 1 \qquad \varrho > 0$$

and inequality (41) imply the assertion, if η is chosen to be equal to $\varepsilon^{1/2}$.

3. The Poisson kernel.

In this section, and from now on, we denote, as in lemma 3, the variables by (t, x), where $t \in R$, $x = (x_1, \ldots x_n) \in R^n$.

Now, consider the real analytic function

$$P(t, x) = 2 D_\zeta E(t, x) + a_0 E(t, x) \quad (t, x) \neq (0, 0) \tag{42}$$

where $\zeta = (a_{00}, .., a_{0n})$ and $D_\zeta = \sum_{j=1}^{n} a_{0j} \frac{\partial}{\partial x_j} + a_{00} \frac{\partial}{\partial t}$ is the differentiation along the conormal direction ζ. From (7) and (11) it follows that

$$P(t, x) = \frac{2c^n t}{r(t, x)(\det A)^{1/2}} l_n'[cr(t, x)] \exp\left(\tilde{a}_0 t + \sum_{j=1}^{n} \tilde{a}_j x_j\right) = \tag{43}$$

$$= \frac{c^{n+1} t \exp\left(\tilde{a}_0 t + \sum_{j=1}^{n} \tilde{a}_j x_j\right)}{(2\pi)^{\frac{n+1}{2}} (\det A)^{1/2}} \int_0^{+\infty} s^{\frac{n-1}{2}} \exp\left[-\frac{sc^2}{2} r^2(t, x) - \frac{1}{2s}\right] ds$$

where

$$\tilde{a} = -\frac{1}{2} A^{-1} a. \tag{44}$$

It is easy to recognize from (43) that P has the sign of t.

From formulas (18), (19), (20), (43), inequalities (14), (15) and known estimates for Bessel functions (see, for instance, [10], appendix), we infer

$$|P(t, x)| \leq \frac{\Gamma\left(\frac{n+1}{2}\right) \nu^{\frac{n+1}{2}}}{2\pi^{\frac{n+1}{2}} (\det A)^{1/2}} \cdot \tag{45}$$

$$\cdot \frac{|t| \exp[-\alpha (t^2 + |x|^2)^{1/2}]}{(t^2 + |x|^2)^{\frac{n+1}{2}}} \left[1 + \frac{c}{C_{n+2} \nu^{1/2}} (t^2 + |x|^2)^{1/2}\right]^{n/2} n \geq 2$$

where α is defined by (18).

(45) establishes that $P \in L^1(R^{n+1})$, $P(t, \cdot) \in L^1(R^n)$ for all $t \in R$ and $P(\cdot, x) \in L^1(R)$ for all $x \in R^n - \{0\}$. Moreover, observe that (13), (20), (43), imply that

$$P(t, x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}} (\det A)^{1/2}} \frac{t}{[r(t, x)]^{n+1}} \{1 + 0[(t^2 + x^2)^{1/2}]\} \text{ as } (t, x) \rightarrow (0, 0)$$

We notice that, when $L = \Delta$, P coincides with $\dfrac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \cdot \dfrac{t}{(t^2 + x^2)^{\frac{n+1}{2}}}$, the usual Poisson kernel.

Finally, we point out that the Fourier transform of $P(t, \cdot)$ is given by

$$[\mathcal{F}_x P(t, \cdot)](\xi) = \psi(t, \xi) \sqrt{H(\xi)}\ sgn\, t \quad t \neq 0 \tag{46}$$

Indeed, from (42) and lemma 3, recalling (17) and (25), we get

$$[\mathcal{F}_x P(t, \cdot)](\xi) = -2a_{00} \frac{\partial \psi}{\partial t}(t, \xi) - \alpha(\xi)\psi(t, \xi) = \psi(t, \xi)\sqrt{H(\xi)}\ sgn\, t$$

Obviously P is a solution of the equation $Lu = 0$ in $R^{n+1} - \{0\}$.

Then, consider the convolution

$$v(t, x) = \int_{R^n} P(t, x - z)\, g(z)\, dz \quad t > 0 \tag{47}$$

the properties of which are stated in the following lemmas:

LEMMA 5. *If* $g \in L^p(R^n)(1 < p < +\infty)$, *then*

i) $$\| v(t, \cdot)\|_{L^p(R^n)} \leq \psi(t, 0)\sqrt{H(0)}\, \| g \|_{L^p(R^n)} =$$
$$= \| g \|_{L^p(R^n)} \exp\left\{\frac{t}{2a_{00}}\left[-a_0 - (a_0^2 + 4a_{00}h^2)^{1/2}\right]\right\}$$

ii) $$\| v(t, \cdot) - g \|_{L^p(R^n)} \to 0 \quad as\ t \to 0+$$

LEMMA 6. *If* $g \in W^{s,p}(R^n)\ (0 < s < 1,\ 1 < p < +\infty)$, *then*

i) $$\left(\int_0^{+\infty} t^{p(1-s)-1}\, dt \int_{R^n} \left|\frac{\partial v}{\partial t}(t, x)\right|^p dx\right)^{1/p} \leq$$
$$\leq C_1\left[\,|z_1(0)|^s \| g \|_{L^p(R^n)} + |g|_{W^{s,p}(R^n)}\right]$$

ii) $$\left(\int_0^{+\infty} t^{p(1-s)-1}\, dt \int_{R^n} \left|\frac{\partial v}{\partial x_j}(t, x)\right|^p dx\right)^{1/p} \leq C_2\, |g|_{W^{s,p}(R^n)}\ .$$

where

$$| g |_{W^{s,p}(R^n)} = \left(\int_{R^n} dz \int_{R^n} \frac{| g(x - z) - g(x) |^p}{| z |^{n+ps}} dx \right)^{1/p}$$

and C_1 and C_2 are constants independent of g.

Moreover, v is an analytic function in the half-space R_+^{n+1} and is a solution of the equation $Lu = 0$.

In particular, if $s = 1 - \frac{1}{p}$, $v \in W^{1,p}(R_+^{n+1})$.

