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ON EQUIANGULAR LINES IN 17 DIMENSIONS AND THE CHARACTERISTIC POLYNOMIAL OF A SEIDEL MATRIX

GARY R. W. GREAVES AND PAVLO YATSYNA

ABSTRACT. For e a positive integer, we find restrictions modulo 2^e on the coefficients of the characteristic polynomial $\chi_S(x)$ of a Seidel matrix S . We show that, for a Seidel matrix of order n even (resp. odd), there are at most $2^{\binom{e-2}{2}}$ (resp. $2^{\binom{e-2}{2}+1}$) possibilities for the congruence class of $\chi_S(x)$ modulo $2^e\mathbb{Z}[x]$. As an application of these results we obtain an improvement to the upper bound for the number of equiangular lines in \mathbb{R}^{17} , that is, we reduce the known upper bound from 50 to 49.

1. INTRODUCTION

For a matrix M , we define the characteristic polynomial of M as $\chi_M(x) = \det(xI - M)$. A symmetric $\{0, \pm 1\}$ -matrix having 0 on the diagonal and all other entries equal to ± 1 is called a **Seidel matrix**. Much like adjacency matrices, Seidel matrices describe the adjacency of a graph. Indeed, a Seidel matrix S of order n corresponds to a graph on n vertices where vertex i is adjacent to vertex j if and only if $S_{ij} = -1$. We call this graph the **underlying graph** of S . Throughout, $\mathbf{1}$ denotes the all-ones vector and $J = \mathbf{1}\mathbf{1}^\top$ denotes the all-ones matrix, of size appropriate to the context in which it is used. Let A be the adjacency matrix of the underlying graph of a Seidel matrix S . Then the matrices A and S are related by the equation $S = J - I - 2A$.

Both the matrices S and A are symmetric matrices with integer entries. Therefore, the eigenvalues of both are necessarily real and also algebraic integers (they are the zeros of the characteristic polynomial, which is monic with integer coefficients). However, the question of which eigenvalues that satisfy these necessary conditions can be eigenvalues of these matrices have very different answers.

We call an algebraic integer **totally real** if all its (Galois) conjugates are real. Hoffman [15] conjectured and Estes [6] showed that (also see Salez [25]) every totally real algebraic integer is an eigenvalue of an adjacency matrix of a graph. In other words, for all totally real algebraic integers α , there exists a graph with adjacency matrix A such that the minimal polynomial of α divides the characteristic polynomial $\chi_A(x)$ of A . On the other hand, Haemers (see [10]) showed that the characteristic polynomial of a Seidel matrix is congruent to $\chi_{J-I}(x)$ modulo $2\mathbb{Z}[x]$. Suppose J is of order n , then $\chi_{J-I}(x) = (x - n + 1)(x + 1)^{n-1}$. Hence, modulo $2\mathbb{Z}[x]$, we have $\chi_{J-I}(x) \equiv (x + 1)^n$ if n is even and $\chi_{J-I}(x) \equiv x(x + 1)^{n-1}$ if n is odd. Observe that, since $\mathbb{Z}[x]/2\mathbb{Z}[x]$ is a UFD, the polynomial $x^2 - 2$ cannot be a

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factor of a Seidel matrix. Hence, in particular, the totally real algebraic integer $\sqrt{2}$ cannot be an eigenvalue of any Seidel matrix.

We further develop this modular restriction on the coefficients of the characteristic polynomial of a Seidel matrix that was observed by Haemers. Our main contribution is to characterise the congruence classes of the characteristic polynomial of a Seidel matrix modulo $2^e\mathbb{Z}[x]$ for e a positive integer. Further, we give an improvement to the upper bound for the number of equiangular lines in \mathbb{R}^{17} .

Seidel matrices were introduced by Van Lint and Seidel [21] as a tool for studying systems of lines in Euclidean space, any two of which make the same angle. Such a system of lines is called an **equiangular** line system. A classical problem from 1948 [12] is, given a positive integer d , to find the maximum cardinality $N(d)$ of an equiangular line system in \mathbb{R}^d . This problem has recently been enjoying lots of attention, for example, the last two years have produced the publications [2, 3, 10, 11, 17, 20, 24, 29, 30, 31]. However, the value of $N(d)$ remains unknown for d as small as 14. One contribution of this paper is to provide an improvement to the upper bound of $N(17)$, that is, to show that $N(17) \leq 49$ (see Remark 5.10).

