Mathematical
Inequalities
& Applications
Volume 4, Number 3 (2001), 397–428

ON EMBEDDINGS BETWEEN CLASSICAL LORENTZ SPACES

M. CARRO, L. PICK, J. SORIA AND V. D. STEPANOV

Abstract. Let $p \in (0,\infty)$, let v be a weight on $(0,\infty)$ and let $\Lambda^p(v)$ be the classical Lorentz space, determined by the norm $\|f\|_{\Lambda^p(v)} := (\int_0^\infty (f^*(t))^p v(t)\,dt)^{1/p}$. When $p \in (1,\infty)$, this space is known to be a Banach space if and only if v is non-increasing, while it is only equivalent to a Banach space if and only if $\Lambda^p(v) = \Gamma^p(v)$, where $\|f\|_{\Gamma^p(v)} := (\int_0^\infty (f^{**}(t))^p v(t)\,dt)^{1/p}$. We may thus conclude that, for $p \in (1,\infty)$, the space $\Lambda^p(v)$ is equivalent to a Banach space if and only if the norm of a function f in it can be expressed in terms of f^{**}. We study the question whether an analogous assertion holds when $p = 1$. Motivated by this problem, we consider general embeddings between four types of classical and weak Lorentz spaces, namely, $\Lambda^p(v)$, $\Lambda^{p,\infty}(v)$, $\Gamma^p(v)$, $\Gamma^{p,\infty}(v)$, where $\Lambda^{p,\infty}(v)$ and $\Gamma^{p,\infty}(v)$ are certain weak analogues of the spaces $\Lambda^p(v)$ and $\Gamma^p(v)$, respectively. We present a unified approach to these embeddings, based on rearrangement techniques. We survey all the known results and prove new ones. Our main results concern the embedding $\Gamma^{p,\infty}(v) \hookrightarrow \Lambda^q(w)$ which had not been characterized so far. We apply our results to the characterization of associate spaces of classical and weak Lorentz spaces and we give a characterization of fundamental functions for which the endpoint Lorentz space and the endpoint Marcinkiewicz space coincide.

Mathematics subject classification (2000): 46E30, 26D15.
Key words and phrases: Classical Lorentz spaces, continuous embeddings, weighted inequalities for non-increasing functions.

REFERENCES

[1] M. ARIÑO AND B. MUCKENHOUPT, *Maximal functions on classical Lorentz spaces and Hardy's inequality with weights for nonincreasing functions*, Trans. Amer. Math. Soc. **320** (1990), 727–735.
[2] S. BARZA, L. E. PERSSON AND J. SORIA, *Sharp weighted multidimensional integral inequalities for monotone functions*, Math. Nachr **210** (2000), 43–58.
[3] C. BENNETT AND K. RUDNICK, *On Lorentz-Zygmund spaces*, Dissert. Math. **175** (1980), 1–72.
[4] C. BENNETT AND R. SHARPLEY, *Interpolation of Operators*, Academic Press, Princeton, 1988.
[5] J. BERGH AND J. LÖFSTRÖM, *Interpolation Spaces. An Introduction*, Springer-Verlag, Berlin, 1976.
[6] D. W. BOYD, *The Hilbert transform on rearrangement-invariant spaces*, Canad. J. Math. **19** (1967), 599-616.
[7] M. CARRO, A. GARCÍA DEL AMO AND J. SORIA, *Weak-type weights and normable Lorentz spaces*, Proc. Amer. Math. Soc. **124** (1996), 849–857.
[8] M. CARRO AND J. SORIA, *Weighted Lorentz spaces and the Hardy operator*, J. Funct. Analysis **112** (1993), 480–494.
[9] M. CARRO AND J. SORIA, *Boundedness of some integral operators*, Canad. J. Math. **45** (1993), 1155–1166.
[10] M. CARRO AND J. SORIA, *The Hardy-Littlewood maximal function and weighted Lorentz spaces*, J. London Math. Soc. **55** (1997), 146–158.
[11] M. L. GOL'DMAN, *On integral inequalities on a cone of functions with monotonicity properties*, Soviet Math. Dokl. **44** (1992), 581–587.

[12] M. L. GOL'DMAN, *On integral inequalities on the set of functions with some properties of monotonicity*, Function spaces, Differential Operators and Nonlinear Analysis, Teubner Texte Zur Math. 133, Leipzig, 1993, pp. 274–279.
[13] M. L. GOL'DMAN, H. P. HEINIG AND V. D. STEPANOV, *On the principle of duality in Lorentz spaces*, Canad. J. Math. **48** (1996), 959–979.
[14] I. HALPERIN, *Function spaces*, Canad. J. Math. **5** (1953), 273–288.
[15] H. P. HEINIG AND L. MALIGRANDA, *Weighted inequalities for monotone and concave functions*, Studia Math. **116** (1995), 133–165.
[16] S. LAI, *Weighted norm inequalities for general operators on monotone functions*, Trans. Amer. Math. Soc. **340** (1993), 811–836.
[17] G. G. LORENTZ, *On the theory of spaces* Λ, Pacific J. Math. **1** (1951), 411–429.
[18] C. J. NEUGEBAUER, *Weighted norm inequalities for averaging operators of monotone functions*, Publ. Math. **35** (1991), 429–447.
[19] E. SAWYER, *Boundedness of classical operators on classical Lorentz spaces*, Studia Math. **96** (1990), 145–158.
[20] G. SINNAMON, *Spaces defined by level functions and their duals*, Studia Math. **111** (1994), 19–52.
[21] G. SINNAMON AND V. D. STEPANOV, *The weighted Hardy inequality: new proofs and the case* $p = 1$, J. London Math. Soc. **54** (1996), 89–101.
[22] J. SORIA, *Lorentz spaces of weak-type*, Quart. J. Math. Oxford **49** (1998), 93–103.
[23] V. D. STEPANOV, *The weighted Hardy's inequality for nonincreasing functions*, Trans. Amer. Math. Soc. **338** (1993), 173–186.
[24] V. D. STEPANOV, *Integral operators on the cone of monotone functions*, J. London Math. Soc. **48** (1993), 465–487.
[25] V. D. STEPANOV, *On integral operators on the cone of monotone functions, and on imbeddings of Lorentz spaces*, Soviet Math. Dokl. **43** (1991), 620–623.