REMARK 4. From Lemmas 5 and 6 it follows easily that, if $g \in W^{s,p}(R^n)$ $(0 < s < 1, 1 < p < +\infty)$, $v(0+, \cdot) = g$, where the left side denotes the trace of v, defined in the usual way.

We shall premise the proof of lemma 5 with

LEMMA 7. *The function P has the following properties:*

i) $$\int_{R^n} P(t, x)\, dx = \psi(t, 0) \sqrt{H(0)} \text{ sgn} t =$$

$$= \begin{cases} \exp\left\{\frac{t}{2a_{00}}[- a_0 - (a_0^2 + 4a_{00} h^2)^{1/2}]\right\} & t > 0 \\ -\exp\left\{\frac{t}{2a_{00}}[- a_0 + (a_0^2 + 4a_{00} h^2)^{1/2}]\right\} & t < 0 \end{cases}$$

ii) $$\int_{|x| > \lambda} | P(t, x) |\, dx \leq C \frac{| t |}{\lambda}$$

where λ is any positive number and the constant C is independent of t and λ.

PROOF OF LEMMA 7. i) follows from (46), substituting $\xi = 0$. Estimate (45) implies the chain of inequalities

$$| P(t, x) | \leq C' \frac{| t |}{(t^2 + | x |^2)^{\frac{n+1}{2}}} \leq C' \frac{| t |}{| x |^{n+1}}$$

where C' is a constant; hence ii) is easily obtained by integration.

Now, we can prove lemma 5: we observe that i) is an immediate consequence of Young's inequality on convolution and i) in lemma 7.

In order to show ii) we proceed as follows:

$$\| v(t,\cdot) - g \|_{L^p(R^n)} \leq \| v(t,\cdot) - \psi(t,0)\sqrt{H(0)}\, g \|_{L^p(R^n)} + $$
$$+ [1 - \psi(t,0)\sqrt{H(0)}]\, \| g \|_{L^p(R^n)}$$

It is evident, from the definition (22) of ψ, that it is enough to prove that

$$\| v(t,\cdot) - \psi(t,0)\sqrt{H(0)}\, g \|_{L^p(R^n)} \to 0 \quad \text{as} \quad t \to 0+$$

Making use again of $i)$ in lemma 5, we get

$$v(t,x) - \psi(t,0)\sqrt{H(0)}\, g(x) = \int_{R^n} P(t,z)[g(x-z) - g(x)]\, dz.$$

From Minkowski's inequality and ii) in lemma 7 we infer that

$$\| v(t,.) - \psi(t,0)\sqrt{H(0)}\, g \|_{L^p(R^n)} \leq \int_{R^n} |P(t,z)| \left(\int_{R^n} | g(x-z) - g(x) |^p\, dx \right)^{1/p} dz \leq$$

$$\leq w(\lambda) \int_{|z| \leq \lambda} | P(t,z) |\, dz + 2 \| g \|_{L^p(R^n)} \int_{|z| > \lambda} | P(t,z) |\, dz \leq$$
$$\leq \psi(t,0)\sqrt{H(0)}\, w(\lambda) + 2C \| g \|_{L^p(R^n)} \frac{|t|}{\lambda}$$

where λ is any positive number and $w(\lambda) = \operatorname*{Sup}_{|z| \leq \lambda} \left(\int_{R^n} |g(x-z) - g(x)|^p\, dx \right)^{1/p}$ tends to 0 as λ tends to $0+$: hence ii) follows at once.

PROOF OF LEMMA 6. The analyticity of v is a consequence of the estimates

$$(48) \qquad | D^\gamma P(t,x) | \leq [(|\gamma| + 1)!]\, C^{|\gamma|+2} (t^2 + |x|^2)^{\frac{-n-|\gamma|}{2}} \exp[-\theta\delta(t^2 + x^2)^{1/2}]$$

where δ is defined by (33) and $0 < \theta < 1$: they follow from (42) and lemma 4. Clearly v is a solution of the equation $Lv(t,x) = 0$ $t > 0$.

For the proof of i) and ii) it is necessary to observe that (46) asserts that $\frac{\partial P}{\partial t}(t,\cdot)$, $\frac{\partial P}{\partial x_j}(t,\cdot)$ belong to $L^1(R^n)$ $(j = 1, \dots, n)$ for all $t \neq 0$. Moreover

$$(49) \qquad \int_{R^n} \frac{\partial P}{\partial x_j}(t,x)\, dx = 0 \quad \text{for all } t \neq 0 \quad (j = 1, \dots, n).$$

For, from the equation

$$\left[\mathcal{F}_x \frac{\partial P}{\partial x_j}(t,\cdot)\right](\xi) = i\xi_j [\mathcal{F}_x P(t,\cdot)](\xi) = i\xi_j \, \psi(t,\xi) \sqrt{H(\xi)} \operatorname{sgn} t$$

(49) follows, substituting $\xi = 0$. Hence

$$\frac{\partial v}{\partial x_j}(t,x) = \int_{R^n} \frac{\partial P}{\partial z_j}(t,z)\, g(x-z)\, dz = \int_{R^n} \frac{\partial P}{\partial z_j}(t,z)\,[g(x-z) - g(x)]\, dz \tag{50}$$

$$(j = 1, \dots, n)$$

From the identity

$$v(t,x) = \psi(t,0)\sqrt{H(0)}\, g(x) + \int_{R^n} P(t,z)\,[g(x-z) - g(x)]\, dz$$

we get

$$\frac{\partial v}{\partial t}(t,x) = \frac{\partial \psi}{\partial t}(t,0)\sqrt{H(0)}\, g(x) + \int_{R^n} \frac{\partial P}{\partial t}(t,z)\,[g(x-z) - g(x)]\, dz \tag{51}$$

From (48), (49), (50) and Minkowski's inequality we infer that

$$\left\| \frac{\partial v}{\partial x_j}(t,\cdot) \right\|_{L^p(R^n)} \le C' \int_{R^n} \frac{L(z)}{(t^2 + |z|^2)^{\frac{n+1}{2}}}\, dz \tag{52}$$

$$\left\| \frac{\partial v}{\partial t}(t,\cdot) \right\|_{L^p(R^n)} \le \left| \frac{\partial \psi}{\partial t}(t,0) \right| \sqrt{H(0)}\, \| g \|_{L^p(R^n)} + C' \int_{R^n} \frac{L(z)}{(t^2 + |z|^2)^{\frac{n+1}{2}}}\, dz \tag{53}$$

where $L(z) = \left(\int_{R^n} | g(x-z) - g(x) |^p \, dx \right)^{1/p}$ and C' is a constant.