In Table 1 below, we give the currently known (including the improvement from this paper) values or lower and upper bounds for $N(d)$ for d at most 23.

d	2	3	4	5	6	7–13	14	15	16	17	18	19	20	21	22	23
$N(d)$	3	6	6	10	16	28	28–29	36	40–41	48–49	56–60	72–75	90–95	126	176	276

TABLE 1. Bounds for the sequence $N(d)$ for $2 \leq d \leq 23$. A single number is given in the cases where the exact number is known. The improvement from this paper, $N(17) \leq 49$, is included. A word of caution: some tables in the literature present the lower bounds for $N(d)$ as if they were the exact value of $N(d)$.

The plan of the paper is as follows. In Section 2, we establish congruences for trace powers of the adjacency matrix of a graph. Next, in Section 3, we apply use these congruences to establish modular restrictions on the coefficients of the characteristic polynomial of a Seidel matrix. In Section 4, we turn our attention to necessary conditions for the existence of a system of 50 equiangular lines in \mathbb{R}^{17} . We establish connections between the existence of equiangular line systems and totally positive algebraic integers with small trace. In Section 5, we show that the necessary conditions from Section 4 cannot be satisfied. Finally, we conclude by remarking on how the approach presented in this paper can be applied to $N(d)$ for $d \neq 17$.

2. A CONGRUENCE FOR THE TRACE OF POWERS OF THE ADJACENCY MATRIX OF A GRAPH

In this section we apply Burnside's lemma to find a congruence modulo $2N$ for a weighted sum of traces of powers of a graph-adjacency matrix.

Let Γ be a graph and let \mathbf{x} be a closed walk of length N in Γ ; we write $\mathbf{x} = x_0x_1 \dots x_{N-1}$ where x_i is adjacent to x_{i+1} for each $i \in \{0, \dots, N-1\}$ with indices reduced modulo N . There is a natural correspondence between the vertices of the closed walk \mathbf{x} and the vertices of a regular N -gon. Hence, under this correspondence, we consider the dihedral group D_N of order $2N$ acting on the set of

closed N -walks of Γ . Let $N \geq 3$ and write $D_N = \langle r, s \mid r^N, s^2, (rs)^2 \rangle$. For $g \in D_N$, we denote by $\text{fix}_\Gamma(g)$ the set of closed N -walks of Γ fixed by g .

Lemma 2.1. *Let Γ be a graph with adjacency matrix A and let $N \geq 3$. Then*

- (i) $|\text{fix}_\Gamma(r^k)| = \text{tr}(A^{\text{gcd}(k,N)})$, for all $k \in \mathbb{Z}$;
- (ii) $|\text{fix}_\Gamma(r^{2k}s)| = 0$, for all $k \in \mathbb{Z}$;
- (iii) $|\text{fix}_\Gamma(r^{2k+1}s)| = \begin{cases} \mathbf{1}^\top A^{N/2} \mathbf{1}, & \text{if } N \text{ is even} \\ 0, & \text{if } N \text{ is odd} \end{cases}$ (for all $k \in \mathbb{Z}$).

Proof. Let $\mathbf{x} = x_0 x_1 \dots x_{N-1}$ be a closed N -walk that is fixed by some element $g \in D_N$. Observe that, if c is a cycle of the group element g then, for each i and j in c , we have $x_i = x_j$.

First suppose that $g = r^k$ for some $k \in \mathbb{Z}$. Then g has order $m = N/\text{gcd}(k, N)$. Therefore, g consists of N/m cycles each of length m . It follows that, for all $i \in \{0, \dots, N/m - 1\}$, we have $x_{im} x_{im+1} \dots x_{im+m-1} = x_0 x_1 \dots x_{m-1}$. Hence $\text{fix}_\Gamma(g)$ consists of closed N/m -walks repeated m times. Since $\text{tr}(A^{N/m})$ is equal to the number of closed walks of length $N/m = \text{gcd}(k, N)$, we have established (i).

Now suppose N is odd and $g = r^k s$ for some $k \in \mathbb{Z}$. In this case, g consists of $\lfloor N/2 \rfloor$ cycles of length 2. It follows that one pair of adjacent vertices of \mathbf{x} must be equal, but there are no such closed walks since Γ has no loops. Whence we have (ii) and (iii) for N odd.

In the remainder of the proof we assume that N is even. Suppose that $g = r^{2k}s$ for some $k \in \mathbb{Z}$. In this case, g consists of $N/2$ cycles of length 2. Then two pairs of adjacent vertices of \mathbf{x} must be equal, but there are no such closed walks since Γ has no loops. This yields (ii).

Finally, suppose that $g = r^{2k+1}s$ for some $k \in \mathbb{Z}$. In this case, g consists of $N/2 - 1$ cycles of length 2. Without loss of generality, we can assume that x_0 and $x_{N/2}$ are fixed by g . Then, for each $i \in \{1, \dots, N/2 - 1\}$, we must have $x_i = x_{N-i}$. Hence $\text{fix}_\Gamma(g)$ consists of closed N -walks made up of an $N/2$ -walk together with its inverse. \square

For a positive integer a , we use $\varphi(a)$ to denote Euler's totient function of a . The following result is the main result of this section, which follows from Lemma 2.1 via a straightforward application of Burnside's lemma (that $\frac{1}{|D_N|} \sum_{g \in D_N} |\text{fix}_\Gamma(g)|$ is an integer).

