ON ENTIRE FUNCTIONS OF EXPONENTIAL TYPE AND INDICATORS OF ANALYTIC FUNCTIONALS

BY

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0. Summary

We shall be concerned with the indicator p of an analytic functional μ on a complex manifold U:

$$p(\varphi) = \overline{\lim_{t\to+\infty}} \frac{1}{t} \log |\mu(e^{t\varphi})|,$$

where φ is an arbitrary analytic function on U. More specifically, we shall consider the smallest upper semicontinuous majorant $p^{\mathfrak{F}}$ of the restriction of p to a subspace $\mathfrak{F}$ of the analytic functions. An obvious problem is then to characterize the set of functions $p^{\mathfrak{F}}$ which can occur as regularizations of indicators. In the case when $U = \mathbf{C}^n$ and $\mathfrak{F}$ is the space of all linear functions on $\mathbf{C}^n$, this set can be described more easily as the set of functions

$$\lim_{\theta\to\zeta} \overline{\lim_{t\to+\infty}} \frac{1}{t} \log |u(t\theta)| \tag{0.1}$$

of n complex variables $\zeta \in \mathbf{C}^n$ where u is an entire function of exponential type in $\mathbf{C}^n$. We shall prove that a function in $\mathbf{C}^n$ is of the form (0.1) for some entire function u of exponential type if and only if it is plurisubharmonic and positively homogeneous of order one (Theorem 3.4). The proof is based on the characterization given by Fujita and Takeuchi of those open subsets of complex projective n-space which are Stein manifolds.

Our objective in Sections 4 and 5 is to study the relation between properties of $p^{\mathfrak{F}}$ and existence and uniqueness of $\mathfrak{F}$-supports of μ, i.e. carriers of μ which are convex with respect to $\mathfrak{F}$ in a certain sense and which are minimal with this property (see Section 1 for defini-

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tions). An example is that under certain regularity conditions, $p^{\mathfrak{J}}$ is convex if and only if μ has only one $\mathfrak{J}$-support.

Section 2 contains a result on plurisubharmonic functions in infinite-dimensional linear spaces and approximation theorems for homogeneous plurisubharmonic functions in $\mathbf{C}^n$.

The author's original proof of Theorem 3.1 was somewhat less direct than the present one (see the remark at the end of Section 3). It was suggested by Professor Lars Hörmander that a straightforward calculation of the Levi form might be possible. I wish to thank him also for other valuable suggestions and several discussions on the subject.

Notation. The complement of a set A with respect to some bigger set which is understood from the context is denoted by $\complement A$. We write $A\setminus B$ for $A \cap \complement B$. The Cartesian product of n copies of A is denoted by A^n.

The interior of a set A in a topological space is denoted by A°, its closure by $\bar{A}$, and its boundary by ∂A. $\mathbf{N}$, $\mathbf{R}$, $\mathbf{C}$ stand for the set of non-negative integers, real numbers, and complex numbers, respectively. The sets $[-\infty, +\infty[= \mathbf{R} \cup \{-\infty\}$ and $[-\infty, +\infty] = \mathbf{R} \cup \{-\infty, +\infty\}$ shall be equipped with their natural topologies so that, for instance, $[-\infty, +\infty]$ is compact. We use a bar to denote complex conjugation of complex numbers and complex-valued functions. The differential operators $\partial/\partial z_k$ and $\partial/\partial \bar{z}_k$ are defined for functions of n complex variables by

$$\frac{\partial}{\partial z_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} - i\,\frac{\partial}{\partial y_k}\right), \quad \frac{\partial}{\partial \bar{z}_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} + i\,\frac{\partial}{\partial y_k}\right),$$

where $z_k = x_k + iy_k$; $x_k, y_k \in \mathbf{R}$, $k = 1, \ldots, n$. The space of complex-valued linear forms on $\mathbf{C}^n$ is denoted by $\mathfrak{L}$ or $\mathfrak{L}(\mathbf{C}^n)$ and the value of $\zeta \in \mathfrak{L}$ at a point $z \in \mathbf{C}^n$ is written $\zeta(z) = \langle z, \zeta\rangle$. There is sometimes no advantage in identifying $\mathfrak{L}$ with $\mathbf{C}^n$; on other occasions, however, we shall do so by means of the formula $\langle z, \zeta\rangle = \sum_1^n z_j \zeta_j$. The norm in $\mathbf{C}^n$ (and in $\mathfrak{L}$ when we use coordinates there) will always be Euclidean: $|z| = (\sum_1^n z_j \bar{z}_j)^{1/2}$.

1. Basic definitions

Let U be a complex analytic manifold. We shall denote by $\mathcal{A}(U)$ the space of all analytic functions in U equipped with the topology of uniform convergence on all compact subsets of U. A continuous complex-valued linear form μ on $\mathcal{A}(U)$ or, in other words, an element of the dual space $\mathcal{A}'(U)$, is called an analytic functional in U. If $\mu \in \mathcal{A}'(U)$, the continuity means that for some compact set $K \subset U$ and some constant C we have

$$|\mu(\varphi)| \leq C \sup_K |\varphi| \tag{1.1}$$

for all $\varphi \in \mathcal{A}(U)$.

It is convenient to have a name for sets which are limits of decreasing sequences of compact sets for which (1.1) holds:

Definition 1.1. *A carrier of an analytic functional* $\mu \in \mathcal{A}'(U)$ *is a compact subset* K *of* U *such that for every neighborhood* L *of* K *there is a constant* C *such that* $|\mu(\varphi)| \leq C \sup_L |\varphi|$ *for all* $\varphi \in \mathcal{A}(U)$.

In Martineau [8] the word carrier is used with a different meaning. However, Martineau's definition and Definition 1.1 coincide if U is a Stein manifold and K is holomorph convex and this will usually be the case in what follows.

The main part of this paper is devoted to a study of carriers of an analytic functional μ which are minimal with respect to inclusion in the family of all those carriers of μ which are convex in a certain sense. We proceed to define these convexity properties.

Definition 1.2. *Let* M *be a subset of* U. *We define the supporting function* H_M *of* M *by*

$$H_M(\varphi) = \sup_{z \in M} \operatorname{Re} \varphi(z), \quad \varphi \in \mathcal{A}(U).$$

(*The supremum over the empty set is defined as* $-\infty$.)

With obvious conventions on infinite values, H_M is convex and positively homogeneous in $\mathcal{A}(U)$, that is

$$H_M(t\varphi) = tH_M(\varphi), \quad H_M(\varphi+\psi) \leq H_M(\varphi) + H_M(\psi), \quad t>0, \varphi, \psi \in \mathcal{A}(U).$$

If M is non-empty and relatively compact in U, H_M is in addition real-valued and continuous. In the special case when $U = \mathbf{C}^n$, the restriction of H_M to the linear functions $\mathfrak{L}$ in $\mathbf{C}^n$ is the usual supporting function of M and it is well-known that every convex and positively homogeneous real-valued function in $\mathfrak{L}$ is such a restriction. In general, however, the restrictions of the supporting functions to a subspace $\mathfrak{F}$ of $\mathcal{A}(U)$ form a proper subset of the convex and positively homogeneous functions in $\mathfrak{F}$.

Definition 1.3. *Let* $\mathfrak{F}$ *be an arbitrary subset of* $\mathcal{A}(U)$. *We define the* $\mathfrak{F}$*-hull* $h_{\mathfrak{F}} M$ *of a set* $M \subset U$ *by*

$$h_{\mathfrak{F}} M = \{z \in U; \text{ for all } \varphi \in \mathfrak{F}, \operatorname{Re} \varphi(z) \leq H_M(\varphi)\}.$$

If $h_{\mathfrak{F}} M = M$ *we say that* M *is* $\mathfrak{F}$*-convex. The manifold* U *is called* $\mathfrak{F}$*-convex if* $h_{\mathfrak{F}} K$ *is compact for every compact set* $K \subset U$.

In other words, $h_{\mathfrak{F}} M$ is the largest subset of U whose supporting function coincides in $\mathfrak{F}$ with that of M. If $\mathfrak{F} \subset \mathcal{G} \subset \mathcal{A}(U)$ and $M \subset N \subset U$ it is obvious that $h_{\mathcal{G}} M \subset h_{\mathfrak{F}} N$. We also have $h_{\mathfrak{F}} h_{\mathcal{G}} M = h_{\mathfrak{F}} M = h_{\mathcal{G}} h_{\mathfrak{F}} M$ if $\mathfrak{F} \subset \mathcal{G}$; in particular $h_{\mathfrak{F}} M$ is $\mathcal{G}$-convex. Another property of the $\mathcal{J}$-hull is that $h_{\mathcal{J}}(\bigcap M_i) \subset \bigcap h_{\mathcal{J}} M_i$; this implies that the intersection of a family of $\mathcal{J}$-convex sets is $\mathcal{J}$-convex.

If $U = \mathbf{C}^n$ and $\mathcal{J} = \mathcal{L}$, the space of all linear functions on $\mathbf{C}^n$, $h_{\mathcal{J}} M = h_{\mathcal{L}} M$ is the closed convex hull of M. More generally, if $\mathcal{J}$ is a finite-dimensional complex linear subspace of $\mathcal{A}(U)$ we can choose a basis $\alpha_1, \ldots, \alpha_m$ in $\mathcal{J}$, and then $h_{\mathcal{J}} M$ is the inverse image under $\alpha = (\alpha_1, \ldots, \alpha_m)$ of the closed convex hull of $\alpha(M)$. Another important special case is when $\mathcal{J} = \mathcal{A}(U)$. Then the $\mathcal{J}$-hull is the usual holomorph convex hull. In fact, if z is in the holomorph convex hull of M we have $|\varphi(z)| \leqslant \sup_M |\varphi|$ for all $\varphi \in \mathcal{A}(U)$, in particular with $\varphi = e^{\psi}$, $e^{\mathrm{Re}\,\psi(z)} \leqslant \sup_M e^{\mathrm{Re}\,\psi}$ so that $z \in h_{\mathcal{A}(U)} M$. On the other hand, if $z \in h_{\mathcal{A}(U)} M$ and $\varphi \in \mathcal{A}(U)$, $|t| = 1$, we obtain $\mathrm{Re}\, t\varphi(z) \leqslant \sup_M \mathrm{Re}\, t\varphi \leqslant \sup_M |\varphi|$ and hence $|\varphi(z)| \leqslant \sup_M |\varphi|$ which means that z belongs to the holomorph convex hull of M. The same proof shows that the real parts in Definitions 1.2 and 1.3 can be replaced by absolute values without affecting $h_{\mathcal{J}} M$ if $\mathcal{J}$ is a subalgebra of $\mathcal{A}(U)$ and in addition either $\mathcal{J}$ is closed or M is relatively compact in U.

Definition 1.4. *A compact subset K of U is called an $\mathcal{J}$-support of $\mu \in \mathcal{A}'(U)$ if K is an $\mathcal{J}$-convex carrier of μ and K is minimal with respect to this property, that is, $h_{\mathcal{J}} L \supset K$ for every carrier L of μ such that $L \subset K$.*

By the Zorn lemma, μ has an $\mathcal{J}$-support if (and only if) μ is carried by some $\mathcal{J}$-convex compact set. This is always the case on an $\mathcal{J}$-convex manifold.

The supporting functions of the $\mathcal{J}$-supports of a functional are closely connected with the growth properties of a generalized Laplace transform $\hat{\mu}$ of μ. This relation is the subject of Section 5; we prefer, however, to define $\hat{\mu}$ now.

Definition 1.5. *Let $\mu \in \mathcal{A}'(U)$. The (generalized) Laplace transform $\hat{\mu}$ of μ is defined by $\hat{\mu}(\varphi) = \mu(e^{\varphi})$, $\varphi \in \mathcal{A}(U)$, and the indicator of μ (or of $\hat{\mu}$) is $p(\varphi) = \overline{\lim}_{t \to +\infty} \log |\hat{\mu}(t\varphi)|^{1/t}$, $t > 0$.*

If $U = \mathbf{C}^n$, the restriction of $\hat{\mu}$ to $\mathcal{L}$ is the usual Laplace (or Fourier-Borel) transform of μ which is an entire function of exponential type in $\mathcal{L}$. The name indicator or indicator function is also usually reserved for the restriction of p to $\mathcal{L}$.

Now suppose that K is a carrier of μ. If L is a neighborhood of K there is a constant C such that for all $\varphi \in \mathcal{A}(U)$,

$$|\hat{\mu}(t\varphi)| = |\mu(e^{t\varphi})| \leqslant C \sup_L \exp \mathrm{Re}\, t\varphi,$$

hence if $t > 0$,

$$\frac{1}{t} \log |\hat{\mu}(t\varphi)| \leqslant \frac{1}{t} \log C + H_L(\varphi).$$

Taking the upper limit as $t \to +\infty$, we get

$$p(\varphi) \leqslant H_L(\varphi).$$

Since L is an arbitrary neighborhood of K we finally obtain

$$p(\varphi) \leqslant H_K(\varphi) \tag{1.2}$$

for all $\varphi \in \mathcal{A}(U)$. In Section 5 we shall prove under certain assumptions on a linear subspace $\mathfrak{F}$ of $\mathcal{A}(U)$ that conversely μ is carried by an $\mathfrak{F}$-convex compact set K if (1.2) holds for all $\varphi \in \mathfrak{F}$.

2. Plurisubharmonic functions

Let Ω be an open set in a linear topological space $\mathfrak{F}$ over the complex field $\mathbf{C}$. A function F in Ω with values in $[-\infty, +\infty[$ is called plurisubharmonic (in symbols $F \in \mathfrak{P}(\Omega)$) if F is upper semicontinuous (i.e., $\{\varphi \in \Omega;\ F(\varphi) < c\}$ is open for every real c) and the function $t \to F(\varphi + t\psi)$ is subharmonic for all $\varphi, \psi \in \mathfrak{F}$ in the open subset of $\mathbf{C}$ where it is defined. This means that $F \in \mathfrak{P}(\Omega)$ if and only if F is upper semicontinuous and

$$F(\varphi) \leqslant \frac{1}{\pi} \int_D F(\varphi + t\psi)\, d\lambda(t) \tag{2.1}$$

for all $\varphi, \psi \in \mathfrak{F}$ such that $\varphi + D\psi \subset \Omega$, where $D = \{t \in \mathbf{C};\ |t| \leqslant 1\}$ and $d\lambda$ is the Lebesgue measure in $\mathbf{C}$.

In Section 5, $\mathfrak{F}$ will always be a subspace of $\mathcal{A}(U)$ with the topology induced by the latter space (U is a complex analytic manifold). If $\mu \in \mathcal{A}'(U)$, then $\hat{\mu}$ is analytic in $\mathcal{A}(U)$ (that is, $\hat{\mu}(\varphi + t\psi) = \mu(e^{\varphi + t\psi})$ is analytic in $t \in \mathbf{C}$ for all $\varphi, \psi \in \mathcal{A}(U)$) and also continuous; therefore $\log |\hat{\mu}| \in \mathfrak{P}(\mathcal{A}(U))$. From this basic example we can construct other plurisubharmonic functions by means of the following theorem which extends a result of Ducateau [1, Proposition 11].

THEOREM 2.1. *Let $\mathfrak{F}$ be a complex linear topological space such that there exists a countable base for the neighborhoods of the origin. Let Ω be an open set in $\mathfrak{F}$ and $(F_\iota)_{\iota \in I}$ a family of plurisubharmonic functions in Ω indexed by a directed set I which is cofinal with a sequence (in the applications I will be either the integers or the reals with their natural order). Suppose that $(F_\iota)_{\iota \in I}$ is bounded from above on every compact set in Ω. Then the upper regularization F^* of $F = \overline{\lim}_{\iota \in I} F_\iota$ is plurisubharmonic in Ω. The analogous conclusion holds for the upper regularization of $G = \sup_{\iota \in I} F_\iota$ without any restriction or structure on the index set.*

Here the upper regularization F^* is the smallest upper semi-continuous majorant of F with values in $[-\infty, +\infty[$, i.e., $F^*(\varphi) = \overline{\lim}_{\psi \to \varphi} F(\psi)$.