Now we show that

$$I = \left[\int_0^{+\infty} t^{p(1-s)-1} \left(\int_{R^n} \frac{L(z)}{(t^2 + z^2)^{\frac{n-1}{2}}}\, dz \right)^p dt \right]^{1/p} \le C'' \, | g |_{W^{s,p}(R^n)} \tag{54}$$

where C'' is a constant depending only on n, p, s. For, Minkowski and

Hölder's inequalities yield the following chain of inequalities

$$I = \left[\int_0^{+\infty}\left(\int_{R^n} t^{-s-\frac{1}{p}} \frac{L(tz)}{(1+z^2)^{\frac{n+1}{2}}}\, dz\right)^p dt\right]^{1/p} \leq$$

$$\leq \int_{R^n}\left(\int_0^{+\infty} t^{-sp-1} L^p(tz)\, dt\right)^{1/p} \frac{dz}{(1+|z|^2)^{\frac{n+1}{2}}} \leq$$

$$\leq \int_{R^n}\left(\int_0^{+\infty} r^{-sp-1} L^p\left(r\frac{z}{|z|}\right) dr\right)^{1/p} \frac{|z|^s}{(1+|z|^2)^{\frac{n+1}{2}}}\, dz =$$

$$= \int_0^{+\infty} \frac{\varrho^{s+n-1}}{(1+\varrho^2)^{\frac{n+1}{2}}}\, d\varrho \int_{|\zeta|=1}\left(\int_0^{+\infty} r^{-sp-1} L^p(r\zeta)\, \mathrm{d}r\right)^{1/p} \mu(d\zeta) \leq$$

$$\leq \omega_n^{1-\frac{1}{p}}\left(\int_0^{+\infty} \frac{\varrho^{s+n-1}}{(1+\varrho^2)^{\frac{n+1}{2}}}\, d\varrho\right)\cdot\left(\int_{|\zeta|=1} \mu(d\zeta)\int_0^{+\infty} r^{-sp-1} L^p(r\zeta)\, dr\right)^{1/p} =$$

$$= C''\left(\int_{R^n} \frac{L^p(z)}{|z|^{n+sp}}\, dz\right)^{1/p} = C''\, |g|_{W^{s,p}(R^n)}$$

where $\mu(d\zeta)$ denotes the Lebesgue measure on $|\zeta| = 1$ and

$$C'' = \omega_n^{1-\frac{1}{p}} \int_{R^n} \frac{\varrho^{s+n-1}}{(1+\varrho^2)^{\frac{n+1}{2}}}\, d\varrho .$$

Now, ii) is an immediate consequence of (52) and (54) with $C_2 = C''C'$, while i) follows from Minkowski's inequality with measure $t^{p(1-s)-1}\, dt$, applied to (51) and from the equation

$$\left(\int_0^{+\infty} t^{p(1-s)-1}\left|\frac{\partial\psi}{\partial t}(t, o)\sqrt{H(0)}\right|^p dt\right)^{1/p} = \left(\int_0^{+\infty} t^{p(1-s)-1} \exp(\mathrm{tp} z_1(0))\, dt\right)^{1/p} |z_1(0)| =$$

$$= p^{s-1}\,\Gamma(p-ps)\,|z_1(0)|^s .$$

The constant C_1 is given by max $[p^{s-1}\Gamma(p-ps), C_2]$.

4. The Neumann problem.

Consider the function

$$N(t,x) = -\int_t^{+\infty} P(s,x)\,ds. \tag{55}$$

Since $P(t,x) > 0$, if $t > 0$, it follows that

$$N(t,x) < 0 \quad t > 0 \tag{56}$$

Moreover, the following estimate holds:

$$|N(t,x)| \le C_\theta (t^2 + |x|^2)^{-\frac{n-1}{2}} \exp[-\theta\alpha(t^2+|x|^2)^{1/2}] \quad t \ge 0 \tag{57}$$

where α is defined by (18), $0 < \theta < 1$ and C_θ is a constant. For, (45) implies that, for every $0 < \theta < 1$, there exists a constant C'_θ such that

$$|P(t,x)| \le C'_\theta \frac{t}{(t^2+|x|^2)^{\frac{n+1}{2}}} \exp[-\theta\alpha(t^2+|x|^2)^{1/2}] \qquad t > 0$$

Hence

$$|N(t,x)| \le C'_\theta \int_t^{+\infty} \frac{s}{(s^2+|x|^2)^{\frac{n+1}{2}}} \exp[-\theta\alpha(s^2+|x|^2)^{1/2}]\,ds \le$$

$$\le C'_\theta \exp[-\theta\alpha(t^2+|x|^2)^{1/2}] \int_t^{+\infty} \frac{s}{(s^2+|x|^2)^{\frac{n+1}{2}}}\,ds =$$

$$= \frac{C'_\theta}{n-1}(t^2+|x|^2)^{\frac{1-n}{2}} \exp[-\theta\alpha(t^2+|x|^2)^{1/2}]$$

From (57) one recognizes easily $N \in L^1(R^{n+1}_+)$ and $N(t,\cdot) \in L^1(R^n)$ for all $t > 0$. Moreover, the Fourier transform of $N(t,\cdot)$ is given by

$$[\mathcal{F}_x N(t,\cdot)](\xi) = \frac{\exp(tz_1(\xi))}{z_1(\xi)} \qquad t \ge 0 \tag{58}$$

This follows from (46) and the fact that $P \in L^1(R^{n+1})$.

We remark also that, for (48), N is a real analytic function, that satisfies the equation $LN = 0$ in R^{n+1}_+.