Lemma 2.2 (cf. [14, Corollary 5a]). *Let Γ be a graph with adjacency matrix A and let $N \geq 4$ be an even integer. Then*

$$\sum_{d \mid N} \varphi(N/d) \text{tr}(A^d) + \frac{N}{2} \mathbf{1}^\top A^{N/2} \mathbf{1} \equiv 0 \pmod{2N}.$$

Note that we also have a similar congruence when N is odd, which also follows from Lemma 2.1.

Lemma 2.3 (cf. [14, Corollary 5a]). *Let Γ be a graph with adjacency matrix A and let $N \geq 3$ be an odd integer. Then*

$$\sum_{d \mid N} \varphi(N/d) \text{tr}(A^d) \equiv 0 \pmod{2N}.$$

A graph Γ is called an **Euler graph** if the degree of each of its vertices is even. Let S be a Seidel matrix. We define the **switching class** of S to be the set of underlying graphs of Seidel matrices of the form DSD where D is a diagonal matrix with diagonal entries equal to ± 1 .

Later we will need the following result due to Seidel [13, 26].

Theorem 2.4. *Let S be a Seidel matrix of order n odd. Then there exists a unique Euler graph in the switching class of S .*

The next result is a standard result from linear algebra.

Lemma 2.5. *Let A be a symmetric integer matrix whose diagonal entries are all even. Then $\mathbf{1}^\top A^i \mathbf{1}$ is even for all integers $i \geq 1$.*

We now establish congruences for the number of walks of a given length in an Euler graph.

Lemma 2.6. *Let Γ be an Euler graph with adjacency matrix A . Then $\mathbf{1}^\top A \mathbf{1} \equiv 0 \pmod{2}$ and $\mathbf{1}^\top A^i \mathbf{1} \equiv 0 \pmod{4}$ for all $i \geq 2$.*

Proof. By Lemma 2.5, we have that $\mathbf{1}^\top A \mathbf{1}$ is even. Since Γ is an Euler graph, we can write $A \mathbf{1} = 2\mathbf{v}$ for some integer vector \mathbf{v} . Hence, for $i \geq 2$, we have $\mathbf{1}^\top A^i \mathbf{1} = 4\mathbf{v}^\top A^{i-2} \mathbf{v}$, which is divisible by 4. \square

Using Lemma 2.2 together with Lemma 2.6, we obtain the following lemma.

Lemma 2.7. *Let Γ be an Euler graph with adjacency matrix A and let $N \geq 4$ be an even integer. Then*

$$\mathrm{tr}(A^N) \equiv - \sum_{\substack{d \mid N \\ d \neq N}} \varphi(N/d) \mathrm{tr}(A^d) \pmod{2N}.$$

3. RELATIONS FOR THE COEFFICIENTS OF CHARACTERISTIC POLYNOMIALS MODULO POWERS OF 2

3.1. A relation for characteristic polynomials. In this section we establish a relation between the characteristic polynomial of a Seidel matrix S and the characteristic polynomial of a graph in the switching class of S . If Γ is a graph with adjacency matrix A then its Seidel matrix has the form $S = J - I - 2A$. The characteristic polynomial $\chi_S(x)$ of S can be written as $\chi_S(x) = \chi_{J-2A}(x+1)$. With this in mind, we instead consider the relation between $\chi_A(x)$ and $\chi_{J-2A}(x)$.

Lemma 3.1. *Let A be a matrix of order n . Write $\chi_{J-2A}(x) = \sum_{i=0}^n a_i x^{n-i}$ and $\chi_A(x) = \sum_{i=0}^n b_i x^{n-i}$. Then*

$$a_r = (-2)^r \left(b_r + \frac{1}{2} \sum_{i=1}^r b_{r-i} \mathbf{1}^\top A^{i-1} \mathbf{1} \right).$$

Proof. By the matrix determinant lemma,

$$\chi_{J-2A}(x) = \chi_{-2A}(x) - \mathbf{1}^\top \mathrm{adj}(xI + 2A) \mathbf{1}.$$

Write $\chi_{-2A}(x) = \sum_{i=0}^n c_i x^{n-i}$. The adjugate matrix can be written [7, p. 38] as

$$\mathrm{adj}(xI + 2A) = \sum_{i=0}^{n-1} (-2A)^{n-1-i} \sum_{j=0}^i x^{i-j} c_j.$$

Note that we have $c_i = (-2)^i b_i$ for all $i \in \{0, \dots, n\}$. The result then follows by equating coefficients. \square

Now we record a couple of corollaries to Lemma 3.1. First, a surprisingly strong restriction on $\chi_{J-2A}(x)$ where A is the adjacency matrix of a graph of order n even.