Proof of Theorem 2.1. Let φ and ψ be fixed elements of $\mathcal{J}$ such that $\varphi + D\psi \subset \Omega$. Take any $\chi \in \mathcal{J}$ so near the origin that $\varphi + \chi + D\psi \subset \Omega$. Since (F_ι) is bounded from above on the compact set $\varphi + \chi + D\psi$, Fatou's lemma can be applied to the family of functions $t \to F_\iota(\varphi + \chi + t\psi)$ on D which gives

$$F(\varphi + \chi) \leqslant \overline{\lim_{\iota \in I}} \frac{1}{\pi} \int_D F_\iota(\varphi + \chi + t\psi)\, d\lambda(t)$$

$$\leqslant \frac{1}{\pi} \int_{*D} F(\varphi + \chi + t\psi)\, d\lambda(t) \leqslant \frac{1}{\pi} \int_D F^*(\varphi + \chi + t\psi)\, d\lambda(t), \tag{2.2}$$

where $\int_*$ denotes the Lebesgue lower integral. In view of our assumption on $\mathcal{J}$ there is a sequence $(\chi_j)_{j \in \mathbf{N}}$ tending to 0 such that $\overline{\lim}_{j \to +\infty} F(\varphi + \chi_j) = F^*(\varphi)$. Also F^* is bounded from above on $\varphi + D\psi$, hence on a neighborhood of this set so that the sequence of functions $D \ni t \to F^*(\varphi + \chi_j + t\psi)$ is bounded from above. Applying Fatou's lemma to this sequence we obtain from (2.2)

$$F^*(\varphi) = \overline{\lim_{j \to +\infty}} F(\varphi + \chi_j) \leqslant \overline{\lim_{j \to +\infty}} \frac{1}{\pi} \int_D F^*(\varphi + \chi_j + t\psi)\, d\lambda(t) \leqslant \frac{1}{\pi} \int_D F^*(\varphi + t\psi)\, d\lambda(t),$$

i.e., we have proved that (2.1) holds for F^*.

The proof for G^* is similar except that the inequality corresponding to (2.2) is trivial:

$$G(\varphi + \chi) \leqslant \frac{1}{\pi} \int_D G^*(\varphi + \chi + t\psi)\, d\lambda(t).$$

The theorem is proved.

If G is plurisubharmonic in a linear space $\mathcal{J}$ satisfying the assumptions of Theorem 2.1 and if

$$G(\varphi) \leqslant C + q(\varphi), \quad \varphi \in \mathcal{J},$$

for some constant C and some continuous seminorm q in $\mathcal{J}$ we define

$$F(\varphi) = \overline{\lim_{t \to +\infty}} G(t\varphi)/t.$$

The family of functions $G(t\varphi)/t$, $t \geqslant 1$, is bounded from above by $|C| + q(\varphi)$ so Theorem 2.1 shows that $F^* \in \mathcal{P}(\mathcal{J})$. In particular, if $\mathcal{J}$ is an arbitrary subspace of $\mathcal{A}(U)$ where U is a complex manifold which is countable at infinity, then the upper regularization of the restriction to $\mathcal{J}$ of the indicator of an analytic functional $\mu \in \mathcal{A}'(U)$ is plurisubbarmonic and positively homogeneous of order one in $\mathcal{J}$.

We shall need the following result of Hardy and Rogosinski [3, Theorem 5] concerning (pluri-)subharmonic functions in one variable which are positively homogeneous.

THEOREM 2.2. *Let $\Omega \subset \mathbf{C}$ be a connected open set such that $0 \notin \Omega$ and $t\zeta \in \Omega$ if $\zeta \in \Omega$, $t>0$. Suppose that $F \in \mathcal{P}(\Omega)$ is positively homogeneous of order ϱ, i.e.*

$$F(t\zeta) = t^{\varrho} F(\zeta), \quad \zeta \in \Omega, t>0.$$

Then F is convex with respect to the functions in Ω which are harmonic and positively homogeneous of order ϱ in the sense that if $b-a<\varrho^* = \inf(2\pi, \pi|\varrho|^{-1})$, the open set

$$\{\zeta \in \Omega; |\zeta| = 1, a < \arg \zeta < b, F(\zeta) < H(\zeta)\} \tag{2.3}$$

is connected for every such harmonic function H defined in the sector $a < \arg \zeta < b$. If $\varrho = 1$, F is convex in the usual sense in Ω.

Proof. The harmonic functions which are positively homogeneous of order $\varrho \neq 0$ are given locally by

$$H(\zeta) = A \operatorname{Re} (\zeta/a)^{\varrho} = Ar^{\varrho} \cos \varrho(\varphi - \varphi_0),$$

where $\zeta = re^{i\varphi}$, $a = e^{i\varphi_0}$ and we have defined ζ^{ϱ} e.g. when ζ is not $\leqslant 0$. If $\varrho = 0$, we have

$$H(\zeta) = A\varphi + B.$$

It is easy to see that there is exactly one such function which assumes given real values at two given points ζ_1, ζ_2 satisfying $0 < \arg \zeta_2 - \arg \zeta_1 < \varrho^*$.

Now suppose that $b - a < \varrho^*$ and that (2.3) is not connected for some harmonic function H. This means that we can find points ζ_1, ζ_3 such that $a < \arg \zeta_1 < \arg \zeta_3 < b$, $F(\zeta_1) < H(\zeta_1)$, $F(\zeta_3) < H(\zeta_3)$ and the set

$$K_H = \{\zeta; |\zeta| = 1, \arg \zeta_1 < \arg \zeta < \arg \zeta_3, F(\zeta) \geqslant H(\zeta)\}$$

is contained in Ω and not empty. Let H_ε be the harmonic function which is positively homogeneous of order ϱ and assumes the values $H(\zeta_j) + \varepsilon$ at ζ_j, $j = 1, 3$. Choose $\varepsilon \geqslant 0$ so that K_{H_ε} is non-empty but $F(\zeta) \leqslant H_\varepsilon(\zeta)$ for all ζ satisfying $\arg \zeta_1 < \arg \zeta < \arg \zeta_3$. Let ζ_2 be the point in K_{H_ε} with least argument. Taking a small disk with center at ζ_2 we conclude that $F - H_\varepsilon$ is $\leqslant 0$ in the whole disk, $=0$ at its center but <0 somewhere in the disk. Clearly this violates the mean value property (2.1). This proves the theorem. We have in particular proved that F cannot assume the value $-\infty$ without being $-\infty$ everywhere in Ω. It is also obvious that F is continuous.

Remark. If $\varrho=1$ and $\Omega=\mathbf{C}\setminus\{0\}$ we can extend F to a function in $\mathcal{P}(\mathbf{C})$. For the continuity implies that $|F|\leqslant C$ on the unit circle, hence $|F(\zeta)|\leqslant C|\zeta|$, $\zeta\neq 0$, which proves that F becomes continuous if we define $F(0)=0$. Moreover, if $\zeta\neq 0$ we obtain

$$0=\lim_{s\to 0+} 2sF(i\zeta)\leqslant \lim_{s\to 0+}(F(is\zeta+\zeta)+F(is\zeta-\zeta))=F(\zeta)+F(-\zeta),$$

which means that F is convex on every straight line through 0, hence everywhere.

We note also that the theorem gives a bound from below for the second differences of the functions considered. If $\delta>0$ is sufficiently small (how small depends only on ϱ) we obtain for all $\eta\in[-\delta,\delta]$

$$\Delta_F^2=F(1+i\eta)-2F(0)+F(1-i\eta)\geqslant -C\eta^2\sup(|F(1+i\eta)|,|F(1-i\eta)|),$$

where the constant C depends on ϱ and δ but not on F or η. To prove this estimate we let $H(re^{i\varphi})=Ar^{\varrho}\cos\varrho(\varphi-\varphi_0)$ be a harmonic function homogeneous of order $\varrho\neq 0$. A calculation shows that when $\zeta=1+i\eta=re^{i\varphi}$,

$$\frac{\partial^2 H}{\partial\eta^2}(re^{i\varphi})=\frac{\varrho(1-\varrho)}{r^2}H(re^{i\varphi_1}),$$

where $\varphi_1=\varphi(1-2/\varrho)$. Hence

$$\sup_{I_0}\left|\frac{\partial^2 H}{\partial\eta^2}\right|\leqslant|\varrho(1-\varrho)|\sup_{I_1}|H|,$$

where $I_0=\{1+i\eta;\eta\in[-\eta_0,\eta_0]\}$ and I_1 is the arc $I_1=\{re^{i\varphi(1-2/\varrho)};\ re^{i\varphi}\in I_0\}$ which occupies an angle $|1-2/\varrho|$ times that of I_0 viewed from the origin. Now it is easy to see that when $\arg(1+i\eta_0)$ and $|1-2/\varrho|\arg(1+i\eta_0)$ are smaller than $\varepsilon<\varrho^*/2$, $\sup_{I_1}|H|$ can be estimated by the value of $|H|$ at $1+i\eta_0$ and $1-i\eta_0$:

$$\sup_{I_1}|H|\leqslant C_{\varrho,\varepsilon}\sup(|H(1+i\eta_0)|,|(H(1-i\eta_0)|),$$

where the constant is independent of H and η_0. Hence

$$\sup_{I_0}\left|\frac{\partial^2 H}{\partial\eta^2}\right|\leqslant C\sup(|H(1+i\eta_0)|,|H(1-i\eta_0)|)$$

if $\varrho\neq 0$ and η_0 is small. If $\varrho=0$ it is easy to prove this inequality directly. We therefore have proved in particular that

$$H(1+i\eta_0)-2H(1)+H(1-i\eta_0)\geqslant -C\eta_0^2\sup(|H(1+i\eta_0)|,|H(1-i\eta_0)|)$$

and changing notation we obtain the desired estimate for Δ_H^2. Now if F is only subharmonic we choose H harmonic such that $H(1+i\eta)=F(1+i\eta)$, $H(1-i\eta)=F(1-i\eta)$. Then $H(1)\geqslant F(1)$ by the convexity so that

$$\Delta_F^2 \geqslant \Delta_H^2 \geqslant -C\eta^2 \sup\,(|H(1+i\eta)|, |H(1-i\eta)|) = -C\eta^2 \sup\,(|F(1+i\eta)|, \ |F(1-i\eta)|),$$

which is the required conclusion.

Corollary 2.3. *Let $F\in \mathfrak{P}(\mathfrak{J}\setminus\{0\})$ be positively homogeneous of order 1 where $\mathfrak{J}$ is an arbitrary complex linear topological space and suppose that F is not identically $-\infty$. Then F is finite everywhere. Moreover F is bounded in a neighborhood of the origin if F can be extended to a plurisubharmonic function in $\mathfrak{J}$. Conversely F admits such an extension if F is bounded from above near $0\in\mathfrak{J}$.*

Proof. Suppose that $F(\psi)=-\infty$ for some $\psi\in\mathfrak{J}\setminus\{0\}$. Since the restriction of F to $\{t\psi;\ t\in \mathbf{C},\ t\neq 0\}$ is convex, we must also have $F(-\psi)=-\infty$. The open set $\Omega=\{\varphi\in\mathfrak{J};\ F(\varphi)<-1\}$ is therefore a neighborhood of $\{\psi,\ -\psi\}$. But if $\varphi,\ -\varphi\in\Omega$ we obtain $F(\varphi)=-\infty$ for otherwise $0\leqslant F(\varphi)+F(-\varphi)<-2$, a contradiction. Therefore $F=-\infty$ in the open set $\Omega\cap-\Omega$ which means that $F=-\infty$ everywhere contrary to our assumption. We have proved that F assumes only real values.

Now if F can be extended to a function in $\mathfrak{P}(\mathfrak{J})$ we must obviously have $F(0)=0$, hence $F<1$ in a neighborhood ω of the origin. When $\varphi\in\omega\cap-\omega$ we therefore obtain $F(\varphi)<1$ and $F(\varphi)\geqslant -F(-\varphi)>-1$, hence $|F(\varphi)|<1$. Conversely, if $F(\varphi)\leqslant C$ when φ is small we get an upper semicontinuous function if we define $F(0)=0$ and the remark following the proof of Theorem 2.2 shows that F becomes a subharmonic function on every complex one-dimensional affine subspace.

Corollary 2.4. *Let $F\in\mathfrak{P}(\mathfrak{J}\setminus\{0\})$ be positively homogeneous of order 1 in a finite-dimensional complex linear space $\mathfrak{J}$. Then F can be extended to a plurisubharmonic function in $\mathfrak{J}$.*

Proof. By the preceding corollary it is sufficient to prove that F is bounded from above near 0. But by definition F is bounded from above on ∂K if K is a compact neighborhood of $0\in\mathfrak{J}$; hence F is bounded from above on $\{t\varphi;\, 0\leqslant t\leqslant 1,\, \varphi\in\partial K\}\supset K$ in view of the homogeneity.

In Section 3 we will need an approximation theorem for functions which are either positively homogeneous of order one or complex homogeneous of order zero. We prefer, however, to prove the following result for arbitrary ϱ.

Theorem 2.5. *Let Ω be an open set in $\mathbf{C}^n$ such that $t\zeta\in\Omega$ if $\zeta\in\Omega$ and $t>0$ and suppose that $F\in\mathfrak{P}(\Omega)$ is positively homogeneous of order ϱ, i.e.*

$$F(t\zeta) = t^\varrho F(\zeta), \quad \zeta \in \Omega, t > 0,$$

and that F is not identically $-\infty$ in any component of Ω. Then there exists a sequence (F_j) of functions in $\mathbf{C}^n \setminus \{0\}$ with the following properties:

(i) $F_j \in C^\infty(\mathbf{C}^n \setminus \{0\})$;

(ii) *F_j is positively homogeneous of order ϱ in $\mathbf{C}^n \setminus \{0\}$;*

(iii) *$F_j \in \mathcal{P}(\Omega_j)$ where Ω_j is an open subset of Ω such that $t\zeta \in \Omega_j$ if $\zeta \in \Omega_j$, $t > 0$, and every compact set $K \subset \Omega$ is contained in Ω_j when $j > j_K$ for some index j_K;*

(iv) *(F_j) converges to F in the sense that for every compact set K contained in Ω and every continuous majorant g of F,* sup (F_j, g) *tends to g and* inf (F_j, F) *tends to F uniformly on K, provided the ranges are given the natural uniform structure of $[-\infty, +\infty[$.*

The corresponding conclusion holds if we replace everywhere "positively homogeneous" by "complex homogeneous", i.e. if we require instead that $t\zeta \in \Omega$ and $F(t\zeta) = |t|^\varrho F(\zeta)$ if $\zeta \in \Omega$, $t \in \mathbf{C} \setminus \{0\}$, and similarly for Ω_j and F_j.

When F is complex homogeneous of order zero the theorem is most naturally regarded as an approximation theorem for plurisubharmonic functions in open subsets of projective $(n-1)$-space.

Note that when $\Omega = \mathbf{C}^n$ or $\Omega = \mathbf{C}^n \setminus \{0\}$ the sets Ω_j are all equal to Ω, at least from some index on. When $\varrho = 1$, of course, the cases $\Omega = \mathbf{C}^n$ and $\Omega = \mathbf{C}^n \setminus \{0\}$ are the same by Corollary 2.4.

The function $|\zeta|^\varrho$ is plurisubharmonic if $\varrho \geq 0$ or $n = 1$. Therefore the approximation from below can be improved in these cases: by adding $\varepsilon_j |\zeta|^\varrho$ to F_j we can arrange that $F_j(\zeta) \geq F(\zeta)$ when ζ belongs to a given compact set K in Ω and $F(\zeta) \geq -C$, $j \geq j_{K,C}$.

Proof of Theorem 2.5. Let Γ be the unitary group in $\mathbf{C}^n$, i.e. the group of all complex linear maps of $\mathbf{C}^n$ onto itself which preserve $|\zeta|$. Since $\gamma \in \Gamma$ is analytic, the function $\gamma^{-1}(\Omega) \ni \zeta \to F(\gamma(\zeta))$ is also plurisubharmonic. Let ω_δ be the set of all points ζ, $|\zeta| = 1$, with distance to $\{\theta \notin \Omega; |\theta| = 1\}$ greater than $\delta \geq 0$, and let $\Omega_\delta = \{t\zeta; t > 0, \zeta \in \omega_\delta\}$ (if $\Omega \supset \mathbf{C}^n \setminus \{0\}$ we take all Ω_δ equal to Ω). Obviously Ω_δ and Ω share the same homogeneity properties. Define $F_\delta(\zeta) = F(\zeta)$ when $\zeta \in \Omega_\delta$, $F_\delta(\zeta) = 0$ otherwise. We shall define regularizations of F by

$$G_\delta(\zeta) = \int_\Gamma F_\delta(\gamma(\zeta)) k(\gamma) d\gamma, \quad \zeta \in \mathbf{C}^n \setminus \{0\},$$

where $d\gamma$ is the Haar measure on Γ, and k is infinitely differentiable on Γ with $k \geq 0$ and

$\int_\Gamma k(\gamma)d\gamma = 1$. ($G_\delta(0) = F(0)$ if $0 \in \Omega$.) Then it follows that G_δ is C^∞ on the unit sphere, hence in $\mathbf{C}^n \setminus \{0\}$. Also G_δ inherits the homogeneity of F. Note that $F_\delta(\gamma(\zeta)) = F(\gamma(\zeta))$ if $\zeta \in \Omega_{\delta+\delta'}$, $\gamma \in \operatorname{supp} k$, and the support of k is contained in

$$\{\gamma \in \Gamma;\ \|\gamma - e\| \leqslant \delta'\} \tag{2.4}$$

where $e \in \Gamma$ is the unit element of Γ, i.e., if $k(\gamma) = 0$ if $|\gamma(\zeta) - \zeta| > \delta'$ for some ζ, $|\zeta| = 1$. If $\zeta + D\theta \in \Omega_{\delta+\delta'}$ we therefore obtain with the notation of (2.1)

$$\int_D G_\delta(\zeta + t\theta)d\lambda(t) = \int_\Gamma k(\gamma)d\gamma \int_D F_\delta(\gamma(\zeta) + t\gamma(\theta))d\lambda(t) \geqslant \pi G_\delta(\zeta)$$

since the order of integration can always be inverted when the integrand is semicontinuous. This means that G_δ is plurisubharmonic in $\Omega_{\delta+\delta'}$ (in fact in Ω_ε for some $\varepsilon < \delta + \delta'$).