In particular, if $L = \sum_{r,j=1}^{n} a_{rj} \frac{\partial^2}{\partial x_r \, \partial x_j} + \frac{\partial^2}{\partial t^2} + \sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j} - h^2$, (i. e., if the coefficients a_{0j} $(j = 1 \dots n)$ and a_0 vanish), then P coincides with $2 \frac{\partial E}{\partial t}$ and, hence, N with $2E$.

Consider the convolution

$$u(t, x) = \int_{R^n} N(t, x - z) \, g(z) \, dz \qquad t > 0 \tag{59}$$

The following lemma holds:

LEMMA 8. *If* $g \in W^{1-\frac{1}{p}, p}(R^n)$, *then*

i) $u \in W^{2,p}(R_+^{n+1})$;

ii) $\| u \|_{W^{2,p}(R_+^{n+1})} \leq C \| g \|_{W^{1-\frac{1}{p}, p}(R^n)}$,

where C is a constant independent of g:

iii) $u(0+, \cdot) = N(0, \cdot) * g$;

iv) $\frac{\partial u}{\partial t}(0+, \cdot) = g$

Moreover, u is an analytic function in the half-space R_+^{n+1} and is a solution of the equation $Lu = 0$.

PROOF. The analyticity of u follows, as before, from estimates for the kernel: they are

$$| D^\gamma N(t, x) | \leq \tag{60}$$

$$\leq [(|\gamma| + 1)!] \, C_\theta^{|\gamma|+2} (t^2 + |x|^2)^{\frac{1-n-|\gamma|}{2}} \exp[-\theta\delta (t^2 + |x|^2)^{1/2}] \text{ if } |\gamma| \geq 1$$

where δ is defined by (33) and $0 < \theta < 1$. For the derivation of (60) we use the formula

$$D^\gamma N(t, x) = \frac{\partial^{|\gamma|} N}{\partial t^{\gamma_0} \partial x_1^{\gamma_1} \dots \partial x_n^{\gamma_n}}(t, x) = \int_t^{+\infty} \frac{\partial^{|\gamma|} P}{\partial s^{\gamma_0} \partial x_1^{\gamma_1} \dots \partial x_n^{\gamma_n}}(s, x) \, ds$$

and estimates (48) for the derivatives of P.

Clearly $Lu(t, x) = 0$ if $t > 0$.

To show i) it suffices, for the usual reasons of density, to suppose $g \in C_0^\infty(R^n)$. Young's inequality and (58) imply

$$\| u(t,\cdot) \|_{L^p(R^n)} \leq \| N(t,\cdot) \|_{L^1(R^n)} \| g \|_{L^p(R^n)} = \frac{\exp(t z_1(0))}{|z_1(0)|} \| g \|_{L^p(R^n)} :$$

hence

$$\| u \|_{L^p(R^{n+1}_+)} \leq p^{-1/p} | z_1(0) |^{-1-\frac{1}{p}} \| g \|_{L^p(R^n)} \tag{61}$$

In order to prove that the second derivatives of u with respect to x are in $L^p(R^{n+1}_+)$, we use the eqnation

$$\frac{\partial^2 u}{\partial x_r \partial x_j} = (\mathcal{F}_x^{-1} \varphi_j \mathcal{F}_x) \frac{\partial v}{\partial x_r} \qquad (r, j = 1, 2, \ldots, n), \tag{62}$$

where v is defined by (47) and

$$\varphi_j(\xi) = \frac{i\xi_j}{z_1(\xi)} \quad (j = 1, 2, \ldots n).$$

(62) is a consequence of (46), (58) and i) in lemma 5: for

$$\frac{\partial^2 u}{\partial x_r \partial x_j} = -\mathcal{F}_x^{-1}[\xi_r \xi_j \mathcal{F}_x N \mathcal{F}_x g] = \mathcal{F}_x^{-1}[i \xi_r \varphi_j \mathcal{F}_x P \mathcal{F}_x g] = (\mathcal{F}_x^{-1} \varphi_j \mathcal{F}_x) \frac{\partial v}{\partial x_r}$$

Since the functions φ_j are multipliers of type (p, p) for every $1 < p < +\infty$ (see proposition 2 in appendix) and the functions $\frac{\partial v}{\partial x_r}$, for ii) in lemma 6 with $s = 1 - \frac{1}{p}$, satisfy the inequality

$$\left\| \frac{\partial v}{\partial x_r} \right\|_{L^p(R^{n+1}_+)} \leq C_2 | g |_{W^{1-\frac{1}{p}, p}(R^n)} \qquad (r = 1, 2, \ldots n),$$

then, there exists a constant C' such that

$$\left\| \frac{\partial^2 u}{\partial x_r \partial x_j} \right\|_{L^p(R^{n+1}_+)} \leq C' | g |_{W^{1-\frac{1}{p}, p}(R^n)}. \tag{63}$$

Since

$$\frac{\partial u}{\partial t}(t, x) = \int_{R^n} \frac{\partial N}{\partial t}(t, x - z) g(z) dz = \int_{R^n} P(t, x - z) g(z) dz = v(t, x), \tag{64}$$

lemma 6, with $s = 1 - \frac{1}{p}$, implies

$$\left\| \frac{\partial^2 u}{\partial t^2} \right\|_{L^p(R_+^{n+1})} \leq C_1 \left[| z_1(0) |^{1-\frac{1}{p}} \| g \|_{L^p(R^n)} + | g |_{W^{1-\frac{1}{p},p}(R^n)} \right] \tag{65}$$

$$\left\| \frac{\partial^2 u}{\partial x_j \, \partial t} \right\|_{L^p(R_+^{n+1})} \leq C_2 \, | g |_{W^{1-\frac{1}{p},p}(R^n)} . \tag{66}$$

Therefore, all the derivatives of u are in $L^p(R_+^{n+1})$: moreover, the inequalities (61), (63), (65), (66) show that $u \in W^{2,p}(R_+^{n+1})$ and there exists a constant C independent of g such that

$$\| u \|_{W^{2,p}(R_+^{n+1})} \leq C \, \| g \|_{W^{1-\frac{1}{p},p}(R^n)} .$$

Thus, also ii) is proved.

iv) is an immediate consequence of (64) and remark 4 after lemma 6, while iii) follows from the property

$$\| N(t, \cdot) - N(0, \cdot) \|_{L^1(R^n)} \to 0 \qquad \text{as} \quad t \to 0+$$

that implies

$$\| N(t, .) * g - N(0, \cdot) * g \|_{L^p(R^n)} \to 0 \quad \text{as} \ t \to 0+ .$$

Hence the trace of $N(t, \cdot) * g$ must coincide with $N(0, \cdot) * g$.