Corollary 3.2. *Let A be the adjacency matrix of a graph of order n even and write $\chi_{J-2A}(x) = \sum_{i=0}^n a_i x^{n-i}$. Then 2^r divides a_r for all $r \in \{0, \dots, n\}$.*

Proof. By Lemma 3.1, it suffices to show that $\mathbf{1}^\top A^{i-1} \mathbf{1}$ is even for all $i \geq 1$. By Lemma 2.5, for all $i \geq 2$, we have that $\mathbf{1}^\top A^{i-1} \mathbf{1}$ is even, and, for $i = 1$, we have $\mathbf{1}^\top A^{i-1} \mathbf{1} = n$, which is also even. \square

Remark 3.3. *Let A be an adjacency matrix of a graph of order n . It is clear that the trace of $J-2A$ equals n and the trace of $(J-2A)^2$ equals n^2 . Write $\chi_{J-2A}(x) = \sum_{i=0}^n a_i x^{n-i}$. Obviously $a_0 = 1$. Furthermore, using Newton's identities, we see that $a_1 = -n$ and $a_2 = 0$.*

Denote by \mathcal{C}_n the set of all Seidel matrices of order n . Given a positive integer e , define the set $\mathcal{P}_{n,e} = \{\chi_S(x) \bmod 2^e \mathbb{Z}[x] \mid S \in \mathcal{C}_n\}$. Using Remark 3.3 together with Corollary 3.2 allows us to obtain the following.

Corollary 3.4. *Let n be an even integer and e be a positive integer. Then the cardinality of $\mathcal{P}_{n,e}$ is at most $2^{\binom{e-2}{2}}$.*

Clearly, if n is small compared to e then the cardinality of $\mathcal{P}_{n,e}$ will be strictly less than $2^{\binom{e-2}{2}}$. Indeed, for $n = 2$ the cardinality of $\mathcal{P}_{n,e}$ is 1 for all e . However, it is straightforward to check that, for small values of e ($e \leq 7$), there exist even n giving equality in the bound in Corollary 3.4. We conjecture that, for all $e \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that $|\mathcal{P}_{n,e}| = 2^{\binom{e-2}{2}}$ for all even $n \geq N$.

By Corollary 3.4, there is only one congruence class modulo $2\mathbb{Z}[x]$ for characteristic polynomials of Seidel matrices of even order. Since $J - I$ is a Seidel matrix with $\chi_{J-I}(x) = (x - (n-1))(x+1)^{n-1}$, we note the following corollary.

Corollary 3.5 (cf. [10, Lemma 2.2]). *Let S be a Seidel matrix of order n even. Then $\chi_S(x) \equiv (x+1)^n \bmod 2\mathbb{Z}[x]$.*

Empirically, we observed that the analogous bound for $\mathcal{P}_{n,e}$ when n is odd should be $2^{\binom{e-2}{2}+1}$. We continue in pursuit of this bound, which culminates in Corollary 3.13.

Note the following well-known result from linear algebra.

Lemma 3.6. *Let A be a symmetric integer matrix of order n odd whose diagonal entries are all even. Then $\det A$ is even.*

We will need a corollary of this lemma.

Corollary 3.7. *Let A be a symmetric integer matrix of order n whose diagonal entries are all zero and write $\chi_A(x) = \sum_{i=0}^n b_i x^{n-i}$. Then b_r is even for all odd r .*

We now establish a result similar to Corollary 3.2.

Lemma 3.8. *Let A be an adjacency matrix of a graph of order n and write $\chi_{J-2A}(x) = \sum_{i=0}^n a_i x^{n-i}$. Then 2^r divides a_r for all r even.*

Proof. Let $\chi_A(x) = \sum_{i=0}^n b_i x^{n-i}$. Using Lemma 3.1, it suffices to show that $\sum_{i=1}^r b_{r-i} \mathbf{1}^\top A^{i-1} \mathbf{1}$ is even. By Lemma 2.5, for all $i \geq 2$, we have that $\mathbf{1}^\top A^{i-1} \mathbf{1}$ is even. And, by Corollary 3.7, the coefficient b_{r-1} is even for all r even. \square

3.2. An application to Euler graphs. Let S be a Seidel matrix of order n odd. By Theorem 2.4, we may assume that the underlying graph of S is an Euler graph. We therefore focus on Euler graphs Γ .