Now for every $\zeta \in \Omega_{\delta+\delta'}$ and every $\varepsilon > 0$ we have $F_\delta(\gamma(\theta)) \leqslant F_\delta(\zeta) + \varepsilon$ if θ is near ζ and the support of k lies near e. Choosing a sequence (k_j) of functions on Γ with supports shrinking to $\{e\}$ and a sequence (δ_j), $\delta_j \searrow 0$, we obtain a sequence $(F_j) = (G_{\delta_j})$ such that for arbitrary $\zeta \in \Omega$

$$\overline{\lim_{j,\, m \to +\infty}} \sup_{|\theta - \zeta| \leqslant 1/m} F_j(\theta) \leqslant F(\zeta).$$

This proves that $\sup(F_j, g)$ tends to g uniformly on every compact set which is contained in Ω if g is a continuous majorant of F.

It remains only to study the approximation from below. For this purpose we suppose in addition to the properties of k already mentioned that k is a function of the trace of γ, $k(\gamma) = k_0(\operatorname{tr} \gamma)$, where $k_0 \in C_0^\infty(\mathbf{C})$ and $k_0(t) = k_0(\bar{t})$. Since the sets $\{\gamma \in \Gamma;\ |n - \operatorname{tr} \gamma| < \varepsilon\}$ form a fundamental system of neighborhoods of $e \in \Gamma$ it is still possible to find functions of this kind with supports arbitrarily close to e. We then have $k(\alpha\beta) = k(\beta\alpha)$ and $k(\bar{\alpha}) = k(\alpha^{-1}) = k(\alpha)$ for all $\alpha, \beta \in \Gamma$ if $\bar{\alpha}$ denotes the element of Γ obtained by taking the complex conjugates of the entries of the matrix determined by α in any given coordinate system. Choose coordinates in such a way that $\zeta = (0, \ldots, 0, 1)$, let H be the real hyperplane $\{\theta \in \mathbf{C}^n;\ \operatorname{Re} \theta_n = 0\}$ and denote by $d\sigma$ the Lebesgue measure in H. Then if the support of k is sufficiently small (e.g. if $k(\gamma) = 0$ when $|\gamma(\theta) - \theta| > |\theta|$ for some θ) there is one and only one function h in H such that

$$\int_\Gamma f(\gamma(\zeta))k(\gamma)d\gamma = \int_H f(\zeta + \theta)h(\theta)d\sigma(\theta) \tag{2.5}$$

for all continuous f which are homogeneous of order zero; in particular $\int_H h(\theta)d\sigma(\theta) = 1$. We denote the coordinates in $\mathbf{C}^n$ by $\theta = (\theta', \theta_n)$ where $\theta' \in \mathbf{C}^{n-1}$, $\theta_n \in \mathbf{C}$ and in H by $\theta = (\theta', i\eta)$,

$\eta \in \mathbf{R}$. We claim that $h(\alpha(\theta)) = h(\theta)$, $\theta \in H$, for every $\alpha \in \Gamma$ of the form $\alpha(\theta) = (\alpha'(\theta'), \pm\theta_n)$, $\theta \in \mathbf{C}^n$, where α' is a unitary map in $\mathbf{C}^{n-1}$; in other words, $h(\theta', i\eta)$ depends only on $|\theta'|$ and $|\eta|$. In fact, $d\sigma$ is invariant under such maps α and if we assume in addition that $\alpha(\zeta) = \zeta$ we obtain

$$\int_H f(\zeta+\theta)\, h(\alpha(\theta))\, d\sigma(\theta) = \int_\Gamma f(\alpha^{-1}\gamma(\zeta))k(\gamma)d\gamma = \int_\Gamma f(\gamma(\zeta))k(\alpha\gamma)d\gamma$$

by the left invariance of $d\gamma$, and then from $k(\alpha\gamma) = k(\gamma\alpha)$ by the right invariance of $d\gamma$,

$$\int_\Gamma f(\gamma(\zeta))k(\alpha\gamma)d\gamma = \int_\Gamma f(\gamma\alpha^{-1}(\zeta))k(\gamma)d\gamma = \int_H f(\zeta+\theta)h(\theta)\,d\sigma(\theta),$$

where the last equality follows from the fact that $\alpha^{-1}(\zeta) = \zeta$. To prove the assertion for arbitrary α of the indicated form it is now sufficient to consider only the map defined by $\alpha(\theta) = (\theta', -\theta_n)$. Then $h(\alpha(\theta)) = h(\bar{\theta})$, $\theta \in H$, by what we have just proved, and we obtain since ζ is real in the coordinate system and $k(\gamma) = k(\bar{\gamma})$,

$$\int_H f(\zeta+\theta)h(\alpha(\theta))\,d\sigma(\theta) = \int_H f(\overline{\zeta+\theta})h(\theta)d\sigma(\theta) = \int_\Gamma f(\bar{\gamma}(\zeta))k(\gamma)d\gamma$$
$$= \int_\Gamma f(\gamma(\zeta))k(\bar{\gamma})d\gamma = \int_H f(\zeta+\theta)h(\theta)\,d\sigma(\theta).$$

This proves that $h(\theta) = h_0(|\theta'|, |\eta|)$.

We now apply (2.5) to the function $f(\xi) = |\xi|^{-\varrho}F(\xi)$ and get if $\zeta \in \Omega_{\delta+\delta'}$ and the support of k is contained in (2.4)

$$G_\delta(\zeta) = \int_H F(\zeta+\theta)\,(1+|\theta|^2)^{-\varrho/2}\,h(\theta)\,d\sigma(\theta) = \int_{-\infty}^{+\infty} d\eta \int_{\mathbf{C}^{n-1}} h(\theta)\,(1+|\theta|^2)^{-\varrho/2}F(\zeta+\theta)\,d\lambda(\theta'),$$

where $d\lambda(\theta')$ is the Lebesgue measure in $\mathbf{C}^{n-1}$. Now we can use the plurisubharmonicity of F and the fact that $h(\theta)$ depends only on $|\theta'|$ for fixed η to conclude that

$$G_\delta(\zeta) \geqslant \int_{-\infty}^{+\infty} F(\zeta(1+i\eta))\,h_1(|\eta|)\,d\eta,$$

where

$$h_1(\eta) = \int_{\mathbf{C}^{n-1}} h(\theta)\,(1+|\theta|^2)^{-\varrho/2}d\lambda(\theta')$$

and

$$\int_{-\infty}^{+\infty} h_1(\eta)\,d\eta = \int_H h(\theta)\,(1+|\theta|^2)^{-\varrho/2}d\sigma(\theta). \tag{2.6}$$

We shall prove that for given δ, $\varepsilon > 0$, $G_\delta(\zeta) \geqslant F(\zeta) - \varepsilon$ when the support of k and thus of h is small. Moreover, we want this to hold uniformly when $|F(\zeta)| \leqslant C$ and $\zeta \in \omega_{\delta+\delta'}$ for given

$C, \delta' > 0$. If we take C so large that $F \leqslant C$ in $\omega_{\delta+\delta'}$ this will imply that $\sup(-C, G_\delta) \geqslant \sup(-C, F) - \varepsilon$ in $\omega_{\delta+\delta'}$ which is equivalent to the formulation given in (iv). Suppose, therefore, that $|F(\zeta)| \leqslant C$ and that $\zeta \in \omega_{\delta+\delta'}$. Writing $f(\eta) = F(\zeta(1+\eta))$ we obtain

$$G_\delta(\zeta) \geqslant \int_{-\infty}^{+\infty} f(\eta)\, h_1(\eta)\, d\eta = \int_0^{+\infty} (f(\eta) + f(-\eta))\, h_1(\eta)\, d\eta$$
$$= f(0) \int_{-\infty}^{+\infty} h_1(\eta)\, d\eta + \int_0^{+\infty} (f(\eta) - 2f(0) + f(-\eta))\, h_1(\eta)\, d\eta.$$

The observation following the proof of Theorem 2.2 shows that when the support of k lies in the set (2.4) and η is in the support of h_1, the second difference occurring in this formula can be estimated by $f(\eta) - 2f(0) + f(-\eta) \geqslant -C_1\eta^2$ where C_1 is a constant which depends on C, ϱ and δ' but not on ζ or F. Hence

$$G_\delta(\zeta) \geqslant f(0) + f(0)\left(\int_{-\infty}^{+\infty} h_1(\eta)\, d\eta - 1\right) - C_1 \int_0^{+\infty} \eta^2 h_1(\eta)\, d\eta$$
$$\geqslant F(\zeta) - C\left|\int_{-\infty}^{+\infty} h_1(\eta)\, d\eta - 1\right| - C_1 \int_0^{+\infty} \eta^2 h_1(\eta)\, d\eta,$$

which proves the assertion since $\int_{-\infty}^{+\infty} h_1(\eta) d\eta$ tends to 1 when the support of h_1 shrinks to $\{0\}$ in view of (2.6). The proof is complete.

If $\varrho = 1$ it is clear that the second difference is non-negative so the proof can be somewhat shortened in this case.

3. An existence theorem for entire functions of exponential type

We have seen that the upper regularization of the function

$$\overline{\lim_{s \to +\infty}} \frac{1}{s} \log |\mu(e^{s\zeta})| \tag{3.1}$$

defined for all linear forms ζ on $\mathbf{C}^n$ is plurisubharmonic if $\mu \in \mathcal{A}'(\mathbf{C}^n)$ (Theorem 2.1). Throughout this section we identify the space of linear functions on $\mathbf{C}^n$ with $\mathbf{C}^n$ itself by the formula $\zeta(z) = \langle z, \zeta \rangle = \sum_1^n z_j \zeta_j$. We shall now prove that, conversely, given a positively homogeneous function $F \in \mathcal{P}(\mathbf{C}^n)$, there exists an analytic functional μ in $\mathbf{C}^n$ such that the upper regularization of the function (3.1) is F. This will generalize a result of Lelong [7] who has proved Theorem 3.4 under the extra assumption that F is complex homogeneous, i.e., $F(t\zeta) = |t|\, F(\zeta)$ for all $\zeta \in \mathbf{C}^n$, $t \in \mathbf{C}$.

A crucial step in the proof is Theorem 3.3 which is a consequence of the following result.

THEOREM 3.1. *Let F be a positively homogeneous and plurisubharmonic function in $\mathbf{C}^n$. Then the open set*

$$\Omega_F = \{\zeta \in \mathbf{C}^n;\ \textit{for some}\ t \in \mathbf{C},\ F(t\zeta) < \operatorname{Re} t\} \tag{3.2}$$

is connected and pseudo-convex.

Note that it is trivial that Ω_F is pseudo-convex if F is complex homogeneous for then $\Omega_F = \{\zeta \in \mathbf{C}^n;\ F(\zeta) < 1\}$.

Proof of Theorem 3.1. We first prove that Ω_F is connected by joining an arbitrary point in Ω_F to the origin by means of a curve in Ω_F. Let $\zeta \in \Omega_F$ and take a non-real t such that $F(t\zeta) < \operatorname{Re} t$. This is possible because the set $\{t \in \mathbf{C};\ F(t\zeta) < \operatorname{Re} t\}$ is open and not void. If $s \geqslant 0$ we then have $F(t\zeta) < \operatorname{Re} t(1 + s\bar{t})$ which means that $\zeta/(1+s\bar{t}) \in \Omega_F$. The continuous curve $\{\zeta/(1+s\bar{t});\ s \in [0, +\infty]\}$ fulfills our requirements.

Next we shall prove that Ω_F is pseudo-convex if F is sufficiently regular. Later the extra hypothesis will be removed.

Thus suppose that F is three times continuously differentiable outside the origin and that $F(\zeta) - \varepsilon|\zeta|$ is plurisubharmonic for some $\varepsilon > 0$. Define $G(\tau, \zeta) = F(\tau\zeta) - \operatorname{Re} \tau$, $\tau \in \mathbf{C}$ $\zeta \in \mathbf{C}^n$ and $g(\zeta) = \inf_{|\tau|=1} G(\tau, \zeta)$. Then $G \in C^3(\{(\tau, \zeta) \in \mathbf{C}^{1+n};\ \tau\zeta \neq 0\})$ is plurisubharmonic in $\mathbf{C}^{1+n}$ and, by Theorem 2.2 and the remark following its proof, convex in τ for fixed ζ. Moreover, g is obviously continuous in $\mathbf{C}^n$ and $\Omega_F = \{\zeta \in \mathbf{C}^n;\ g(\zeta) < 0\}$. We claim that g is also twice continuously differentiable in the set $\omega_\varepsilon = \{\zeta \in \mathbf{C}^n;\ \zeta \neq 0 \text{ and } g(\zeta) < \varepsilon|\zeta|\}$ where $\varepsilon > 0$ is so small that $F(\zeta) - \varepsilon|\zeta|$ is plurisubharmonic. It is clear that ω_ε is a neighborhood of $\partial\Omega_F$ for $0 \in \Omega_F$.

Let $\zeta \in \omega_\varepsilon$. This means that $\tau \to G(\tau, \zeta) - \varepsilon|\tau\zeta|$ which is a convex and positively homogeneous function has a negative minimum on the unit circle. But such a function can attain its minimum on the unit circle more than once only if it is $\geqslant 0$. Thus there is a unique point τ such that $|\tau| = 1$ and $G(\tau, \zeta) = g(\zeta)$; we define a continuous function α in ω_ε by means of the equation $\alpha(\zeta) = \tau$.

Since G is positively homogeneous in τ for fixed ζ we have the Euler identity

$$\tau\frac{\partial G}{\partial \tau} + \bar{\tau}\frac{\partial G}{\partial \bar{\tau}} = G \tag{3.3}$$

and, taking $\partial/\partial\tau$ of this equation,

$$\tau\frac{\partial^2 G}{\partial \tau^2} + \bar{\tau}\frac{\partial^2 G}{\partial \tau\,\partial \bar{\tau}} = 0. \tag{3.4}$$

We also note that $\partial^2 G/\partial\tau\partial\bar{\tau} \geqslant \varepsilon|\zeta|\partial^2|\tau|/\partial\tau\partial\bar{\tau} = \varepsilon|\zeta|/4$ when $|\tau| = 1$, for $G(\tau, \zeta) - \varepsilon|\tau\zeta|$ is plurisubharmonic in τ by assumption.

When $\tau \to G(\tau, \zeta)$ attains its minimum on the unit circle, that is, when $\tau = \alpha(\zeta)$, we get

$$\tau \frac{\partial G}{\partial \tau} - \bar{\tau} \frac{\partial G}{\partial \bar{\tau}} = 0. \tag{3.5}$$

Define $H(\tau, \zeta) = \operatorname{Im}(\tau \partial G(\tau, \zeta)/\partial \tau)$. To prove that α is smooth we shall apply the implicit function theorem to the C^2 function $h(s, \zeta) = H(e^{is}, \zeta)$, $s \in \mathbf{R}$. Obviously one of the solutions s of $h(s, \zeta) = 0$ satisfies $\alpha(\zeta) = e^{is}$. From (3.3) and (3.4) we get

$$\frac{\partial h}{\partial s} = \operatorname{Im}\left(ie^{is} \frac{\partial G}{\partial \tau} - 2i \frac{\partial^2 G}{\partial \tau \partial \bar{\tau}}\right) = \frac{1}{2} G - 2 \frac{\partial^2 G}{\partial \tau \partial \bar{\tau}} < \frac{1}{2} \varepsilon |\zeta| - 2\varepsilon |\zeta|/4 = 0$$

when $\zeta \in \omega_\varepsilon$. This proves that each solution s of $h(s, \zeta) = 0$ is locally a C^2 function of ζ, hence that $\alpha \in C^2(\omega_\varepsilon)$. As a consequence, $g(\zeta) = G(\alpha(\zeta), \zeta)$ is twice continuously differentiable in ω_ε.