At last we state the following

LEMMA 9. *If* $(f, g) \in L^p(R_+^{n+1}) \times W^{1-\frac{1}{p},p}(R^n)$ $(1 < p < +\infty)$, *the Neumann problem*

$$\begin{cases} Lu = f \\ u \in W^{2,p}(R_+^{n+1}) \\ \dfrac{\partial u}{\partial t}(0+, \cdot) = g \end{cases} \tag{67}$$

L being defined by (6), *admits a unique solution u given by*

$$u(t, x) = \int_{R^n} N(t, x - z) \left[g(z) - \frac{\partial w}{\partial t}(0, z) \right] dz + w(t, x) \tag{68}$$

where

$$w(t,x)=\int_{R_+^{n+1}} E(t-s,x-z)f(s,z)\,ds\,dz$$

and $\frac{\partial w}{\partial t}(0,\cdot)$ *stands for the trace of* $\frac{\partial w}{\partial t}$.

PROOF. Lemma 2 and remark 3 show that $w\in W^{2,p}(R_+^{n+1})$ and satisfies the equation $Lw=f$. Hence, for lemma 8, $u\in W^{2,p}(R_+^{n+1})$ and satisfies the equation $Lu=f$ and the condition $\frac{\partial u}{\partial t}(0+,\cdot)=g$.

Now we prove that u is the unique solution of (67). Let u be an arbitrary function in $C_0^\infty(\overline{R_+^{n+1}})$: put

$$Lu=f$$

$$\frac{\partial u}{\partial t}(0+,\cdot)=g$$

$$[\mathcal{F}_x u(t,\cdot)](\xi)=v(t,\xi)$$

$$[\mathcal{F}_x f(t,\cdot)](\xi)=\widehat{f}(t,\xi)$$

$$\mathcal{F}_x g(\xi)=\widehat{g}(\xi).$$

Then v is a solution of the problem

$$\text{(70)}\qquad \begin{cases} a_{00}\dfrac{\partial^2 v}{\partial t^2}+\alpha(\xi)\dfrac{\partial v}{\partial t}-\beta(\xi)\,v=\widehat{f}(t,\xi)\\ \dfrac{\partial v}{\partial t}(0,\xi)=\widehat{g}(\xi)\end{cases}$$

where $\alpha(\xi)$ and $\beta(\xi)$ are defined respectively by (25) and (26).

Since $v\in L^2(R_+^{n+1})$, from (70) we infer that v can be represented as follows:

$$v(t,\xi)=c(\xi)\exp(tz_1(\xi))-\int_0^{+\infty}\psi(t-s,\xi)\widehat{f}(s,\xi)\,ds,$$

where $z_1(\xi)$ is the root with real negative part of equation (23), $\psi(t,\xi)$ is defined by (22) and $c(\xi)$ is a suitable ξ — function, that is determined by imposing the condition $\frac{\partial v}{\partial t}(0, \xi) = \widehat{g}(\xi)$. We get

$$c(\xi) = \frac{\widehat{g}(\xi)}{z_1(\xi)} + \frac{1}{z_1(\xi)} \int_0^{+\infty} \frac{\partial \psi}{\partial t}(-s, \xi)\widehat{f}(s, \xi)\, ds :$$

hence

$$v(t, \xi) = \frac{\exp(tz_1(\xi))}{z_1(\xi)} \widehat{g}(\xi) + \frac{\exp(tz_1(\xi))}{z_1(\xi)} \int_0^{+\infty} \frac{\partial \psi}{\partial t}(-s, \xi)\widehat{f}(s, \xi)\, ds + \\ - \int_0^{+\infty} \psi(t-s, \xi)\widehat{f}(s, \xi)\, ds. \tag{71}$$

From the formula

$$\left[\mathcal{F}_x \frac{\partial E}{\partial t}(t, \cdot)\right](\xi) = - \frac{\partial \psi}{\partial t}(t, \xi),$$

that is a consequence of lemmas 3 and 4, we infer

$$\int_0^{+\infty} \frac{\partial \psi}{\partial t}(-s, \xi)\widehat{f}(s, \xi)\, ds = \tag{72}$$

$$= - \int_0^{+\infty} ds \int_{R^n} \exp[-i(x, \xi)]\, dx \int_{R^n} \frac{\partial E}{\partial t}(-s, x-z) f(s, z)\, dz =$$

$$= - \int_{R^n} \exp[-i(x, \xi)]\, dx \int_{R^{n+1}_+} \frac{\partial E}{\partial t}(-s, x-z) f(s, z)\, ds\, dz =$$

$$= - \int_{R^n} \exp[-i(x, \xi)] \frac{\partial w}{\partial t}(0, x)\, dx$$

w being defined by (69).

Moreover, for (22)

$$\int_0^{+\infty} \psi(t-s, \xi)\widehat{f}(s, \xi)\, ds = \tag{73}$$

$$= - \int_0^{+\infty} ds \int_{R^n} \exp\left[- i (x, \xi)\right] dx \int_{R^n} E(t - s, x - z) f(s, z)\, dz =$$

$$= - \int_{R^n} \exp\left[- i (x, \xi)\right] w(t, x)\, dx.$$

(58), (71), (72), (73) imply that u is of the form (68): since $C_0^\infty(\overline{R_+^{n+1}})$ is dense in $W^{2,p}(R_+^{n+1})$ the uniqueness is proved.

5. Proof of the theorem.

The proof of the theorem follows easily from lammas 10 and 11 stated below.