Corollary 3.9. *Let Γ be an Euler graph of order n odd and let A be its adjacency matrix. Write $\chi_{J-2A}(x) = \sum_{i=0}^n a_i x^{n-i}$ and $\chi_A(x) = \sum_{i=0}^n b_i x^{n-i}$. Then, for all $r \in \{1, \dots, \frac{n-1}{2}\}$, we have*

$$\begin{aligned} b_{2r} &\equiv \frac{-a_{2r+1}}{2^{2r}n} \pmod{4}; \\ b_{2r-1} &\equiv \frac{a_{2r+1} + a_{2r} + a_{2r-1}a_3}{2^{2r-1}} \pmod{4}. \end{aligned}$$

Proof. Using Lemma 2.6, Lemma 3.1, and Corollary 3.7, we have that

$$\begin{aligned} (1) \quad &\frac{a_{2r+1}}{2^{2r-1}} \equiv -2nb_{2r} \pmod{8}; \\ (2) \quad &\frac{na_{2r}}{2^{2r-1}} \equiv 2nb_{2r} + b_{2r-1}n^2 + nb_{2r-2}\mathbf{1}^\top A\mathbf{1} \pmod{4}; \\ (3) \quad &a_3 \equiv -4b_2n \pmod{8}. \end{aligned}$$

Furthermore, since $b_1 = 0$, we can write $a_2 = 4b_2 + 2\mathbf{1}^\top A\mathbf{1}$. By Remark 3.3, we have $a_2 = 0$, then using (3), we find that $a_3 \equiv 2n\mathbf{1}^\top A\mathbf{1} \equiv 2\mathbf{1}^\top A\mathbf{1} \pmod{8}$. Hence,

$$\frac{a_{2r-1}a_3}{2^{2r-1}} \equiv -nb_{2r-2}\mathbf{1}^\top A\mathbf{1} \pmod{4}.$$

Since, by Lemma 3.8, 2^r divides a_{2r} , using (1) and (2), we can write

$$b_{2r-1} \equiv \frac{a_{2r+1} + a_{2r} + a_{2r-1}a_3}{2^{2r-1}} \pmod{4},$$

as required. \square

For a nonzero integer a , the **2-adic valuation** $\nu_2(a)$ of a is the multiplicity of 2 in the prime factorisation of a . The 2-adic valuation of 0 is defined to be ∞ . For an integer b relatively prime to a , the 2-adic valuation $\nu_2(a/b)$ of a/b is defined as $-\nu_2(b)$ if b is even and $\nu_2(a)$ otherwise. Denote by $B_2(a)$ the number of 1s in the binary expansion of a .

Lemma 3.10. *Let s and t be integers satisfying $s \equiv t \pmod{4}$. Then for all positive integers m , we have $s^m \equiv t^m \pmod{2^{\nu_2(m)+2}}$.*

Proof. Write $s = t + 4u$ for some integer u . Since $s^m = t^m + \sum_{i=1}^m \binom{m}{i} (4u)^i t^{m-i}$, it suffices to show that

$$(4) \quad \nu_2 \left(\binom{m}{i} \right) + 2i \geq \nu_2(m) + 2, \quad \text{for all } i \in \{1, \dots, m\}.$$

By a theorem of Kummer [18], we have $\nu_2 \left(\binom{m}{i} \right) = B_2(m-i) + B_2(i) - B_2(m)$. Hence, (4) becomes $B_2(m-1) \leq 2i - 3 + B_2(m-i) + B_2(i)$ for all $i \in \{1, \dots, m\}$.

Observe that the number of 1s in the binary expansion of $m-i$ is at least $B_2(m-1) - B_2(i-1)$. Thus, we have $B_2(m-1) \leq B_2(m-i) + B_2(i-1)$. Finally, it is straightforward to verify the inequality $B_2(i-1) \leq 2i - 3 + B_2(i)$ for all $i \in \{1, \dots, m\}$. \square

We will also need the next lemma in preparation for the subsequent result.

Lemma 3.11. *Let l be a positive integer and let m_1, m_2, \dots, m_l be nonnegative integers having a positive sum. Let $m \in \{m_i \mid m_i \neq 0\}$. Then*

$$\nu_2 \left(\frac{(m_1 + m_2 + \dots + m_l - 1)!}{m_1! m_2! \dots m_l!} \right) \geq -\nu_2(m).$$

Proof. Without loss of generality, assume that $m = m_1$ and let $s = \sum_{i=2}^l m_i$. Observe that

$$\frac{(m_1 + m_2 + \dots + m_l - 1)!}{m_1! m_2! \dots m_l!} = \frac{(s + m_1 - 1) \dots (s + 1)}{m_1!} \frac{s!}{m_2! \dots m_l!}$$

Clearly the right hand side is the product of $1/m$ and multinomial coefficients. \square

Next we show a congruence between coefficients of the characteristic polynomial of $J - 2A$ where A is the adjacency matrix of an Euler graph.