Furthermore, the gradient of g never vanishes when $g = 0$. For we have in view of (3.3) and (3.5)

$$\frac{\partial g}{\partial \zeta_j} = \frac{\partial G}{\partial \tau} \frac{\partial \alpha}{\partial \zeta_j} + \frac{\partial G}{\partial \bar{\tau}} \frac{\partial \bar{\alpha}}{\partial \zeta_j} + \frac{\partial G}{\partial \zeta_j} = \frac{\partial G}{\partial \zeta_j} = \alpha \frac{\partial F}{\partial \zeta_j}. \tag{3.6}$$

Also
$$0 = \frac{\partial G}{\partial \tau} = \sum \zeta_j \frac{\partial F}{\partial \zeta_j} - \frac{1}{2} \tag{3.7}$$

when $\tau = \alpha(\zeta)$ which proves that $\sum \zeta_j \partial g/\partial \zeta_j = \frac{1}{2}\alpha(\zeta)$, in particular the gradient is non-zero.

Now an open set $\Omega = \{\zeta \in \mathbf{C}^n;\, g(\zeta) < 0\}$ where g is twice continuously differentiable and has a non-vanishing gradient whenever $g = 0$ is pseudo-convex if and only if the Levi form $L(g)$ of g satisfies

$$L(g) = \sum_{j,k=1}^{n} \frac{\partial^2 g}{\partial \zeta_j \partial \bar{\zeta}_k} s_j \bar{s}_k \geq 0 \tag{3.8}$$

for all points on the boundary of Ω (which is $\{\zeta;\, g(\zeta) = 0\}$) and for all $s \in \mathbf{C}^n$ such that

$$\sum_{j=1}^{n} s_j \frac{\partial g}{\partial \zeta_j} = 0. \tag{3.9}$$

(See Theorem 2.6.12 in Hörmander [5].)

We shall express the Levi form of g in terms of that of F. First note that the homogeneity of F gives

$$F = \sum \frac{\partial F}{\partial \zeta_k} \zeta_k + \sum \frac{\partial F}{\partial \bar{\zeta}_k} \bar{\zeta}_k.$$

Hence
$$0 = \sum_k \frac{\partial^2 F}{\partial \zeta_j \partial \zeta_k} \zeta_k + \sum_k \frac{\partial^2 F}{\partial \zeta_j \partial \bar{\zeta}_k} \bar{\zeta}_k$$

which can be written

$$H(F;\, s,\, \zeta) + L(F;\, s,\, \zeta) = 0 \tag{3.10}$$

where H and L are, respectively, bilinear and sesquilinear forms:

$$H(F;\, s,\, t) = \sum \frac{\partial^2 F}{\partial \zeta_j \partial \zeta_k} s_j t_k, \qquad L(F;\, s,\, t) = \sum \frac{\partial^2 F}{\partial \zeta_j \partial \bar{\zeta}_k} s_j \bar{t}_k,$$

and the argument for the derivatives is ζ. Taking $\partial/\partial\bar{\zeta}_k$ of $\partial g(\zeta)/\partial\zeta_j = \alpha\partial F(\alpha\zeta)/\partial\zeta_j$ we get with this notation

$$L(g;\, s) = \sum s_j \frac{\partial F}{\partial \zeta_j} \sum \bar{s}_k \frac{\partial \alpha}{\partial \bar{\zeta}_k} + L(F;\, s) + \alpha^2 L(F;\, s,\, \alpha\zeta) \sum \bar{s}_k \frac{\partial \bar{\alpha}}{\partial \bar{\zeta}_k} + H(F;\, s,\, \alpha\zeta) \sum \bar{s}_k \frac{\partial \alpha}{\partial \bar{\zeta}_k},$$

where the argument for the derivatives of F is $\alpha\zeta$. ($L(F;\, s) = L(F;\, s,\, s)$ and analogously for $H(F;\, s)$.) In view of (3.6) and (3.9), the first term on the right-hand side is zero. Also $\alpha\bar{\alpha} = 1$ implies

$$\alpha \frac{\partial \bar{\alpha}}{\partial \bar{\zeta}_k} + \bar{\alpha} \frac{\partial \alpha}{\partial \bar{\zeta}_k} = 0 \tag{3.11}$$

so that we obtain
$$L(g;\, s) = L(F;\, s) + 2\bar{s}_0 \alpha^2 L(F;\, s,\, \alpha\zeta) \tag{3.12}$$

if $s_0 = \sum_1^n s_j \partial\alpha/\partial\zeta_j$. It remains to express s_0 in terms of F only. If we operate with $\partial/\partial\bar{\zeta}_k$ on (3.7) we get using (3.11)

$$-\bar{s}_0 H(F;\, \alpha\zeta) + \bar{s}_0 L(F;\, \alpha\zeta) + \bar{\alpha}^2 L(F;\, \alpha\zeta,\, s) = 0,$$

the argument for the derivatives being $\alpha\zeta$. Hence by (3.10)

$$2\bar{s}_0 L(F;\, \alpha\zeta) + \bar{\alpha}^2 L(F;\, \alpha\zeta,\, s) = 0.$$

This gives an expression for s_0 which inserted into (3.12) yields

$$L(g;\, s) = \frac{L(F;\, s)\, L(F;\, \alpha\zeta) - L(F;\, s,\, \alpha\zeta)\, L(F;\, \alpha\zeta,\, s)}{L(F;\, \alpha\zeta)}.$$

(Note that $L(F;\, \alpha\zeta) \geqslant \frac{1}{4}\varepsilon|\zeta| > 0$.) Thus the Cauchy–Schwarz inequality shows that $L(g;\, s) \geqslant 0$ when (3.9) is fulfilled; we have proved that Ω_F is pseudo-convex in the special case we have treated so far.

Now suppose that F satisfies the assumptions in the theorem and let (F_j) be a sequence of smooth positively homogeneous and plurisubharmonic functions converging to F in the sense of Theorem 2.5. (If $F = -\infty$ there is nothing to prove.) It is of course no restriction to assume that $F_j \geqslant F$ and that $F_j(\zeta) - \varepsilon_j|\zeta|$ is plurisubharmonic for some $\varepsilon_j > 0$ so by what we have just proved it follows that

$$\Omega_{F_j} = \{\zeta \in \mathbf{C}^n;\ \text{for some } t \in \mathbf{C},\ F_j(t\zeta) < \operatorname{Re} t\}$$

is pseudo-convex for every j. Now a pseudo-convex set Ω is characterized by the fact that $-\log d(\zeta, \complement\Omega)$ is plurisubharmonic where it is $<+\infty$. (We use the notation $d(\zeta, \complement\Omega) = \inf_{\theta \notin \Omega} |\theta - \zeta|$.) If therefore Ω_j are pseudo-convex open sets the function

$$f(\zeta) = -\varliminf_{j \to +\infty} \log d(\zeta, \complement\,\Omega_j)$$

is plurisubharmonic wherever it is $<+\infty$ and continuous in $\mathbf{C}^n$ if its range is given the usual topology of $[-\infty, +\infty]$. Hence the set where $f(\zeta) < +\infty$ is pseudo-convex (see e.g. Hörmander [5, Theorem 2.6.7]).

To prove that Ω_F is pseudo-convex it is thus sufficient to prove that $\varliminf_{j \to +\infty} d(\zeta, \complement\Omega_{F_j}) > 0$ if and only if $\zeta \in \Omega_F$. Suppose $\zeta \in \Omega_F$. Then $F(t\zeta) < \operatorname{Re} t$ *for some* $t \in \mathbf{C}$, hence, keeping t fixed, we find $\varepsilon > 0$ such that $F(t\theta) \leqslant \operatorname{Re} t - \varepsilon$ for all θ in a bounded neighborhood ω of ζ. By the properties of the sequence (F_j) there is an index j_0 such that $F_j(t\theta) < \operatorname{Re} t$ for all $j \geqslant j_0$ and all $\theta \in \omega$. This proves that $\omega \subset \Omega_{F_j}$ when $j \geqslant j_0$, hence $\varliminf_{j \to +\infty} d(\zeta, \complement\,\Omega_{F_j}) > 0$. On the other hand, $F \leqslant F_j$ for all j so that $\Omega_F \supset \Omega_{F_j}$, therefore $\zeta \notin \Omega_F$ implies $\varliminf_{j \to +\infty} d(\zeta, \complement\,\Omega_{F_j}) = 0$. We have proved that Ω_F is pseudo-convex.

We shall now consider open subsets of the complex projective n-space $P_n(\mathbf{C})$. We denote by π the canonical projection $\mathbf{C}^{n+1} \setminus \{0\} \to P_n(\mathbf{C})$ and define a hyperplane $H_a = \{\zeta \in \mathbf{C}^{n+1}; \sum \bar{a}_j \zeta_j = 1\}$ for every $a \in \mathbf{C}^{n+1}$, $|a| = 1$. If F' is a function in $\omega' \subset P_n(\mathbf{C})$, $F(\zeta) = \pi^* F'(\zeta) = F'(\pi(\zeta))$ defines a function $F = \pi^* F'$ in $\pi^{-1}(\omega')$. Conversely, if F is given in $\pi^{-1}(\omega')$ and is complex homogeneous of order zero, $F'(\pi(\zeta)) = F(\zeta)$ defines a function F' in ω'.

THEOREM 3.2. *Let ω' be an open set in $P_n(\mathbf{C})$, $\omega' \neq P_n(\mathbf{C})$, and define $\omega = \pi^{-1}(\omega') \subset \mathbf{C}^{n+1} \setminus \{0\}$. Then ω' is a Stein manifold if and only if $H_a \cap \omega$ is a pseudo-convex open subset of H_a for every $a \in \mathbf{C}^{n+1}$, $|a| = 1$.*

Proof. If ω' is a Stein manifold there exists a continuous plurisubharmonic function G' in ω' such that $\{\zeta \in \omega'; G'(\zeta) < c\}$ is relatively compact in ω' for every real c (see e.g. Theorem 5.1.6 in Hörmander [5]). Then $G = \pi^* G'$ is plurisubharmonic in ω and $\{\zeta \in H_a \cap \omega; G(\zeta) + |\zeta|^2 < c\}$ is relatively compact in $H_a \cap \omega$ for every c. Hence $H_a \cap \omega$ is pseudo-convex in any of the senses of this word, see Theorem 2.6.7 in Hörmander [5].

To prove the converse we shall use the solution of the Levi problem given by Grauert, see Theorem 5.2.10 in Hörmander [5]. We shall thus construct a function $G' \in C^\infty(\omega')$ which is strictly plurisubharmonic and tends to infinity at the boundary of ω', i.e. $\{\zeta \in \omega'; G'(\zeta) < c\}$ is relatively compact in ω' for every real c.

By our assumption on $H_a \cap \omega$, the function $-\log d_a$ is plurisubharmonic in $H_a \cap \omega$ if d_a denotes the distance to the boundary of ω in H_a:

$$d_a(\zeta) = \inf (|\zeta - \theta|; \theta \in H_a \backslash \omega), \quad \zeta \in H_a \cap \omega,$$

(it may happen that $d_a = +\infty$). Let

$$D_a(\zeta) = d_a(\zeta)(1 + |\zeta|^2)^{-1/4}, \quad \zeta \in H_a \cap \omega,$$

and extend d_a and D_a to points in $\omega_a = \{\zeta \in \omega; \sum \bar{a}_j \zeta_j \neq 0\}$ by

$$d_a(\zeta) = d_a(\zeta / \Sigma \bar{a}_j \zeta_j), \quad D_a(\zeta) = D_a(\zeta / \Sigma \bar{a}_j \zeta_j), \quad \zeta \in \omega_a.$$

Then $-\log d_a$ and $-\log D_a$ are plurisubharmonic and complex homogeneous of order zero in ω_a. Define D and d in ω as the infimum of D_a and d_a, respectively:

$$D(\zeta) = \inf (D_a(\zeta); |a| = 1, \omega_a \ni \zeta), \qquad d(\zeta) = \inf (d_a(\zeta); |a| = 1, \omega_a \ni \zeta).$$

We claim that $-\log D$ is continuous and plurisubharmonic in ω.

First of all, let us note that $D(\zeta)$ is never $+\infty$. For if $\zeta \in \omega$ we take a $\theta \notin \omega \cup \{0\}$ which exists by assumption and choose a so that $|a| = 1$, $\Sigma \bar{a}_j \zeta_j \neq 0$, $\Sigma \bar{a}_j \theta_j \neq 0$. Then $\zeta / \Sigma \bar{a}_j \zeta_j \in H_a \cap \omega$, $\theta / \Sigma \bar{a}_j \theta_j \in H_a \backslash \omega$ and

$$D(\zeta) \leqslant D_a(\zeta) \leqslant d_a(\zeta) \leqslant \left| \frac{\zeta}{\sum \bar{a}_j \zeta_j} - \frac{\theta}{\sum \bar{a}_j \theta_j} \right| < +\infty.$$

It is also easy to see that $D(\zeta)$ is always >0. Let A, $0 < A \leqslant \pi/2$, be the angular distance between a point $\zeta \in \omega$ and $\complement \omega$. Using the notation $x = |\zeta / \Sigma \bar{a}_j \zeta_j| \geqslant 1$ we obtain $d_a(\zeta) \geqslant x \sin A$ if $\zeta \in \omega_a$ and hence

$$D_a(\zeta) = d_a(\zeta)(1 + x^2)^{-1/4} \geqslant x(1 + x^2)^{-1/4} \sin A \geqslant 2^{-1/4} \sin A,$$

which proves the assertion.

We shall now describe $D(\zeta)$ locally as an infimum of $D_a(\zeta)$ where a varies over a set independent of ζ. To this end we shall first prove that

$$d_a(\zeta) \geqslant \frac{|\zeta| \, d(\zeta)}{2} \text{ when } \zeta \in H_a \cap \omega.$$

If $d_a(\zeta) = +\infty$ there is nothing to prove so we suppose that $d_a(\zeta) = |\zeta - \theta|$ for some $\theta \in H_a \backslash \omega$. Let B be the angle between the rays determined by ζ and θ, $0 < B < \pi$. Obviously $d_a(\zeta) = |\zeta - \theta| \geqslant |\zeta| \sin B$. Now define $b = (\zeta + \theta)/|\zeta + \theta|$ if $0 < B \leqslant \pi/2$ and $b = (\zeta - \theta)/|\zeta - \theta|$ if $\pi/2 < B < \pi$. Then in view of the fact that θ and $-\theta$ both belong to $\complement \omega$

$$d(\zeta) \leqslant d_b(\zeta) \leqslant 2 \inf \left(\tan \frac{B}{2},\ \cot \frac{B}{2}\right).$$

Hence
$$\frac{d_a(\zeta)}{d(\zeta)} \geqslant \frac{|\zeta| \sin B}{2 \inf \left(\tan \frac{B}{2},\ \cot \frac{B}{2}\right)} = |\zeta| \sup \left(\cos^2 \frac{B}{2},\ \sin^2 \frac{B}{2}\right) \geqslant \frac{|\zeta|}{2}.$$

We therefore obtain if $\zeta \in H_a \cap \omega$

$$\frac{D_a(\zeta)}{D(\zeta)} \geqslant \frac{D_a(\zeta)}{d(\zeta)} \geqslant \frac{|\zeta| D_a(\zeta)}{2\, d_a(\zeta)} = \frac{|\zeta|}{2\,(1+|\zeta|^2)^{1/4}}.$$

Since the right hand side is $\geqslant \varepsilon > 1$ when $|\zeta| \geqslant 5$ and since $D(\zeta) > 0$ we find that

$$D(\zeta) = \inf_a \left(D_a(\zeta);\ |a| = 1,\ \frac{D_a(\zeta)}{D(\zeta)} < \varepsilon\right) = \inf_a \left(D_a(\zeta);\ |a| = 1,\ |\zeta| \leqslant C \left|\sum \bar{a}_j \zeta_j\right|\right),\ \zeta \in \omega,$$

for any constant $C \geqslant 5$. Hence

$$D(\theta) = \inf_a \left(D_a(\theta);\ |a| = 1,\ |\zeta| \leqslant 6 \left|\Sigma \bar{a}_j \zeta_j\right|\right)$$

for points θ near ζ. This proves that $-\log D$ is continuous and plurisubharmonic since it is locally a supremum of plurisubharmonic functions

$$\tfrac{1}{4} \log \left(1 + \left|\zeta / \sum \bar{a}_j \zeta_j\right|^2\right) - \log d_a(\zeta).$$

It is also obvious that $-\log D$ tends to $+\infty$ at the boundary of ω except at 0.