Notations: we denote by E^+, P^+, N^+, respectively the fundamental solution and the Poisson kernels related with the operator L^+ in (1). The functions E^- and P^- are analogously defined with regard to L^- in (1); while N^- is defined as follows:

$$N^-(t, x) = - \int_{-\infty}^{t} P^-(s, x)\, ds. \tag{74}$$

Moreover, the functions $\alpha^\pm, \beta^\pm, H^\pm$ are connected with $L^\pm$, according to formulas (24), (25), (26); z_1^+ and z_2^- denote the roots of the equations

$$a_{00}^\pm z^2 + \alpha^\pm(\xi) z - \beta^\pm(\xi) = 0$$

respectively with negative and positive real parts.

LEMMA 10. *Let $u \in W^{2,p}(R^{n+1})$ $(1 < p < +\infty)$ be a solution of* (1). *The assertions stated below are true:*

i) *the following representation formula holds:*

$$u(t, x) = \begin{cases} \displaystyle\int_{R^n} N^+(t, x - z)\left[g(z) - \frac{\partial w^+}{\partial t}(0, z)\right] dz + w^+(t, x) & t > 0 \\ \displaystyle\int_{R^n} N^-(t, x - z)\left[g(z) - \frac{\partial w^-}{\partial t}(0, z)\right] dz + w^-(t, x) & t < 0 \end{cases} \tag{75}$$

where g is the trace of the normal derivative of u on the hyperplane $t = 0$

and

$$w^{\pm}(t, x) = \int_{R_{\pm}^{n+1}} E^{\pm}(t-s, x-z) f(s, z)\, ds\, dz\,; \tag{76}$$

ii) $g = \dfrac{\partial u}{\partial t}(0, \cdot)$ *is a solution belonging to* $W^{1-\frac{1}{p}, p}(R^n)$ *of the integral equation*

$$\int_{R^n} N(x-z)\, g(z)\, dz = (Uf)(x) \tag{77}$$

where

$$N(x) = N^{-}(0, x) - N^{+}(0, x) \tag{78}$$

and

$$(Uf)(x) = \int_{R^n} N^{-}(0, z) \frac{\partial w^{-}}{\partial t}(0, z)\, dz - \tag{79}$$

$$- \int_{R^n} N^{+}(0, z) \frac{\partial w^{+}}{\partial t}(0, z)\, dz - w^{-}(0, z) + w^{+}(0, z).$$

Vice versa, if there exists $g \in W^{1-\frac{1}{p}, p}(R^n)\,(1 < p < +\infty)$ *that satisfies equation* (77)*, then the function defined by* (75) *belongs to* $W^{2,p}(R^{n+1})$ *and is a solution of* (1).

The operator U*, defined by* (79)*, is bounded from* $L^p(R^{n+1})$ *into* $W^{2-\frac{1}{p}, p}(R^n)\,(1 < p < +\infty)$.

LEMMA 11. *The integral equation*

$$\int_{R^n} N(x-z)\, g(z)\, dz = f(x), \tag{80}$$

where N *is defined by* (78) *and* f *is any given function in* $W^{2-\frac{1}{p}, p}(R^n)$*, admits a unique solution* $g \in W^{1-\frac{1}{p}, p}(R^n)\,(1 < p < +\infty)$*. Moreover,* g *verifies the inequality*

$$\| g \|_{W^{1-\frac{1}{p}, p}(R^n)} \leq C \| f \|_{W^{2-\frac{1}{p}, p}(R^n)}, \tag{81}$$

C being a constant independent of f.

PROOF OF LEMMA 10. Let u be a solution of (1): we denote by u^+ and u^- respectively its restrictions to R_+^{n+1} and R_-^{n+1}. Then u^+ and u^- are solutions of the Neumann problems

$$
(82) \qquad \begin{cases} L^+ u^+ = f \\ u^+ \in W^{2,p}(R_+^{n+1}) \\ \dfrac{\partial u^+}{\partial t}(0+, \cdot) = g \end{cases} \qquad \begin{cases} L^- u^- = f \\ u^- \in W^{2,p}(R_-^{n+1}) \\ \dfrac{\partial u^+}{\partial t}(0-, \cdot) = g \end{cases}
$$

Lemma 9, applied to u^+ and u^-, implies that u can be represented as in (75). Moreover, from the equation $u^+(0+, \cdot) = u^-(0-, \cdot)$, (79) and iii) in lemma 8, it follows easily that g is a solution of (77).

Vice versa, if $g \in W^{1-\frac{1}{p}, p}(R^n)$ is a solution of (77) and we denote as before by u^+ and u^- the restrictions to R_+^{n+1} and R_-^{n+1} of the function u defined by (75), lemma 9 implies that u^+ and u^- are solutions of problems (82). Consequently, u satisfies equation (1): it remains to show that $u \in W^{2,p}(R^{n+1})$. This property follows from (82) and equations $\dfrac{\partial u^+}{\partial t}(0+, \cdot) = \dfrac{\partial u^-}{\partial t}(0-, \cdot)$, $u^+(0+, \cdot) = u^-(0-, \cdot)$: the former is an immediate consequence of iv) in lemma 8, while the latter is nothing else but a rearrengement of (77).

PROOF OF LEMMA 11. The existence of a solution of equation (80) belonging to $W^{1-\frac{1}{p}, p}(R^n)$ follows from the property:

i) the operator G, inverse of the convolution with kernel N, is a bounded operator from $W^{2-\frac{1}{p}, p}(R^n)$ into $W^{1-\frac{1}{p}, p}(R^n)$. G is defined for $f \in C_0^\infty(R^n)$ by the equation

$$
(83) \qquad (Gf)(x) = \frac{1}{(2\pi)^n} \int_{R^n} \exp\,[i(x, \xi)] \frac{\widehat{f}(\xi)}{\widehat{N}(\xi)}\, d\xi\,,
$$

where $\widehat{f}$ and $\widehat{N}$ denote the Fourier transforms of f and N.

The uniqueness of the solution is an obvious consequence of property i) and the following one:

ii) the convolution with kernel N is a bounded operator from $W^{1-\frac{1}{p}, p}(R^n)$ into $W^{2-\frac{1}{p}, p}(R^n)$.