Lemma 3.12. *Let Γ be an Euler graph of order n odd, let A be its adjacency matrix, and suppose that $\chi_{J-2A}(x) = \sum_{i=0}^n a_i x^{n-i}$. Then, for $k \in \{2, \dots, (n-1)/2\}$, we have*

$$a_{2k+1} \equiv \sum_{d \mid 2k} \sum_{\substack{m_1 + 2m_2 + \dots + dm_d = d \\ m_1 \geq 0, \dots, m_d \geq 0 \\ m_{2k} = 0}} C_d(m_1, \dots, m_d) P_d(m_1, \dots, m_d) \pmod{2^{2k+1}},$$

where

$$C_d(m_1, \dots, m_d) := 2^{2k} \frac{d\varphi(2k/d)}{2k} \frac{(m_1 + m_2 + \dots + m_d - 1)!}{m_1! m_2! \dots m_d!}$$

and

$$P_d(m_1, \dots, m_d) := \prod_{j=1}^{d/2} \left(\frac{a_{2j+1}}{2^{2j} n} \right)^{m_{2j}} \prod_{j=1}^{d/2+1} \left(\frac{a_{2j+1} + a_{2j} + a_{2j-1} a_3}{2^{2j-1}} \right)^{m_{2j-1}}.$$

Proof. First write $\chi_A(x) = \sum_{i=0}^n b_i x^{n-i}$. By Newton's identities, we have

$$(5) \quad \text{tr}(A^i) = \sum_{\substack{m_1 + 2m_2 + \dots + im_i = i \\ m_1 \geq 0, \dots, m_i \geq 0}} \frac{i(m_1 + m_2 + \dots + m_i - 1)!}{m_1! m_2! \dots m_i!} \prod_{j=1}^i (-b_j)^{m_j}.$$

Using Lemma 3.1 together with Lemma 2.6, we have $a_{2k+1} \equiv 2^{2k} b_{2k} \pmod{2^{2k+1}}$. Now, writing $b_{2k} = b_{2k} + \text{tr}(A^{2k})/2k - \text{tr}(A^{2k})/2k$, apply Lemma 2.7 to obtain the congruence

$$(6) \quad a_{2k+1} \equiv 2^{2k} \left(b_{2k} + \frac{\text{tr}(A^{2k})}{2k} + \sum_{\substack{d \mid 2k \\ d \neq 2k}} \varphi(2k/d) \frac{\text{tr}(A^d)}{2k} \right) \pmod{2^{2k+1}}.$$

Observe that, using (5), we can write

$$2^{2k} \left(b_{2k} + \frac{\text{tr}(A^{2k})}{2k} \right) = \sum_{\substack{m_1 + \dots + 2km_{2k} = 2k \\ m_1 \geq 0, \dots, m_{2k-1} \geq 0 \\ m_{2k} = 0}} C_{2k}(m_1, \dots, m_{2k-1}, m_{2k}) \prod_{j=1}^{2k-1} (-b_j)^{m_j}.$$

For fixed integers m_1, \dots, m_{2k-1} , define $\nu = \min(\{\nu_2(m_i) \mid m_i \neq 0\})$ and let $m \in \{m_i \mid \nu_2(m_i) = \nu\}$. By Lemma 3.11, we have

$$\nu_2(C_{2k}(m_1, \dots, m_{2k-1}, 0)) \geq 2k - \nu_2(m).$$

Observe that, for odd j , by Corollary 3.7, we have $b_j \equiv -b_j \pmod{4}$. Now, using Corollary 3.9 together with Lemma 3.10, we can write

$$\prod_{j=1}^{2k-1} (-b_j)^{m_j} \equiv P_{2k}(m_1, \dots, m_{2k-1}, 0) \pmod{2^{\nu_2(m)+2}}.$$

Since the 2-adic valuation of $C_{2k}(m_1, \dots, m_{2k-1}, 0)$ is at least $2k - \nu_2(m)$, it follows that $2^{2k} \left(b_{2k} + \frac{\text{tr}(A^{2k})}{2k} \right)$ is congruent modulo 2^{2k+2} to

$$\sum_{\substack{m_1+2m_2+\dots+2km_{2k}=2k \\ m_1 \geq 0, \dots, m_{2k-1} \geq 0 \\ m_{2k}=0}} C_{2k}(m_1, \dots, m_{2k}) P_{2k}(m_1, \dots, m_{2k}).$$

Next, for d a proper divisor of $2k$, using (5), we can write

$$2^{2k} \varphi(2k/d) \frac{\text{tr}(A^d)}{2k} = \sum_{\substack{m_1+2m_2+\dots+dm_d=d \\ m_1 \geq 0, \dots, m_d \geq 0}} C_d(m_1, \dots, m_d) \prod_{j=1}^d (-b_j)^{m_j}.$$