Let us also note that the function $-\log D'$ which is induced by $-\log D$ is strictly plurisubharmonic in ω'. For $f_a(\zeta) = \frac{1}{4} \log \left(1 + |\zeta / \Sigma \bar{a}_j \zeta_j|^2\right)$ is strictly plurisubharmonic in $H_b \cap \omega_a$ since it is the composition of the strictly plurisubharmonic function $\mathbf{C}^{n+1} \ni \theta \to \frac{1}{4} \log (1 + |\theta|^2)$ and the regular analytic map $H_b \cap \omega_a \ni \zeta \to \zeta / \Sigma \bar{a}_j \zeta_j \in \mathbf{C}^{n+1}$. We can even choose a strictly plurisubharmonic function g in a neighborhood $U_b \subset H_b$ of b such that $f_a - g$ is plurisubharmonic in U_b for all a satisfying $|\Sigma \bar{a}_j b_j| \geqslant 1/6$. This implies that for $\zeta \in H_b \cap \omega$ near b,

$$-\log D(\zeta) - g(\zeta) = \sup_a \left(-\log d_a(\zeta) + f_a(\zeta) - g(\zeta);\ |a| = 1 \leqslant 6 |\Sigma \bar{a}_j b_j|\right),$$

where the right hand side is a supremum of plurisubharmonic functions, hence $-\log D$ is strictly plurisubharmonic in H_b near b which means that $-\log D'$ is strictly plurisubharmonic in ω'.

We have thus seen that $-\log D'$ has all the properties we require except that we only know that it is continuous. We shall now regularize it by means of a construction which is patterned after the proof of Theorem 2.6.11 in Hörmander [5]. Let $F=-\log D$ in ω and choose a sequence (F_j) of C^∞ functions in $\mathbf{C}^{n+1}\setminus\{0\}$ with the properties mentioned in Theorem 2.5. Define $\omega_j=\{\zeta\in\omega;\ F(\zeta)<j\}$. By relabeling the functions F_j and adding a small constant we may suppose that $F_j\in\mathcal{P}(\omega_j)$ and that

$$F_j(\zeta)=\int_\Gamma F(\gamma(\zeta))\,k_j(\zeta)\,d\zeta+\varepsilon_j$$

when $\zeta\in\omega_j$. Here ε_j may be chosen so that $F\leqslant F_j\leqslant F+1$ in ω_j which is possible by the remark preceding the proof of Theorem 2.5 since F is continuous. This shows that F_j' (which is defined by $F_j'(\pi(\zeta))=F_j(\zeta)$) is strictly plurisubharmonic in $\omega_j'=\pi(\omega_j)$. If therefore $\chi\in C^\infty(\mathbf{R})$ satisfies $\chi(t)=0$ when $t\leqslant 0$ and $\chi'(t),\ \chi''(t)>0$ when $t>0$, the function $\zeta\to a_j\chi(F_j'(\zeta)-j+2)$, $a_j>0$, is plurisubharmonic in ω_j', strictly plurisubharmonic in $\omega_j'\setminus\bar\omega_{j-2}'$ which is an open neighborhood of $\omega_j'\setminus\omega_{j-1}'$; it is zero when $F_j'(\zeta)\leqslant j-2$ in particular when $\zeta\in\omega_{j-3}'$. Hence

$$G_k'(\zeta)=F_0'(\zeta)+\sum_{j=1}^k a_j\chi(F_j'(\zeta)-j+2)$$

defines an infinitely differentiable function in $\mathbf{C}^{n+1}\setminus\{0\}$ and $G_k'=G_m'$ in ω_j' if $k,m>j+1$. Moreover, the constants a_j can be recursively determined to make G_k' strictly plurisubharmonic and $\geqslant F'$ in ω_k'. The limit $G'=\lim_{k\to+\infty}G_k'$ is defined in ω' and has all desired propeties so the proof is finished.

THEOREM 3.3. *If $F\in\mathcal{P}(\mathbf{C}^n)$ is positively homogeneous of order 1, the open set $\omega_F'=\pi(\omega_F)$ in projective n-space defined by*

$$\omega_F=\{\zeta\in\mathbf{C}^{n+1};\ \textit{for some } t\in\mathbf{C},\ F(t\zeta_1,\dots,t\zeta_n)<\operatorname{Re}t\zeta_{n+1}\}$$

is a connected Stein manifold if (and only if) F is not identically $-\infty$. Moreover ω_F' determines F uniquely: if $F,G\in\mathcal{P}(\mathbf{C}^n)$ are positively homogeneous of order 1 we have $F\leqslant G$ if and only if $\omega_F'\supset\omega_G'$.

Proof. Let H_a be as defined before Theorem 3.2. We shall prove that $H_a\cap\omega_F$ is pseudoconvex in H_a. First suppose that $a_{n+1}\neq 0$. Then

$$H_a\cap\omega_F=\{\zeta\in\mathbf{C}^{n+1};\ \sum_1^{n+1}\bar a_j\zeta_j=1 \text{ and for some } t\in\mathbf{C},$$

$$F(\bar a_{n+1}t\zeta_1,\dots,\bar a_{n+1}t\zeta_n)+\operatorname{Re}\sum_1^n\bar a_jt\zeta_j<\operatorname{Re}t\}$$

is isomorphic to Ω_G where we have defined $G(\zeta)=F(\bar a_{n+1}\zeta)+\operatorname{Re}\Sigma_1^n\bar a_j\zeta_j$, $\zeta\in\mathbf{C}^n$, and Ω_G is given by (3.2) with F replaced by G. Since Ω_G is pseudo-convex by Theorem 3.1 the same is true of $H_a \cap \omega_F$. Also, since Ω_G is connected, $H_a \cap \omega_F$ and ω_F' are connected. We shall now consider $a \in \mathbf{C}^{n+1}$, $a_{n+1}=0$. If $H_a \cap \omega_F = H_a$ there is nothing more to prove. If, on the other hand, $H_a \setminus \omega_F \neq \emptyset$ the distance function $d_a(\zeta)$ defined in the proof of Theorem 3.2 is obviously a uniform limit of distance functions $d_b(\zeta)$, $b_{n+1} \neq 0$, hence $-\log d_a$ is plurisubharmonic which is equivalent to pseudo-convexity. The first part of the theorem now follows from Theorem 3.2.

If $F \leqslant G$ it is obvious that $\omega_F \supset \omega_G$. Conversely, suppose that $F, G \in \mathcal{P}(\mathbf{C}^n)$ are positively homogeneous of order one and that $F(\zeta) > G(\zeta)$ for some $\zeta \in \mathbf{C}^n$. Choose a complex number ζ_{n+1} such that for all $t \in \mathbf{C}$

$$\operatorname{Re} \zeta_{n+1} t \leqslant F(t\zeta), \quad \operatorname{Re} \zeta_{n+1} = F(\zeta) > G(\zeta).$$

This means that $(\zeta_1, \ldots, \zeta_n, \zeta_{n+1}) \in \omega_G \setminus \omega_F$. Hence the inclusion $\omega_G \subset \omega_F$ is impossible unless $F \leqslant G$. The proof is complete.

THEOREM 3.4. *Suppose $F \in \mathcal{P}(\mathbf{C}^n)$ is positively homogeneous of order one. Then there is an entire function u of exponential type such that for all $\zeta \in \mathbf{C}^n$*

$$\varlimsup_{\theta \to \zeta} \varlimsup_{s \to +\infty} \frac{1}{s} \log |u(s\theta)| = F(\zeta); \tag{3.13}$$

equivalently, there exists an analytic functional $\mu \in \mathcal{A}'(\mathbf{C}^n)$ such that the upper regularization of the restriction to $\mathbf{C}$ of its indicator is F.

Proof. If $F = -\infty$ we take $u=0$, $\mu=0$. Otherwise ω_F', defined in Theorem 3.3, is a proper subset of $P_n(\mathbf{C})$, hence a Stein manifold. Therefore there is a function $f_1' \in \mathcal{A}(\omega_F')$ which cannot be continued across the boundary of ω_F', more precisely, we shall require that for every connected open neighborhood V of an arbitrary point on the boundary of ω_F', each component of $V \cap \omega_F'$ contains zeros of f_1' of arbitrarily high order (the construction is given e.g. in Hörmander [5, Theorem 2.5.5] for sets in $\mathbf{C}^n$ but can obviously be extended to Stein manifolds in $P_n(\mathbf{C})$). The set $M = \pi\{\zeta \in \omega_F; \zeta_{n+1}=0\}$ is a closed submanifold of ω_F', thus there is a function $f_2' \in \mathcal{A}(\omega_F')$ which is zero on this set without being zero identically in ω_F' (a consequence of Cartan's Theorem A, see [5, Theorem 7.2.11]). It follows that $f' = f_1' f_2'$ is zero on the submanifold M and cannot be continued beyond ω_F'. Let $f = \pi^* f'$. Since $0 \in \Omega_F$, the function $g(\zeta_1, \ldots, \zeta_n) = f(\zeta_1, \ldots, \zeta_n, 1)$ which is analytic in Ω_F has a power series expansion

$$g(\zeta) = \sum a_k \zeta^k$$

which converges near the origin ($k = (k_1, \ldots, k_n)$, $\zeta^k = \zeta_1^{k_1}, \ldots, \zeta_n^{k_n}$). It is then easy to see that

$$u(\zeta)=\sum a_k \frac{\zeta^k}{|k|!}$$

is entire and of exponential type ($|k|=k_1+\ldots+k_n$). (The relation between g and u has been studied before by Paris [9].) Define $\mu\in\mathcal{A}'(\mathbf{C}^n)$ by $\mu(e^{\langle z,\zeta\rangle})=u(\zeta)$, that is, $\mu(\varphi)=\Sigma a_k b_k/|k|!$ if $\varphi(z)=\Sigma b_k z^k/k!$ where $k!=k_1!\ldots k_n!$. Let p be the indicator of μ restricted to the linear functions, i.e. $p(\zeta)=\overline{\lim}_{s\to+\infty}\log|u(s\zeta)|^{1/s}$.

We recall that the Borel transform of an entire function U of exponential type in one complex variable is given for large $|t|$ by

$$H(t)=\sum_0^\infty A_j t^{-j-1} \quad \text{if} \quad U(\tau)=\sum_0^\infty A_j\frac{\tau^j}{j!}.$$

The corresponding integral representation is

$$H(t)=\int_0^{+\infty} U(s\tau)\,e^{-st\tau}\,\tau ds,$$

where $\tau\in\mathbf{C}$ has to be chosen suitably for every t. It follows from this formula that H can be analytically continued into the complement of the convex compact set

$$K=\{t\in\mathbf{C};\ \text{for all}\ \tau\in\mathbf{C},\ \overline{\lim_{s\to+\infty}}\frac{1}{s}\log|U(s\tau)|\geqslant \operatorname{Re} t\tau\}.$$

Conversely we have

$$U(\tau)=\frac{1}{2\pi i}\int_\Gamma H(t)\,e^{t\tau}\,dt,$$

where Γ is some large circle. This integral representation of U shows immediately that for all $\varepsilon>0$ we have

$$|U(\tau)|\leqslant C_\varepsilon \exp\,(\sup_{t\in L}\operatorname{Re} t\tau+\varepsilon|\tau|)$$

if H is analytic outside a compact convex set $L\subset\mathbf{C}(e^{-\infty}=0)$.

Now let $h_\zeta(t)=g(\zeta/t)/t=t^{-1}f(\zeta_1,\ldots,\zeta_n,t)$ for some fixed $\zeta\in\mathbf{C}^n$. Then h_ζ is the Borel transform of $\tau\to u(\tau\zeta)$ so that

$$u(\tau\zeta)=\frac{1}{2\pi i}\int_\Gamma h_\zeta(t)\,e^{t\tau}\,dt.$$

In view of our choice of f, h_ζ can be analytically continued to every point t such that $(\zeta,t)\in\omega_F$; in particular there is no singularity at the origin if $(\zeta,0)\in\omega_F$. We can therefore choose Γ in any neighborhood of the convex set $\{t\in\mathbf{C};\ \text{for all}\ \tau\in\mathbf{C},\ F(\tau\zeta)\geqslant\operatorname{Re} t\tau\}$ and estimate u by

$$|u(\tau\zeta)|\leqslant C_\varepsilon\exp\,(F(\tau\zeta)+\varepsilon|\tau|)$$

for every $\varepsilon>0$ (ζ is fixed). Hence $p(\zeta)\leqslant F(\zeta)$ and, since ζ is arbitrary, $p^*\leqslant F$.

On the other hand, the integral

$$h_\zeta(t) = \int_0^{+\infty} u(s\tau\zeta)\, e^{-st\tau}\, \tau ds$$

converges absolutely and uniformly for all (ζ, t) satisfying $p(\tau\zeta) \leqslant \operatorname{Re} t\tau - \varepsilon$. It follows that $h_\zeta(\zeta_{n+1})$ is an analytic function of (ζ, ζ_{n+1}) in ω_{p^*}, in particular $\zeta_{n+1} h_\zeta(\zeta_{n+1}) = f(\zeta_1, \ldots, \zeta_n, \zeta_{n+1})$ can be analytically continued to a function in $\mathcal{A}(\omega_{p^*})$. But since ω'_{p^*} is connected by Theorem 3.3 this is possible only if $\omega'_{p^*} \subset \omega'_F$, therefore, by the last part of the same theorem, $p^* \geqslant F$. The proof is complete.

Remark. When F is sufficiently regular, Theorem 3.4 can be proved more directly. For example, if F is Lipschitz continuous we can use Theorem 4.4.3 in Hörmander [5] to find an entire function of exponential type such that the left hand side $p^*(\zeta)$ of (3.13) is $\leqslant F(\zeta)$ for all $\zeta \in \mathbf{C}^n$ with equality at any given point. A category argument then shows that there exist functions $u \in \mathcal{A}(\mathbf{C}^n)$ of exponential type such that $p^* \leqslant F$ and equality holds in a dense set in $\mathbf{C}^n$, hence everywhere since F is continuous and p^* is semicontinuous.

This can be used to give an alternative proof of Theorems 3.3 and 3.4 even for general F. For by tracing the argument in the proof of Theorem 3.4 backwards we can establish that ω'_{F_j} is a Stein manifold if F_j is a regularization of F found by means of Theorem 2.5. Then, using Theorem 3.2 together with the discussion in the two last paragraphs of the proof of Theorem 3.1, we conclude that ω'_F itself is a Stein manifold, that is, we have obtained Theorem 3.3. We therefore have to employ the Borel transformation a second time in order to arrive at the conclusion of Theorem 3.4.

4. Properties of supports of analytic functionals

In the first part of this section we shall study how the supports of an analytic functional behave under analytic mappings. Later on this will be used to prove various properties of the family of all $\mathfrak{F}$-supports of a functional and then, in Section 5, to relate the $\mathfrak{F}$-supports to the indicator. (For definitions of these notions we refer to Section 1.)

Let U and V be complex analytic manifolds and α an analytic map of U into V. We call α regular if its rank is everywhere equal to the dimension of U, that is, if for all choices of local coordinates $z_1, \ldots, z_n$ in U and $w_1, \ldots, w_m$ in V the matrix $(\partial\alpha_j/\partial z_k)$ has rank n where it is defined. The map α is called proper if $\alpha^{-1}(K)$ is compact in U for every compact subset K of V. Finally, a proper, one-to-one and regular map is called an embedding of U into V.

Throughout the rest of the paper we shall use the notation α^* for the map $\mathcal{A}(V) \ni \psi \to \psi \circ \alpha \in \mathcal{A}(U)$. It is also convenient to denote its adjoint by α again, thus $\alpha\mu(\psi) = \mu(\alpha^*\psi) =$

$\mu(\psi \circ \alpha)$ if $\mu \in \mathcal{A}'(U)$. In particular this notation will be employed when U is an open subset of V and $\iota: U \to V$ is the canonical injection.

For the purposes of this section and the next a Stein manifold may be defined as a complex manifold which can be embedded into $\mathbf{C}^m$ for some m.

We will need the following result which is a consequence of Cartan's theorems A and B.

THEOREM 4.1. *Let α be an embedding of a complex manifold U into a Stein manifold V (U is of course Stein, too). Then $\alpha^*: \mathcal{A}(V) \to \mathcal{A}(U)$ is onto, i.e. every $\varphi \in \mathcal{A}(U)$ is of the form $\psi \circ \alpha$ for some $\psi \in \mathcal{A}(V)$. Moreover, the value of ψ at a point $z \notin \alpha(U)$ can be arbitrarily prescribed. It follows also that $\alpha: \mathcal{A}'(U) \to \mathcal{A}'(V)$ is an injection.*

It was shown in Section 1 that $\mathfrak{F}$-convexity has a simple geometric meaning if $\mathfrak{F}$ is a finite-dimensional subspace of the analytic functions. The following approximation theorem will therefore be useful.