Both i) and ii) can be shown by interpolation, using a theorem of Hörmander- Mihlin and well-known properties of interpolation spaces $W^{s,p}(R^n)$ $(1 < s < 2)$. For the sake of brevity we prove i) only.

The quoted interpolation property (see, for instance, [4], p. 399, [5]. chap. VII, § 2, n. 4 or [7] theor. 2.1, 2.8) follows from the assertions:

iii) the operator G, defined by (83), can be extended with a bounded operator from $W^{1,p}(R^n)$ into $L^p(R^n)$;

iv) the operator G can be extended with a bounded operator from $W^{2,p}(R^n)$ into $W^{1,p}(R^n)$.

Clearly, iv) follows from iii), since G commutes with differentiations. Then we focus our attention on iii). Observe that from the equations

$$\widehat{N} = \widehat{N}^{-}(0, \cdot) - \widehat{N}^{+}(0, \cdot)$$

$$\widehat{N}^{-}(0, \xi) = \frac{1}{z_2^{-}(\xi)}$$

$$\widehat{N}^{+}(0, \xi) = \frac{1}{z_1^{+}(\xi)}$$

it follows that

$$\widehat{N}(\xi) = \frac{1}{z_2^{-}(\xi)} - \frac{1}{z_1^{+}(\xi)}$$

where z_1^{+} and z_2^{-} are defined at the beginning of this section.

We have

$$(Gf)(x) = \frac{1}{(2\pi)^n} \int_{R^n} \exp[i(x, \xi)] M(\xi) [(1 - \Delta)^{1/2} f]^{\wedge}(\xi)\, d\xi \qquad f \in C_0^{\infty}(R^n)$$

where

$$M(\xi) = \frac{z_1^{+}(\xi)\, z_2^{-}(\xi)}{[z_1^{+}(\xi) - z_2^{-}(\xi)](1 + |\xi|^2)^{1/2}}$$

and

$$(1 - \Delta)^{1/2} f(x) = \frac{1}{(2\pi)^n} \int_{R^n} \exp[i(x, \xi)] (1 + |\xi|^2)^{1/2} \widehat{f}(\xi)\, d\xi \qquad f \in C_0^{\infty}(R^n).$$

In appendix it is shown that M verifies the following inequalities (that are consequences of the properties of z_1^{+} and z_2^{-}):

$$\text{(84)} \qquad \sup_{\xi \in R^n} |\xi|^{|\gamma|} |D^{\gamma} M(\xi)| \leq C_{\gamma},$$

γ being any multi-index and C_γ some constant depending on γ.

For the theorem of Hörmander-Mihlin inequalities (84) imply that M is a multiplier of type (p, p) for every $1 < p < +\infty$. (For terminology see appendix below).

Moreover, the operator $(1 - \Delta)^{1/2}$ is bounded from $W^{1,p}(R^n)$ into $L^p(R^n)$. This property, that can be easily seen by a further application of the theorem of Hörmander-Mihlin, is a particular case of a theorem of Calderón on spaces of Bessel potentials $L^p_s(R^n)$. Then, property iii) is proved.

Finally, the estimate (81) follows from i).

APPENDIX

For the convenience of the reader we recall, following [3], the definition of multipliers and some criteria that enable to ascertain whether a given function is a multiplier.

DEFINITION. $M^q_p(R^n)\,(p, q \geq 1)$ is the set of Fourier transforms $\widehat{T}$ of all temperate distributions such that

$$\sup_{u \in C_0^\infty(R^n)} \frac{\| T * u \|_{L^q(R^n)}}{\| u \|_{L^p(R^n)}} < +\infty .$$

The elements in $M^q_p(R^n)$ are called multipliers of type (p, q).

THEOREM. ([3], p. 120) *Let $f \in L^\infty(R^n)$ and assume that*

$$\frac{1}{r^n} \int_{\frac{r}{2} \leq |\xi| \leq r} | r^{|\gamma|} D^\gamma f(\xi) |^2 \, d\xi \leq B \qquad 0 < r < +\infty, \quad |\gamma| \leq k,$$

where B is a constant and k is the least integer $> \frac{n}{2}$.

Then f belongs to $M^p_p(R^n)$ for every $1 < p < +\infty$.

In particular we shall use the following corollary (Mihlin's Theorem, see [8]).

COROLLARY 1. *If $f \in C^n(R^n - \{0\})$ and*

$$\sup_{\xi \in R^n} | \xi |^{|\gamma|} | D^\gamma f(\xi) | \leq B \qquad \text{for } |\gamma| \leq n,$$

where B is a constant, then $f \in M^p_p(R^n)$ for every $1 < p < +\infty$.

We prove

PROPOSITION 1. *Let P and Q be two polynomials in (ξ, η) $(\xi \in R^n, \eta \in R^m, n \geq 1, m \geq 1)$ of the same degree q: let $H(\xi) = (H_1(\xi), \dots, H_m(\xi))$ be a vector in R^m with components that are polynomials of degree 2 in ξ and satisfy the following inequalities:*

$$\text{i)} \qquad |H_r(\xi)| \geq C'(1 + |\xi|^2) \qquad (r = 1, \dots, n)$$

$$\text{ii)} \qquad |Q(\xi, H_1(\xi)^{1/2}, \dots, H_m(\xi)^{1/2})| \geq C''(1 + |\xi|^2)^{q/2},$$

C' and C'' being positive constants.

Then the function

$$R(\xi) = \frac{P(\xi, H_1(\xi)^{1/2}, \dots, H_m(\xi)^{1/2})}{Q(\xi, H_1(\xi)^{1/2}, \dots, H_m(\xi)^{1/2})}$$

is a multiplier of type (p, p) for every $1 < p < +\infty$.