Again, for fixed integers m_1, \dots, m_{2k} define $\nu = \min(\{\nu_2(m_i) \mid m_i \neq 0\})$ and let $m \in \{m_i \mid \nu_2(m_i) = \nu\}$. This time, we have the lower bound,

$$\nu_2(C_d(m_1, \dots, m_d)) \geq 2k - \nu_2(m) - 1.$$

Indeed, using Euler's formula, we see that

$$\varphi(2k/d) \frac{d}{2k} = \prod_{\substack{p \mid 2k/d \\ p \text{ prime}}} \frac{p-1}{p}$$

and hence $\nu_2(d\varphi(2k/d)/2k) \geq -1$. Then, by Lemma 3.11, the 2-valuation of $C_d(m_1, \dots, m_d)$ has lower bound $\nu_2(C_d(m_1, \dots, m_d)) \geq 2k - \nu_2(m) - 1$. Using Corollary 3.9 together with Lemma 3.10, we can write

$$\prod_{j=1}^d (-b_j)^{m_j} \equiv P_d(m_1, \dots, m_d) \pmod{2^{\nu_2(m)+2}}.$$

Hence $2^{2k} \varphi(2k/d) \text{tr}(A^d)/2k$ is congruent modulo 2^{2k+1} to

$$\sum_{\substack{m_1+2m_2+\dots+dm_d=d \\ m_1 \geq 0, \dots, m_d \geq 0}} C_d(m_1, \dots, m_d) P_d(m_1, \dots, m_d).$$

The lemma then follows from (6). \square

Now we can bound the number of congruence classes modulo $2^e \mathbb{Z}[x]$ of characteristic polynomials of Seidel matrices of odd order.

Corollary 3.13. *Let n be an odd integer and e be a positive integer. Then the cardinality of $\mathcal{P}_{n,e}$ is at most $2^{\binom{e-2}{2}+1}$.*

Proof. By Remark 3.3, the sets $\mathcal{P}_{n,1}$ and $\mathcal{P}_{n,2}$ both have cardinality 1. Moreover, by Lemma 3.1 and Lemma 3.8, $\mathcal{P}_{n,3}$ has cardinality at most 2. Assume that $\mathcal{P}_{n,e}$ has cardinality at most $2^{\binom{e-2}{2}+1}$.

It suffices to show that each polynomial in $\mathcal{P}_{n,e}$ can be lifted to at most 2^{e-2} polynomials in $\mathcal{P}_{n,e+1}$. Let S be a Seidel matrix of order n . By Theorem 2.4, there is a unique Euler graph Γ in the switching class of S . Let A be the adjacency matrix of Γ . Then $\chi_{J-I-2A}(x) \bmod 2^e\mathbb{Z}[x]$ is an element of $\mathcal{P}_{n,e}$. Write $\chi_{J-2A}(x) = \sum_{i=0}^n a_i x^{n-i}$. Then, by Lemma 3.1, for all $i \geq e+2$ we have $a_i \equiv 0 \pmod{2^{e+1}}$. By Remark 3.3, we have $a_0 = 1$, $a_1 = -n$, and $a_2 = 0$.

Assume that the congruence class of each of a_3, a_4, \dots, a_e modulo 2^e is given. Then, for each of a_3, a_4, \dots, a_e , there are two possibilities for its congruence class modulo 2^{e+1} . Furthermore, if $e+1$ is even then, by Lemma 3.8, the coefficient a_{e+1} is divisible by 2^{e+1} . On the other hand, if $e+1$ is odd, then by Lemma 3.12, the congruence class of a_{e+1} is determined by that of a_3, a_4, \dots, a_e . Therefore, given that the congruence class of $\chi_{J-2A}(x)$ is fixed modulo $2^e\mathbb{Z}[x]$, we have that there are 2^{e-2} possibilities for congruence class of $\chi_{J-2A}(x)$ modulo $2^{e+1}\mathbb{Z}[x]$. \square

Just as for Corollary 3.4, for small values of e ($e \leq 7$), we have checked that there exist odd n giving equality in the bound in Corollary 3.13. The sharpness of this bound for $n = 49$ and $e = 5$ is a crucial ingredient in the proofs of Theorem 5.6 and Theorem 5.7. Furthermore, we conjecture that, for all integers $e \geq 3$, there exists $N \in \mathbb{N}$ such that $|\mathcal{P}_{n,e}| = 2^{\binom{e-2}{2}+1}$ for all odd $n \geq N$.