LEMMA 4.2. *Let U be a complex manifold and $\mathfrak{F}$ a complex linear subspace of $\mathcal{A}(U)$ such that $\mathfrak{F}^m$ contains a proper map $(\alpha_1, \ldots, \alpha_m)$ of U into $\mathbf{C}^m$ for some m. Then every $\mathfrak{F}$-convex compact set $K \subset U$ has a fundamental system of neighborhoods, each of which is compact and convex with respect to a finite-dimensional subspace of $\mathfrak{F}$.*

Proof. Let L be an arbitrary compact neighborhood of K and let $\mathcal{G}$ be the subspace of $\mathfrak{F}$ spanned by $\alpha_1, \ldots, \alpha_m$. Then $h_{\mathcal{G}}L$ is compact for it is the inverse image of the compact set $h_{\mathbf{C}}\alpha(L)$ in $\mathbf{C}^m$. Now for any $z \in h_{\mathcal{G}}L \setminus L^\circ$ there is by assumption an element $\varphi_z \in \mathfrak{F}$ such that $\operatorname{Re} \varphi_z(z) > \sup_K \operatorname{Re} \varphi_z$. Let a_z be real numbers satisfying $\operatorname{Re} \varphi_z(z) > a_z > \sup_K \operatorname{Re} \varphi_z$ and denote by ω_z the neighborhood of z where $\operatorname{Re} \varphi_z > a_z$. Finitely many of the ω_z cover $h_{\mathcal{G}}L \setminus L^\circ$. We denote the corresponding functions φ_z and numbers a_z by φ_j and a_j, $j = 1, \ldots, k$, respectively. Then we have

$$K \subset A \subset L^\circ \cup \complement h_{\mathcal{G}}L,$$

where $A = \{z \in U; \operatorname{Re} \varphi_j(z) \leq a_j, j = 1, \ldots, k\}$. If $\mathcal{H}$ is the space spanned by $\alpha_1, \ldots, \alpha_m, \varphi_1, \ldots, \varphi_k$ we have proved that $A \cap h_{\mathcal{H}}L \subset A \cap h_{\mathcal{G}}L \subset L^\circ$ and hence $h_{\mathcal{H}}(A \cap L) \subset h_{\mathcal{H}}A \cap h_{\mathcal{H}}L = A \cap h_{\mathcal{H}}L \subset L^\circ$ so that $h_{\mathcal{H}}(A \cap L)$ is a neighborhood of K contained in L°. This proves the theorem.

For later reference we list also the following result where both hypothesis and conclusion are weaker than in Lemma 4.2. Recall that we have called U $\mathfrak{F}$-convex if the $\mathfrak{F}$-hull of any compact set is compact. Obviously U is $\mathfrak{F}$-convex if $\mathfrak{F}^m$ contains a proper map of U into $\mathbf{C}^m$ for some m.

LEMMA 4.2′. *Let U be an $\mathfrak{F}$-convex complex manifold where $\mathfrak{F}$ is an arbitrary subset of $\mathcal{A}(U)$. Then every $\mathfrak{F}$-convex compact set K in U has a fundamental system of $\mathfrak{F}$-convex neighborhoods.*

Proof. Let L be an arbitrary compact neighborhood of K. Then $h_{\mathcal{F}} L$ is compact by assumption. We can therefore proceed as in the last proof, replacing both $\mathcal{G}$ and $\mathcal{H}$ by $\mathcal{F}$.

The next two lemmas describe the behavior of $\mathcal{F}$-convexity under analytic mappings.

LEMMA 4.3. *Let U, V be complex manifolds and $\mathcal{G}$ a subset of $\mathcal{A}(V)$, α an analytic map of U into V. Define $\mathcal{F}=\alpha^*\mathcal{G}$. Then*

$$\alpha^{-1}(h_{\mathcal{G}}\alpha(M)) = h_{\mathcal{F}} M \tag{4.1}$$

for every subset M of U and

$$\alpha^{-1}(h_{\mathcal{G}} N) \supset h_{\mathcal{F}}\alpha^{-1}(N) \tag{4.2}$$

for every set $N \subset V$. In particular, $\alpha^{-1}(N)$ is $\mathcal{F}$-convex if N is $\mathcal{G}$-convex. Furthermore, M is $\mathcal{F}$-convex if $\alpha(M)$ is $\mathcal{G}$-convex and α is one-to-one.

Proof. If $a \notin h_{\mathcal{F}} M$ there is a function $\varphi \in \mathcal{F}$ such that $\operatorname{Re}\varphi(a) > \sup_M \operatorname{Re}\varphi$. By assumption $\varphi = \psi\circ\alpha$ for some $\psi \in \mathcal{G}$ so that $\operatorname{Re}\psi(\alpha(a)) > \sup_M \operatorname{Re}\psi\circ\alpha = \sup_{\alpha(M)} \operatorname{Re}\psi$. Hence $\alpha(a) \notin h_{\mathcal{G}}\alpha(M)$, i.e. $a \notin \alpha^{-1}(h_{\mathcal{G}}\alpha(M))$.

Suppose now, on the other hand, that $\alpha(a) \notin h_{\mathcal{G}}\alpha(M)$. Then for some $\psi \in \mathcal{G}$, $\operatorname{Re}\psi(\alpha(a)) > \sup_{\alpha(M)} \operatorname{Re}\psi = \sup_M \operatorname{Re}\psi\circ\alpha$. Since $\psi\circ\alpha \in \mathcal{F}$ this proves that $a \notin h_{\mathcal{F}} M$. This completes the proof of (4.1).

To prove (4.2), finally, it is sufficient to apply (4.1) to $M = \alpha^{-1}(N)$ and use the obvious inclusion $\alpha(\alpha^{-1}(N)) \subset N$.

Lemma 4.3 shows in particular that if α is a one-to-one map of U into V, a set $M \subset U$ is holomorph convex if there exists a holomorph convex set $N \subset V$ such that $M = \alpha^{-1}(N)$, i.e., $\alpha(U) \cap N = \alpha(M)$. We shall now study the converse of this statement.

LEMMA 4.4. *Let α be an embedding of a Stein manifold U into another V. Then*

$$\alpha(h_{\mathcal{A}(U)} M) = h_{\mathcal{A}(V)}\alpha(M) \tag{4.3}$$

for all subsets M of U; in particular M is $\mathcal{A}(U)$-convex if and only if $\alpha(M)$ is $\mathcal{A}(V)$-convex.

Proof. By Theorem 4.1, $\mathcal{A}(U) = \alpha^*\mathcal{A}(V)$ so that Lemma 4.3 gives

$$\alpha^{-1}(h_{\mathcal{A}(V)}\alpha(M)) = h_{\mathcal{A}(U)} M.$$

Hence
$$\alpha(h_{\mathcal{A}(U)} M) = \alpha(\alpha^{-1}(h_{\mathcal{A}(V)}\alpha(M))) = h_{\mathcal{A}(V)}\alpha(M) \cap \alpha(U)$$

so that it suffices to prove that $h_{\mathcal{A}(V)}\alpha(M) \subset \alpha(U)$. But by Theorem 4.1 again, we can to each $b \notin \alpha(U)$ find $\psi \in \mathcal{A}(V)$ such that $\psi(b) > 0$, $\psi = 0$ on $\alpha(U)$, hence $b \notin h_{\mathcal{A}(V)}\alpha(M)$.

Next we study how a mapping affects a carrier of an analytic functional.

LEMMA 4.5. *Let U, V be complex manifolds and $\alpha: U \to V$ an analytic map. If $\mu \in \mathcal{A}'(U)$ is carried by a compact set $K \subset U$, $\alpha\mu$ is carried by $\alpha(K)$.*

Proof. Suppose that ω is a neighborhood of $\alpha(K)$. Then $\alpha^{-1}(\omega)$ is a neighborhood of K so that for some constant C we have $|\mu(\varphi)| \leqslant C \sup_{\alpha^{-1}(\omega)}|\varphi|$. Hence $|\alpha\mu(\psi)| \leqslant C \sup_{\alpha^{-1}(\omega)}|\psi \circ \alpha| \leqslant C \sup_{\omega}|\psi|$. This proves that $\alpha(K)$ carries $\alpha\mu$.

It is not equally trivial to treat the inverse image of a carrier. We first need an auxiliary result.

LEMMA 4.6. *Let V be a Stein manifold and α an embedding of a complex manifold U into V. Then for any analytic functional $\nu \in \mathcal{A}'(V)$ such that $\nu(\psi) = 0$ if $\psi = 0$ on $\alpha(U)$ there is a unique analytic functional $\mu \in \mathcal{A}'(U)$ such that $\nu = \alpha\mu$.*

Proof. We can define μ by $\mu(\varphi) = \nu(\psi)$ if $\varphi = \psi \circ \alpha$. Indeed, every φ is of the form $\psi \circ \alpha$ by Theorem 4.1, and if $\psi_1 \circ \alpha = \psi_2 \circ \alpha$ we have $\psi_1 - \psi_2 = 0$ on $\alpha(U)$ so that $\nu(\psi_1) = \nu(\psi_2)$. This defines μ as a linear form on $\mathcal{A}(U)$. To prove the continuity of μ we note that $\mathcal{A}(U)$ and $\mathcal{A}(V)$ are Fréchet spaces, hence $\alpha^*: \mathcal{A}(V) \to \mathcal{A}(U)$ is an open map since it is onto by Theorem 4.1. For any compact set $L \subset V$ we can therefore find a compact set $K \subset U$ and a constant B such that for every $\varphi \in \mathcal{A}(U)$ there is a $\psi \in \mathcal{A}(V)$ with $\varphi = \alpha^*\psi$ and $\sup_L|\psi| \leqslant B \sup_K|\varphi|$. Hence if $|\nu(\psi)| \leqslant C \sup_L|\psi|$ we get $|\mu(\varphi)| \leqslant BC \sup_K|\varphi|$ so that μ is continuous.

We can now prove that the inverse image of a holomorph convex carrier of $\alpha\mu$ is a carrier of μ. This is a theorem of Martineau [8, Ch. I, Théorème 2.6] but we prefer to formulate a complete proof here.

LEMMA 4.7. *Let α be an embedding of a complex manifold U into a Stein manifold V, let $\mu \in \mathcal{A}'(U)$ and suppose that $\alpha\mu$ is carried by an $\mathcal{A}(V)$-convex compact set $L \subset V$. Then μ is carried by the $\mathcal{A}(U)$-convex set $\alpha^{-1}(L) = \alpha^{-1}(L \cap \alpha(U))$.*

Proof. Put $K = \alpha^{-1}(L)$ so that $\alpha(K) = L \cap \alpha(U)$ and let ω be an arbitrary open neighborhood of K. Now take by Lemma 4.2′ an $\mathcal{A}(V)$-convex compact neighborhood L_1 of L such that $\alpha^{-1}(L_1) \subset \omega$. By assumption there is a constant C such that $|\alpha\mu(\psi)| \leqslant C \sup_{L_1}|\psi|$ for all $\psi \in \mathcal{A}(V)$. Let $V_1 \subset V$ be a Stein manifold containing L_1 but so small that $U_1 = \alpha^{-1}(V_1) \subset \omega$ (Lemma 4.2′) and let $\iota: U_1 \to U$ and $\varkappa: V_1 \to V$ be the inclusion maps. Now a function which is analytic in a neighborhood of L_1 can be approximated uniformly on L_1 by functions in $\mathcal{A}(V)$ so that we may extend $\alpha\mu$ by continuity into an analytic functional ν on V_1 such that $|\nu(\psi_1)| \leqslant C \sup_{L_1}|\psi_1|$, $\psi_1 \in \mathcal{A}(V_1)$. We thus have $\varkappa\nu = \alpha\mu$. But we know even more: By e.g. Theorem 7.2.7 in Hörmander [5] a function $\psi_1 \in \mathcal{A}(V_1)$ such that $\psi_1 = 0$ on $\alpha(U) \cap V_1 = \alpha(U_1)$ can be approximated uniformly on L_1 by functions $\psi \in \mathcal{A}(V)$ satisfying $\psi = 0$ on $\alpha(U)$.

Since $\alpha\mu(\psi)=0$ for such functions ψ, we get $\nu(\psi_1)=0$ for all ψ_1 which are analytic in V_1 and zero on $\alpha(U_1)$. We can therefore apply Lemma 4.6 to prove that $\nu=\alpha_1\varrho$ for some $\varrho\in\mathcal{A}'(U_1)$ for the restriction α_1 of α to U_1 is an embedding of U_1 into V_1. Thus $\alpha\mu=\varkappa\nu=\varkappa\alpha_1\varrho=\alpha\iota\varrho$. In view of Theorem 4.1 the map $\alpha:\mathcal{A}'(U)\to\mathcal{A}'(V)$ is injective so that $\alpha\mu=\alpha\iota\varrho$ implies $\mu=\iota\varrho$. Hence μ is continuous for the topology in $\mathcal{A}(U)$ induced by $\mathcal{A}(U_1)$ which proves the lemma.

We can now prove the following result on the image of $\mathcal{F}$-supports.

THEOREM 4.8. *Let U and V be Stein manifolds, α an embedding of U into V, and $\mathcal{G}$ a subset of $\mathcal{A}(V)$. Define $\mathcal{F}=\alpha^*\mathcal{G}=\{\psi\circ\alpha;\psi\in\mathcal{G}\}$ and suppose that $K\subset U$ is an $\mathcal{F}$-convex compact set such that $h_{\mathcal{G}}\alpha(K)$ is compact. Then K is an $\mathcal{F}$-support of an analytic functional $\mu\in\mathcal{A}'(U)$ if and only if $h_{\mathcal{G}}\alpha(K)$ is a $\mathcal{G}$-support of $\alpha\mu$.*

Proof. Suppose that K is an $\mathcal{F}$-support of μ. Then $h_{\mathcal{G}}\alpha(K)$ is by Lemma 4.5 a $\mathcal{G}$-convex carrier of $\alpha\mu$ and we must show that it is minimal among these sets. Thus let $L\subset h_{\mathcal{G}}\alpha(K)$ be a $\mathcal{G}$-convex carrier of $\alpha\mu$. Then $\alpha^{-1}(L)$ carries μ by Lemma 4.7. But we also obtain from Lemma 4.3 that $\alpha^{-1}(L)\subset\alpha^{-1}(h_{\mathcal{G}}\alpha(K))=h_{\mathcal{F}}K=K$ and that $\alpha^{-1}(L)$ is $\mathcal{F}$-convex. Hence $\alpha^{-1}(L)$ must be equal to K, in particular $\alpha(K)=\alpha(\alpha^{-1}(L))\subset L$ so that $h_{\mathcal{G}}\alpha(K)\subset h_{\mathcal{G}}L=L$. We have thus proved that $h_{\mathcal{G}}\alpha(K)$ is a minimal $\mathcal{G}$-convex carrier of $\alpha\mu$.

Now assume that conversely $h_{\mathcal{G}}\alpha(K)$ is a $\mathcal{G}$-support of $\alpha\mu$. Then $K=h_{\mathcal{F}}K=\alpha^{-1}(h_{\mathcal{G}}\alpha(K))$ carries μ (Lemma 4.7). Next let $M\subset K$ be an $\mathcal{F}$-convex carrier of μ. Then $h_{\mathcal{G}}\alpha(M)$ is included in $h_{\mathcal{G}}\alpha(K)$ and carries $\alpha\mu$ (Lemma 4.5) so it must be equal to $h_{\mathcal{G}}\alpha(K)$. Hence we have proved that $M=h_{\mathcal{F}}M=\alpha^{-1}(h_{\mathcal{G}}\alpha(M))=\alpha^{-1}(h_{\mathcal{G}}\alpha(K))=h_{\mathcal{F}}K=K$ which means that K is an $\mathcal{F}$-support.

COROLLARY 4.9. *Under the assumptions of the theorem, if $L\subset V$ is a $\mathcal{G}$-support of $\alpha\mu$, then $\alpha^{-1}(L)$ is an $\mathcal{F}$-support of μ.*

Proof. Put $K=\alpha^{-1}(L)$. Then K is by Lemma 4.7 a carrier of μ, so that $h_{\mathcal{G}}\alpha(K)$ is a $\mathcal{G}$-convex carrier of $\alpha\mu$ according to Lemma 4.5. Since $L\supset h_{\mathcal{G}}\alpha(K)$ and L is a $\mathcal{G}$-support, $L=h_{\mathcal{G}}\alpha(K)$ and we can apply Theorem 4.8 to get the desired conslusion.