PROOF. We observe that for i) and ii) R is in $C^\infty(R^n)$. Moreover

$$\frac{\partial R}{\partial \xi_j} = \left[\frac{\partial P}{\partial \xi_j} + \frac{1}{2}\sum_{r=1}^{m} \frac{\partial P}{\partial \eta_r}\frac{1}{H_r^{1/2}}\frac{\partial H_r}{\partial \xi_j}\right] Q^{-1} - \left[\frac{\partial Q}{\partial \xi_j} + \frac{1}{2}\sum_{r=1}^{m} \frac{\partial Q}{\partial \eta_r}\frac{1}{H_r^{1/2}}\frac{\partial H_r}{\partial \xi_j}\right] PQ^{-2} =$$

$$= \left(\prod_{r=1}^{m} H_r\right)^{-1} Q^{-2} R_j$$

where

$$R_j = \left[\left(\prod_{r=1}^{m} H_r\right)\frac{\partial P}{\partial \xi_j} + \frac{1}{2}\sum_{r=1}^{m}\frac{\partial P}{\partial \eta_r}\frac{\partial H_r}{\partial \xi_j}\left(\prod_{s \neq r} H_s\right) H_r^{1/2}\right] Q +$$

$$- \left[\left(\prod_{r=1}^{m} H_r\right)\frac{\partial Q}{\partial \xi_j} + \frac{1}{2}\sum_{r=1}^{m}\frac{\partial Q}{\partial \eta_r}\frac{\partial H_r}{\partial \xi_j}\left(\prod_{s \neq r} H_s\right) H_r^{1/2}\right] P$$

is a polynomial in $(\xi, H_1(\xi)^{1/2}, \dots, H_m(\xi)^{1/2})$ of degree less or equal to $2m + 2q - 1$.

CLAIM: for all multi-index γ the following equation holds:

$$\text{(A1)} \qquad D^\gamma R = \left(\prod_{r=1}^{m} H_r\right)^{-|\gamma|} Q^{-1-|\gamma|} R_\gamma,$$

R_γ being a polynomial in $(\xi, H_1(\xi)^{1/2}, \dots, H_m(\xi)^{1/2})$ of degree less or equal to $|\gamma|(2m+q-1)+q$.

The proof proceeds by induction: we suppose that (A1) is true for all γ with $|\gamma| = r$ and we show that it is valid for all γ with $|\gamma| = r+1$, For,

$$\frac{\partial}{\partial \xi_j} D^\gamma R = -|\gamma| \left(\prod_{r=1}^{m} H_r\right)^{-1-|\gamma|} Q^{-1-|\gamma|} R_\gamma \sum_{r=1}^{m} \left(\prod_{s\neq r} H_s\right) \frac{\partial H_r}{\partial \xi_j} +$$

$$-(1+|\gamma|)\left(\prod_{r=1}^{m} H_r\right)^{-|\gamma|} Q^{-2-|\gamma|} R_\gamma \left[\frac{\partial Q}{\partial \xi_j} + \frac{1}{2}\sum_{r=1}^{m} \frac{\partial Q}{\partial \eta_r} \frac{1}{H_r^{1/2}} \frac{\partial H_r}{\partial \xi_j}\right] +$$

$$+\left(\prod_{r=1}^{m} H_r\right)^{-|\gamma|} Q^{-1-|\gamma|} \left[\frac{\partial R_\gamma}{\partial \xi_j} + \frac{1}{2}\sum_{r=1}^{m} \frac{\partial R_\gamma}{\partial \eta_r} \frac{1}{H_r^{1/2}} \frac{\partial H_r}{\partial \xi_j}\right] = \left(\prod_{r=1}^{m} H_r\right)^{-1-|\gamma|} Q^{-2-|\gamma|} R_{\gamma,j}$$

where

$$R_{\gamma,j} = -|\gamma|\, Q R_\gamma \sum_{r=1}^{m} \left(\prod_{s\neq r} H_s\right) \frac{\partial H_r}{\partial \xi_j} -$$

$$-(1+|\gamma|) R_\gamma \left[\left(\prod_{r=1}^{m} H_r\right) \frac{\partial Q}{\partial \xi_j} + \frac{1}{2}\sum_{r=1}^{m} \frac{\partial Q}{\partial \eta_r}\left(\prod_{s\neq r} H_s\right) H_r^{1/2} \frac{\partial H_r}{\partial \xi_j}\right] +$$

$$+\left(\prod_{r=1}^{m} H_r\right) Q \frac{\partial R_\gamma}{\partial \xi_j} + \frac{1}{2} Q \sum_{r=1}^{m} \frac{\partial R_\gamma}{\partial \eta_r}\left(\prod_{s\neq r} H_s\right) H_r^{1/2} \frac{\partial H_r}{\partial \xi_j}.$$

Clearly $R_{\gamma,j}$ is a polynomial in $(\xi, H_1(\xi)^{1/2}), \dots H_m(\xi)^{1/2})$ of degree less or equal to $(1+|\gamma|)(2m+q-1)+q$ for the hypothesis of induction: the proof of (A1) is fulfilled.

From (A1) and i), ii) and the hypothesis on P and Q we infer that there exists a constant C_γ such that

$$\text{(A2)} \qquad |D^\gamma R(\xi)| \leq C_\gamma (1+|\xi|^2)^{-|\gamma|/2} \qquad |\gamma| \geq 0.$$

From (A2) and an application of corollary 1, the assertion follows.

From proposition 1, iii) in remark 1 after lemma 3, (29), the definitions of z_1^+ and z_2^-, the fact that A is a symmetric positive definite matrix and $\alpha^\pm(\xi)$, defined as in (25), are linear functions in ξ, there follows easily:

PROPOSITION 2. The following functions are in $M_p^p(R^n)$ for every $1 < p < +\infty$:

$$\varphi_{rj}(\xi) = \frac{\xi_r \, \xi_j}{(A\,\xi, \xi) - i\,(a, \xi) + h^2} \qquad (r, j = 1, \dots, n)$$

$$\varphi_j^+(\xi) = \frac{i\,\xi_j}{z_j^+(\xi)} \qquad (j = 1, \dots, n)$$

$$\varphi_j^-(\xi) = \frac{i\,\xi_j}{z_2^-(\xi)} \qquad (j = 1, \dots, n)$$

$$M(\xi) = \frac{z_1^+(\xi)\, z_2^-(\xi)}{[z_1^+(\xi) - z_2^-(\xi)]\,(1 + |\xi|^2)^{1/2}}\,.$$

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