COROLLARY 4.10. *We keep the hypotheses in the theorem. Denote by $K_{\mathcal{F}}(L_{\mathcal{G}})$ the intersection of all $\mathcal{F}$-supports of μ (all $\mathcal{G}$-supports of $\alpha\mu$). Then $\alpha^{-1}(L_{\mathcal{G}})=K_{\mathcal{F}}$. In particular, μ has a unique $\mathcal{F}$-support if and only if $\alpha\mu$ has a unique $\mathcal{G}$-support.*

Proof. If K is an $\mathcal{F}$-support of μ, $h_{\mathcal{G}}\alpha(K)$ is a $\mathcal{G}$-support of $\alpha\mu$ so we get $h_{\mathcal{G}}\alpha(K)\supset L_{\mathcal{G}}$, hence $K=h_{\mathcal{F}}K=\alpha^{-1}(h_{\mathcal{G}}\alpha(K))\supset\alpha^{-1}(L_{\mathcal{G}})$ and $K_{\mathcal{F}}\supset\alpha^{-1}(L_{\mathcal{G}})$.

On the other hand, if L is a $\mathcal{G}$-support of $\alpha\mu$, $\alpha^{-1}(L)$ is an $\mathcal{F}$-support of μ, hence $\alpha^{-1}(L)\supset K_{\mathcal{F}}$ and $L\supset h_{\mathcal{G}}\alpha(\alpha^{-1}(L))\supset h_{\mathcal{G}}\alpha(K_{\mathcal{F}})$ which implies $L_{\mathcal{G}}\supset h_{\mathcal{G}}\alpha(K_{\mathcal{F}})$ so that $\alpha^{-1}(L_{\mathcal{G}})\supset\alpha^{-1}(h_{\mathcal{G}}\alpha(K_{\mathcal{F}}))=h_{\mathcal{F}}K_{\mathcal{F}}=K_{\mathcal{F}}$.

Finally, if $K_{\mathcal{F}}$ carries μ, $h_{\mathcal{G}}\alpha(K_{\mathcal{F}})\subset L_{\mathcal{G}}$ carries $\alpha\mu$ so that $\alpha\mu$ has a unique $\mathcal{G}$-support (and then of course $h_{\mathcal{G}}\alpha(K_{\mathcal{F}})=L_{\mathcal{G}}$). Conversely, if $L_{\mathcal{G}}$ carries $\alpha\mu$, $\alpha^{-1}(L_{\mathcal{G}})=K_{\mathcal{F}}$ carries μ.

THEOREM 4.11. *Let U be a complex manifold and $\mathcal{F}$ a linear subspace of $\mathcal{A}(U)$ such that $\mathcal{F}^m$ contains a proper map $(\alpha_1, \ldots, \alpha_m)$ of U into $\mathbf{C}^m$. Then an analytic functional $\mu\in\mathcal{A}'(U)$ has a unique $\mathcal{F}$-support if and only if μ has a unique $\mathcal{G}$-support for every finite-dimensional subspace $\mathcal{G}$ of $\mathcal{F}$ such that $\alpha_1, \ldots, \alpha_m\in\mathcal{G}$.*

Proof. First suppose that μ has a unique $\mathcal{F}$-support K and let $\mathcal{G}$ be any subset of $\mathcal{F}$ such that $h_{\mathcal{G}}K$ is compact. Then it is clear that the $\mathcal{G}$-hull of K is contained in every $\mathcal{G}$-convex carrier of μ since every $\mathcal{G}$-convex set is $\mathcal{F}$-convex. This proves one half of our assertion.

Next assume that μ has two different $\mathcal{F}$-supports K_1 and K_2 and let L_1, L_2 be compact neighborhoods of K_1 and K_2, respectively, such that $K_1\setminus L_2\neq\emptyset$ and $K_2\setminus L_1\neq\emptyset$. In addition we may choose L_1 and L_2 so small that μ is not carried by $L_1\cap L_2$. In fact, since μ is not carried by $K_1\cap K_2$ the same is true of some compact neighborhood L of $K_1\cap K_2$ and then it is sufficient to take L_1, L_2 such that $L_1\cap L_2\subset L$. Now choose finite-dimensional subspaces $\mathcal{G}_1$ and $\mathcal{G}_2$ of $\mathcal{F}$ such that $h_{\mathcal{G}_j}K_j\subset L_j$, $j=1, 2$, (Lemma 4.2). Then $h_{\mathcal{G}}K_j\subset L_j$ if $\mathcal{G}=\mathcal{G}_1+\mathcal{G}_2$ so that μ has two $\mathcal{G}$-convex carriers $h_{\mathcal{G}}K_1$, $h_{\mathcal{G}}K_2$ but is not carried by their intersection. Hence μ cannot have a unique $\mathcal{G}$-support.

With the help of this theorem it follows from Corollary 4.10 that, for example, if μ is an analytic functional on a Stein manifold U with more than one $\mathcal{A}(U)$-support, then U can be embedded into $\mathbf{C}^m$ for some m in such a way that the image of μ has several $\mathcal{L}$-supports.

The following result will be useful later when we shall study the restriction of a functional $\mu\in\mathcal{A}'(U)$ to $\mathcal{A}(V)$ where V is another manifold containing U as an open subset.

THEOREM 4.12. *Let U be a Stein manifold and $\mathcal{F}\subset\mathcal{A}(U)$ a convex cone such that U is $\mathcal{F}$-convex. Suppose that $\mu\in\mathcal{A}'(U)$ has two different $\mathcal{F}$-supports K_0 and K_1. Then for any neighborhood ω of K_0 there exists an $\mathcal{F}$-support $K_2\subset\omega$, $K_2\neq K_0$. In particular, μ has infinitely many $\mathcal{F}$-supports.*

The proof will be divided into a few lemmas.

LEMMA 4.13. *Let $\mathcal{F}$ be a convex cone in $\mathcal{A}(U)$, let K_0, K_1 be subsets of U and $z\in h_{\mathcal{F}}(K_0\cup K_1)$ a fixed point. Define for $t=(t_0, t_1)\in\mathbf{R}^2$*

$$g(t)=g(t_0,t_1)=\sup_{\varphi}\,(\mathrm{Re}\,\varphi(z);\ \varphi\in\mathfrak{J}\ \textit{and}\ \mathrm{Re}\,\varphi\leqslant t_j\ \textit{in}\ K_j,\ j=0,1).$$

Then g has values in $[-\infty,+\infty[$, is positively homogeneous and concave, i.e.,

$$g(\lambda t)=\lambda g(t),\quad \lambda>0,\ t=(t_0,t_1)\in\mathbf{R}^2, \tag{4.4}$$

$$g(s+t)\geqslant g(s)+g(t),\quad s,t\in\mathbf{R}^2. \tag{4.5}$$

Proof. The homogeneity is trivial. It is sufficient to prove (4.5) when both $g(s)$ and $g(t)$ are $>-\infty$. Take arbitrary real numbers $a<g(s)$ and $b<g(t)$ and choose functions $\varphi,\ \psi\in\mathfrak{J}$ such that $\mathrm{Re}\,\varphi(z)\geqslant a$, $\mathrm{Re}\,\psi(z)\geqslant b$ and $\mathrm{Re}\,\varphi\leqslant s_j$, $\mathrm{Re}\,\psi\leqslant t_j$ on K_j, $j=0,1$. Then $\varphi+\psi\in\mathfrak{J}$ and $g(s+t)\geqslant\mathrm{Re}\,(\varphi+\psi)(z)\geqslant a+b$ which proves (4.5).

LEMMA 4.14. *With the notation of the previous lemma, suppose in addition that $z\notin h_{\mathfrak{J}}K_0$. Then there is a constant $q>0$ such that for $t_1<t_0$,*

$$g(t_0,t_1)\leqslant t_0-q(t_0-t_1).$$

Proof. From (4.4) and (4.5) we obtain

$$(s_1-s_0)g(t)+(t_0-t_1)g(s)\leqslant g(t_0s_1-t_1s_0,\ t_0s_1-t_1s_0) \tag{4.6}$$

provided $s_0<s_1$, $t_1<t_0$. Now since $z\in h_{\mathfrak{J}}(K_0\cup K_1)$ we must have $g(t)\leqslant\sup\,(t_0,t_1)$ which together with (4.6) implies

$$(s_1-s_0)g(t)+(t_0-t_1)g(s)\leqslant t_0s_1-t_1s_0.$$

This can be written as $g(t)\leqslant t_0-q(t_0-t_1)$ where $q=(g(s)-s_0)/(s_1-s_0)$. Since we have assumed that $z\notin h_{\mathfrak{J}}K_0$ there is a function $\varphi\in\mathfrak{J}$ such that $\mathrm{Re}\,\varphi(z)>\sup_{K_0}\mathrm{Re}\,\varphi$, thus $g(s_0,s_1)>s_0$ if $s_j=\sup_{K_j}\mathrm{Re}\,\varphi$. This proves that q is >0 if s is conveniently chosen.

We need also the following result to prove Theorem 4.12.

THEOREM 4.15. *Let $\mu\in\mathcal{A}'(U)$ where U is a Stein manifold and let K_0, K_1 be carriers of μ, $L=h_{\mathcal{A}(U)}(K_0\cup K_1)$. Suppose that K is a compact set in U and that there exist two disjoint sets M_0 and M_1 which are closed in $L\setminus K$ and satisfy $L\setminus K=M_0\cup M_1$, $K_j\setminus K\subset M_j$, $j=0,1$. Then μ is carried by K.*

The proof has been given in [6] when U is (an open subset of) $\mathbf{C}^m$. The general case follows easily from this if we embed U into $\mathbf{C}^m$ for some m and use Lemmas 4.4, 4.5 and 4.7. The theorem can also be deduced from Théorème 2.2, Ch. I, in Martineau [8] (using the open case) in the same way as Theorem 2.4 was obtained from Corollary 2.5 in a remark in [6]. It has been pointed out by A. Martineau in a personal communication to the author that either proof of Theorem 4.15 remains valid if the assumption on L is replaced by the weaker condition that $L \supset K_0 \cup K_1$, $H^1(L; O) = 0$ and L has the Runge property.

Proof of Theorem 4.12. By Lemma 4.2′ we can find an $\mathfrak{J}$-convex compact neighborhood M of K_0 contained in ω. Also, since $K_0 \setminus K_1 \neq \emptyset$, there is a function $\varphi \in \mathfrak{J}$ such that $\sup_{K_1} \operatorname{Re} \varphi = t_1 < t_0 = \sup_{K_0} \operatorname{Re} \varphi$. Put $K = \{z \in M; \operatorname{Re} \varphi(z) \leqslant a\}$, where a is determined later, $M_0 = \{z \in L \setminus K; \operatorname{Re} \varphi(z) \geqslant a\}$, and $M_1 = \{z \in L \setminus K; \operatorname{Re} \varphi(z) \leqslant a\}$. Here we have written L for $h_{\mathcal{A}(U)}(K_0 \cup K_1)$. We shall check the assumptions of Theorem 4.15. It is clear that $M_0 \cup M_1 = L \setminus K$ and that M_0 and M_1 are closed in $L \setminus K$. Furthermore $M_0 \supset K_0 \setminus K$ and $M_1 \supset K_1 \setminus K$ if $t_1 \leqslant a$. It remains only to verify that M_0 and M_1 are disjoint if a is conveniently chosen. If $z \in M_0 \cap M_1$ we have in particular $\operatorname{Re} \varphi(z) = a$ so that with the notation of Lemma 4.13, $g(t_0, t_1) \geqslant a$. On the other hand, Lemma 4.14 shows that $g(t_0, t_1) \leqslant t_0 - q(t_0 - t_1)$ for some constant $q > 0$ since $z \notin h_{\mathfrak{J}} K_0$. This proves that $M_0 \cap M_1$ is empty if $(1-q)t_0 + qt_1 < a$. Applying Theorem 4.15 we find that $K \subset M$ carries μ, and since K is $\mathfrak{J}$-convex it contains an $\mathfrak{J}$-support K_2. Obviously $K_2 \neq K_0$ if we take a in the interval $(1-q)t_0 + qt_1 < a < t_0$. This completes the proof of Theorem 4.12.

We conclude this section with a result which relates the property of an analytic functional μ of having a unique $\mathfrak{J}$-support with the same property of the restriction of μ to the space of functions which are analytic in a larger manifold.

LEMMA 4.16. *Let V be a Stein manifold, U an open set in V and L a compact subset of U which is $\mathcal{A}(V)$-convex as a subset of V. Suppose that $\mu \in \mathcal{A}'(U)$ is carried by L. Then an $\mathcal{A}(V)$-convex set $K \subset L$ carries μ if (and only if) K carries $\iota\mu \in \mathcal{A}'(V)$ where ι is the inclusion map of U into V. In particular, if U has the Runge property with respect to V, the $\mathcal{A}(U)$-convex carriers of μ are the same as the $\mathcal{A}(V)$-convex carriers of $\iota\mu$ which are contained in U.*

Proof. Let $\omega \subset U$ be an arbitrary neighborhood of an $\mathcal{A}(V)$-convex set $K \subset L \subset U$. We shall prove that there is a constant C such that

$$|\mu(\varphi)| \leqslant C \sup_{\omega} |\varphi| \tag{4.7}$$

for all $\varphi \in \mathcal{A}(U)$ under the assumption that K carries $\iota\mu$. First take by means of Lemma 4.2′ an $\mathcal{A}(V)$-convex compact neighborhood L_1 of L, $L_1 \subset U$. Then for arbitrary $\varphi \in \mathcal{A}(U)$ and $\psi \in \mathcal{A}(V)$ we have

$$|\mu(\varphi)| \leqslant |\iota\mu(\psi)| + |\mu(\varphi - \iota^* \psi)| \leqslant C \sup_{\omega \cap L_1} |\psi| + C \sup_{L_1} |\varphi - \psi|$$

for some constant C. Hence

$$|\mu(\varphi)| \leqslant C \sup_{\omega} |\varphi| + 2\, C \sup_{L_1} |\varphi - \psi|.$$

But φ is analytic in a neighborhood of the $\mathcal{A}(V)$-convex set L_1 so the infimum of the last term as $\psi \in \mathcal{A}(V)$ varies is zero by the Oka–Weil approximation theorem. This proves (4.7).

THEOREM 4.17. *Let V be a Stein manifold and $\mathcal{G}$ a convex cone in $\mathcal{A}(V)$ such that V is $\mathcal{G}$-convex. Let $\mu \in \mathcal{A}'(U)$, where U is an open subset of V, be carried by some $\mathcal{G}$-convex compact set L which is contained in U. Let $\iota: U \to V$ be the inclusion map. Then $\iota\mu$ has a unique $\mathcal{G}$-support if and only if μ has a unique $\iota^*\mathcal{G}$-support.*

Proof. Define $\mathfrak{F} = \iota^*\mathcal{G}$. First suppose that $\iota\mu$ has a unique $\mathcal{G}$-support K_0. Clearly $K_0 \subset L$ so that K_0 carries μ by Lemma 4.16. On the other hand, if $K \subset U$ carries μ it also carries $\iota\mu$ so that $h_{\mathcal{G}} K \supset K_0$, therefore $h_{\mathfrak{F}} K = h_{\mathcal{G}} K \cap U \supset K_0$. Hence the intersection of all $\mathfrak{F}$-convex carriers of μ contains K_0 which, as we have seen, itself is a carrier. This means that K_0 is the only $\mathfrak{F}$-support of μ.

Next assume that $\iota\mu$ has at least two $\mathcal{G}$-supports. Since $\iota\mu$ is carried by L we can find one $\mathcal{G}$-support $K_0 \subset L$. Let L_1 be a $\mathcal{G}$-convex compact neighborhood of L contained in U (Lemma 4.2′). Then $\iota\mu$ has also a $\mathcal{G}$-support $K_1 \subset L_1$, $K_1 \neq K_0$, by Theorem 4.12. But Lemma 4.16 shows that both K_0 and K_1 carry μ whereas their intersection does not even carry $\iota\mu$. Since K_0 and K_1 are also $\mathfrak{F}$-convex, μ cannot have a unique $\mathfrak{F}$-support.

5. The indicator of an analytic functional

The results in the previous section will now be used to generalize a theorem of Martineau (Theorem 5.1 below). We shall also characterize the set of analytic functionals which have a unique $\mathfrak{F}$-support under certain conditions concerning $\mathfrak{F}$.

THEOREM 5.1. *Let $\mu \in \mathcal{A}'(\mathbf{C}^n)$ and suppose that*

$$p(\zeta) \leqslant H_K(\zeta), \quad \zeta \in \mathfrak{L}, \tag{5.1}$$

where p is the indicator of μ (Definition 1.5) and H_K is the supporting function of a compact set $K \subset \mathbf{C}^n$, $\mathfrak{L}$ being the space of linear functions on $\mathbf{C}^n$. Then μ is carried by $h_{\mathfrak{L}} K$. Conversely, (5.1) holds if μ is carried by $h_{\mathfrak{L}} K$.

This is Théorème 4.1, Ch. II, in Martineau [8]. Other proofs have been given by Ehrenpreis [2] and Hörmander [4], [5].

Before we extend this theorem to more general subspaces of the analytic functions than $\mathfrak{L}$ we draw some immediate conclusions.

THEOREM 5.2. *Let $\mu \in \mathcal{A}'(\mathbf{C}^n)$. Then for every $\zeta \in \mathfrak{C}$ we have*

$$p^{\mathfrak{C}}(\zeta) = \inf_K (H_K(\zeta);\ K \text{ carries } \mu)$$

where $p^{\mathfrak{C}}$ is the upper regularization of the restriction of p to $\mathfrak{C}$.

Proof. We write P for $p^{\mathfrak{C}}$ and introduce complex coordinates in $\mathfrak{C}$. Let θ be a fixed element of $\mathfrak{C}$ with $|\theta| = 1$ and let a be any real number greater than $P(\theta)$. We have to prove that $H_K(\theta) \leqslant a$ for some carrier K of μ. Put $q_s(\zeta) = a \operatorname{Re} \Sigma \zeta_j \bar{\theta}_j + s(|\zeta| - \operatorname{Re} \Sigma \zeta_j \bar{\theta}_j)$ and let M_s be the compact set $\{\zeta \in \mathfrak{C}; |\zeta| = 1,\ q_s(\zeta) \leqslant P(\zeta)\}$. We claim that $\bigcap_{s>0} M_s = \emptyset$. Indeed, $q_s(\theta) = a > P(\theta)$ so that $\theta \notin M_s$. If $\zeta \neq \theta$, $|\zeta| = 1$, we have $q_s(\zeta) \to +\infty$ when $s \to +\infty$ so that $\zeta \notin M_s$ when s is large. This implies that $M_s = \emptyset$ for some $s > 0$, in particular $q_s(\zeta) \geqslant P(\zeta)$ for all $\zeta \in \mathfrak{C}$. Let K_s be the ball with supporting function q_s. Then by Theorem 5.1, K_s carries μ and we have seen that $H_{K_s}(\theta) = a$. Since a was an arbitrary number $> P(\theta)$ this proves that the infimum is everywhere $\leqslant P$. Conversely we trivially have $P \leqslant H_K$ for every carrier K of μ (see the end of Section 1).

COROLLARY 5.3. *If $\mu \in \mathcal{A}'(\mathbf{C}^n)$, μ has a unique $\mathfrak{C}$-support if and only if $p^{\mathfrak{C}}$ is convex.*

Proof. Suppose that $p^{\mathfrak{C}}$ is convex and let K_0 be the convex compact set such that $H_{K_0} = p^{\mathfrak{C}}$. Then Theorem 5.1 shows that a convex compact set K carries μ if and only if $K \supset K_0$ so that K_0 is the smallest convex carrier, hence the only $\mathfrak{C}$-support.

If, on the other hand, μ has a unique $\mathfrak{C}$-support K_0, then $p^{\mathfrak{C}} = \inf (H_K;\ K \text{ carries } \mu) = H_{K_0}$ by the preceding theorem.

It can easily be proved that if $p^{\mathfrak{C}}$ is not convex, all minimal convex majorants of $p^{\mathfrak{C}}$ must be linear in some open set. Hence, according to the corollary, the $\mathfrak{C}$-supports of a functional with several $\mathfrak{C}$-supports must all have edges. This gives a new proof of Theorem 3.1 in [6].

The conclusion of Theorem 5.2 is false in general for $\mu \in \mathcal{A}'(U)$, U a proper subset of $\mathbf{C}^n$. The corollary, on the other hand, can be generalized by means of Theorem 4.17. We shall not do so now, however, since more general statements will follow from the next theorem.

THEOREM 5.4. *Let V be a Stein manifold, U an open subset of V and denote by ι the canonical injection $U \to V$. Suppose that the linear subspace $\mathcal{G}$ of $\mathcal{A}(V)$ contains elements $\alpha_1, \ldots, \alpha_m$ such that $(\alpha_1, \ldots, \alpha_m)$ is an embedding of V into $\mathbf{C}^m$. Further, let $\mu \in \mathcal{A}'(U)$ be carried by some $\mathcal{G}$-convex set $L \subset U$, and let $K \subset L$. Then μ is carried by $h_{\mathcal{G}}K$ if (and only if)*

$$p(\iota^* \psi) \leqslant H_K(\psi),\ \psi \in \mathcal{G}.$$

In particular, we have

COROLLARY 5.5. *If $\mu \in \mathcal{A}'(V)$, where V is a Stein manifold, then μ is carried by $h_{\mathcal{A}(V)}K$ if and only if $p(\psi) \leqslant H_K(\psi)$ for all $\psi \in \mathcal{A}(V)$.*

Another special case is of course Theorem 5.1.

Proof of Theorem 5.4. By Lemma 4.16 it is sufficient to prove that $\iota\mu$ is carried by an arbitrary neighborhood K_1 of K. Choose a finite-dimensional subspace $\mathcal{H}$ of $\mathcal{G}$ such that $h_{\mathcal{H}}K \subset K_1$ (Lemma 4.2). We may of course assume that $\alpha_1, ..., \alpha_m \in \mathcal{H}$. Let $\alpha_1, ..., \alpha_k$ generate $\mathcal{H}$ and define $\alpha = (\alpha_1, ..., \alpha_k)$. Then the hypotheses imply that the indicator q of $\alpha\iota\mu$ satisfies $q(\zeta) \leqslant H_{\alpha(K)}(\zeta)$ if ζ is a linear function on $\mathbf{C}^k$. Hence by Theorem 5.1, $h_{\mathbf{C}}\alpha(K)$ carries $\alpha\iota\mu$ and therefore, in view of Lemma 4.7, $\alpha^{-1}(h_{\mathbf{C}}\alpha(K)) = h_{\mathcal{H}}K \subset K_1$ carries $\iota\mu$. This proves the theorem.

Our next goal is to generalize similarly Theorem 5.2 and Corollary 5.3. We first need a simple lemma on the operation of taking the upper regularization.

LEMMA 5.6. *Let U and V be complex manifolds which are countable unions of compact sets and let $\alpha: U \to V$ be an analytic map. Further, let $\mathcal{G}$ and $\mathcal{F} = \alpha^*\mathcal{G}$ be closed subspaces of $\mathcal{A}(V)$ and $\mathcal{A}(U)$, respectively. Then if $\mu \in \mathcal{A}'(U)$ we have $p_\alpha^{\mathcal{G}}(\psi) = p^{\mathcal{F}}(\psi \circ \alpha)$ for all $\psi \in \mathcal{G}$ where p and p_α denote, respectively, the indicators of μ and $\alpha\mu$, and $p^{\mathcal{F}}$ is the upper regularization of the restriction of p to $\mathcal{F}$, $p_\alpha^{\mathcal{G}}$ being defined similarly.*

Proof. It is clear that $p_\alpha(\psi) = p(\psi \circ \alpha)$, $\psi \in \mathcal{G}$. Let $p^{\mathcal{F}}(\psi \circ \alpha) < c$. Then $p(\varphi_1) < c$ for all $\varphi_1 \in \mathcal{F}$ near $\psi \circ \alpha$, in particular $p_\alpha(\psi_1) = p(\psi_1 \circ \alpha) < c$ for ψ_1 near ψ. This implies that $p_\alpha^{\mathcal{G}}(\psi) \leqslant c$, hence $p_\alpha^{\mathcal{G}}(\psi) \leqslant p^{\mathcal{F}}(\psi \circ \alpha)$.

Conversely, if $p_\alpha^{\mathcal{G}}(\psi) < c$ we have $p(\psi_1 \circ \alpha) = p_\alpha(\psi_1) < c$ for all ψ_1 in a neighborhood ω of ψ. But the restriction of α^* to $\mathcal{G}$ is an open map by Banach's theorem since it maps a Fréchet space $\mathcal{G}$ continuously onto another, $\mathcal{F}$. Hence $\alpha^*\omega$ is a neighborhood of $\psi \circ \alpha$ in $\mathcal{F}$ so that $p^{\mathcal{F}}(\psi \circ \alpha) \leqslant \sup_{\varphi_1 \in \alpha^*\omega} p(\varphi_1) \leqslant c$. This means that $p^{\mathcal{F}}(\psi \circ \alpha) \leqslant p_\alpha^{\mathcal{G}}(\psi)$.

We will use this lemma only when $\mathcal{F}$ and $\mathcal{G}$ have finite dimension so that the hypothesis that they be closed is automatically fulfilled.

THEOREM 5.7. *Let V be a complex manifold and $\mathcal{G} \subset \mathcal{A}(V)$ a linear subspace such that $\mathcal{G}^m$ contains an embedding of V into $\mathbf{C}^m$ for some m. Then for every $\mu \in \mathcal{A}'(V)$ and every $\psi \in \mathcal{G}$ we have*

$$p^{\mathcal{G}}(\psi) = \inf_K \,(H_K(\psi);\, K \text{ carries } \mu),$$

where $p^{\mathcal{G}}$ denotes the upper regularization of the restriction to $\mathcal{G}$ of the indicator of μ.

Proof. We need only prove that the left-hand side is not less than the right-hand side. Let $\alpha = (\alpha_1, ..., \alpha_m)$ be an embedding of V into $\mathbf{C}^m$ and let $\psi \in \mathcal{G}$ be fixed. Then

$\beta=(\alpha_1, \ldots, \alpha_m, \psi)$ embeds V into $\mathbf{C}^{m+1}$ and we denote by $\mathcal{H}$ the subspace of $\mathcal{G}$ spanned by $\alpha_1, \ldots, \alpha_m, \psi$. It is obviously sufficient to prove that

$$p^{\mathcal{H}}(\psi) \geqslant \inf (H_K(\psi); K \text{ carries } \mu)$$

for $p^{\mathcal{H}} \leqslant p^{\mathcal{G}}$ in $\mathcal{H}$. Let ζ be the linear function $z \to z_{m+1}$ in $\mathbf{C}^{m+1}$. Then $H_K(\psi)=H_{\beta(K)}(\zeta)$ and, by Lemma 5.6, $p^{\mathcal{H}}(\psi)=p_\beta^{\mathbf{C}}(\zeta)$ so what we have to prove is

$$p_\beta^{\mathbf{C}}(\zeta) \geqslant \inf (H_{\beta(K)}(\zeta); K \text{ carries } \mu). \tag{5.2}$$

Suppose that $L \subset \mathbf{C}^{m+1}$ is a convex carrier of $\beta\mu$. Then $K=\beta^{-1}(L)$ is in view of Lemma 4.7 a carrier of μ and $\beta(K) \subset L$ which proves that

$$\inf (H_{\beta(K)}(\zeta); K \text{ carries } \mu) \leqslant \inf (H_L(\zeta); L \text{ carries } \beta\mu).$$

But Theorem 5.2 shows that the right-hand side of this inequality is $p_\beta^{\mathbf{C}}(\zeta)$ which proves (5.2).

COROLLARY 5.8. *Under the assumptions of Theorem* 5.7 *we have* $p^{\mathcal{A}(V)}=p^{\mathcal{G}}$ *in* $\mathcal{G}$.

Proof. If K carries μ we obtain $p(\psi) \leqslant H_K(\psi)$, $\psi \in \mathcal{A}(V)$, and therefore $p^{\mathcal{A}(V)}(\psi) \leqslant H_K(\psi)$ so that, by the theorem, $p^{\mathcal{A}(V)}(\psi) \leqslant p^{\mathcal{G}}(\psi)$ for $\psi \in \mathcal{G}$. It is trivial that, conversely, $p^{\mathcal{G}} \leqslant p^{\mathcal{A}(V)}$ in $\mathcal{G}$.

We finally extend Corollary 5.3.

THEOREM 5.9. *Let* V, U, $\mathcal{G}$ *and* $\mu \in \mathcal{A}'(U)$ *satisfy the hypotheses of Theorem* 5.4. *Let* p_ι *denote the indicator of* $\iota\mu \in \mathcal{A}'(V)$. *Then* $p_\iota^{\mathcal{G}}$ *is convex if and only if* μ *has a unique* $\iota^*\mathcal{G}$*-support.*

A special case is, of course,

COROLLARY 5.10. *Let* μ *be an analytic functional on a Stein manifold* V. *Then* μ *has a unique* $\mathcal{A}(V)$*-support if and only if* $p^{\mathcal{A}(V)}$ *is convex.*

Proof of Theorem 5.9. In view of Theorems 4.11 and 4.17, μ has a unique $\iota^*\mathcal{G}$-support if and only if $\iota\mu$ has a unique $\mathcal{H}$-support for all finite dimensional subspaces $\mathcal{H}$ of $\mathcal{G}$ such that $\alpha_1, \ldots, \alpha_m \in \mathcal{H}$. On the other hand, $p_\iota^{\mathcal{G}}$ is obviously convex if and only if its restriction to every such $\mathcal{H}$ is convex, and by Corollary 5.8, $p_\iota^{\mathcal{G}}=p_\iota^{\mathcal{A}(V)}=p_\iota^{\mathcal{H}}$ in $\mathcal{H}$ so that $p_\iota^{\mathcal{G}}$ is convex if and only if $p_\iota^{\mathcal{H}}$ is convex for all $\mathcal{H}$ of the described kind.

It will thus be sufficient to prove that $\iota\mu$ has a unique $\mathcal{H}$-support if and only if $p_\iota^{\mathcal{H}}$ is convex where $\mathcal{H}$ is a finite dimensional space containing $\alpha_1, \ldots, \alpha_m$. Let $\alpha_1, \ldots, \alpha_k$ span $\mathcal{H}$

and define $\alpha = (\alpha_1, \ldots, \alpha_k)$, an embedding of V into $\mathbf{C}^k$. Now Corollary 4.10 shows that $\iota\mu$ has a unique $\mathcal{H}$-support if and only if $\alpha\iota\mu$ has a unique $\mathfrak{L}$-support and by Lemma 5.6 the indicator q of $\alpha\iota\mu$ satisfies $q^{\mathfrak{L}}(\zeta) = p_\iota^{\mathcal{H}}(\zeta \circ \alpha)$, $\zeta \in \mathfrak{L}$, so our assertion is finally reduced to Corollary 5.3.

Added in proof. Theorem 3.4 has been proved independently by A. Martineau (see Séminaire Lelong June 6, 1966). His proof is the same as that sketched in the remark at the end of Section 3 except that a careful adoption of Theorem 4.4.3 in [5] allows him to work without any regularity assumption on F. I am indebted to him for pointing out that Theorem 3.2 is not new; it is due to R. Fujita and A. Takeuchi (J. Math. Soc. Japan, vols. 15 and 16 respectively).

References

[1]. DUCATEAU, C.-F., Fonctions plurisousharmoniques et convexité complexe dans les espaces de Banach. *Sém. Lelong* (Analyse) 1962, No. 2, Secretariat mathématique, Paris 1962.
[2]. EHRENPREIS, L., *The structure of solutions of partial differential equations.* Lecture notes, Stanford University 1961.
[3]. HARDY, G. H. & ROGOSINSKI, W. W., Theorems concerning functions subharmonic in a strip. *Proc. Roy. Soc. London*, A 185 (1946), 1–14.
[4]. HÖRMANDER, L., L^2 estimates and existence theorems for the $\bar{\partial}$ operator. *Acta Math.*, 113 (1965), 89–152.
[5]. ——, *An introduction to complex analysis in several variables.* Van Nostrand, Princeton, N.J., 1966.
[6]. KISELMAN, C. O., On unique supports of analytic functionals. *Ark. Mat.*, 6, 307–318 (1965).
[7]. LELONG, P., Fonctions entières (n variables) et fonctions plurisousharmoniques de type exponentiel. *C. R. Acad. Sci. Paris*, 260 (1965), 1063–1066.
[8]. MARTINEAU, A., Sur les fonctionnelles analytiques et la transformation de Fourier-Borel. *J. Analyse Math.*, 11 (1963), 1–164.
[9]. PARIS, J., Croissance des fonctions de plusieurs variables et domaines d'holomorphie associés. *Acad. Roy. Belg. Bull. Cl. Sci.*, (5) 48 (1962), 29–36.

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