4. ON 50 EQUIANGULAR LINES IN \mathbb{R}^{17}

4.1. From equiangular lines to Seidel matrices. Now we apply our restrictions on the characteristic polynomial of a Seidel matrix to equiangular lines. Suppose we have a system \mathcal{L} of 50 equiangular lines in \mathbb{R}^{17} . Let $\{\mathbf{v}_1, \dots, \mathbf{v}_{50}\}$ be a set of unit spanning vectors for the lines in \mathcal{L} . The inner product of any two distinct vectors $\mathbf{v}_i^\top \mathbf{v}_j = \pm\alpha$ for some α in the open interval $(0, 1)$. The Gram matrix G for this set of vectors has diagonal entries equal to 1 and off-diagonal entries equal to $\pm\alpha$. The matrix $S = (G - I)/\alpha$ is the Seidel matrix corresponding to the system of lines \mathcal{L} . Note that the smallest eigenvalue of S is $-1/\alpha$ with multiplicity $50 - 17 = 33$. By [19, Theorem 3.4.2] together with [21, Lemma 6.1], we must have $1/\alpha = 5$. Thus the smallest eigenvalue of S is -5 with multiplicity 33.

The purpose of this section is to prove the following proposition.

Proposition 4.1. *Suppose S is a Seidel matrix of order 50 having smallest eigenvalue -5 with multiplicity 33. Then the characteristic polynomial $\chi_S(x)$ of S must be one of the three polynomials*

$$\begin{aligned} &(x+5)^{33}(x-9)^{10}(x-11)^5(x^2-20x+95), \\ &(x+5)^{33}(x-9)^{12}(x-11)^4(x-13), \\ &\text{or } (x+5)^{33}(x-7)(x-9)^9(x-11)^7. \end{aligned}$$

Let $\lambda_1, \dots, \lambda_{17}$ be the 17 other eigenvalues of S , which satisfy the inequalities $-5 < \lambda_1 \leq \dots \leq \lambda_{17}$. Observe that S has $\text{tr } S = 0$ and $\text{tr } S^2 = 50 \cdot 49$. Since the smallest eigenvalues of S is -5 with multiplicity 33, we have

$$(7) \quad \sum_{i=1}^{17} \lambda_i = 33 \cdot 5 = 165$$

$$(8) \quad \sum_{i=1}^{17} \lambda_i^2 = 50 \cdot 49 - 33 \cdot 5^2 = 1625.$$

Remark 4.2. Write $\chi_S(x) = \sum_{i=0}^{50} c_i x^{n-i}$. Since $\text{tr } S = 0$ and $\text{tr } S^2 = n(n-1)$, using Newton's identities, we have that $c_0 = 1$, $c_1 = 0$, and $c_2 = -50 \cdot 49/2$.

4.2. Totally positive algebraic integers of small trace. A polynomial $p(x) \in \mathbb{Z}[x]$ (resp. algebraic integer) is called **totally positive** if all of its zeros (resp. conjugates) are positive. The **trace** of a polynomial $p(x) \in \mathbb{Z}[x]$ (resp. algebraic integer) is defined to be the sum of its zeros (resp. conjugates); the trace of $p(x)$ is denoted by $\text{tr } p(x)$. In this section, we turn the problem of determining the unknown eigenvalues λ_i (for $i \in \{1, \dots, 17\}$) of S into a problem about totally positive algebraic integers of small trace.

Putting (7) and (8) together, we have

$$\sum_{i=1}^{17} (\lambda_i - 10)^2 = 25.$$

Define the polynomials $F(x)$ and $G(x)$ by $F(x) := \chi_S(x)/(x+5)^{33} = \prod_{i=1}^{17} (x - \lambda_i)$ and $G(x) := \prod_{i=1}^{17} (x - (\lambda_i - 10)^2)$. By Corollary 3.5, none of the λ_i can be an even (rational) integer. Hence $G(x)$ is a totally positive, monic polynomial in $\mathbb{Z}[x]$ with trace 25. Also note that the difference between the trace and the degree of $G(x)$ is 8. Therefore, each irreducible factor of $G(x)$ must also have its trace minus its degree at most 8.

Denote by $T(d, t)$ the set of irreducible, totally positive, monic, degree- d , integer polynomials of trace t . Each irreducible factor of $G(x)$ must belong to $T(d, t)$ for some d and t where $1 \leq d \leq 17$ and $d \leq t \leq d + 8$.

Lemma 4.3. *Suppose that $f(x) = \prod_{i=1}^d (x - \lambda_i) \equiv (x+1)^d \pmod{2\mathbb{Z}[x]}$. Let μ be an integer and $g(x) = \prod_{i=1}^d (x - (\lambda_i - 2\mu)^2)$. Then $g(x) \equiv (x+1)^d \pmod{2\mathbb{Z}[x]}$.*

Proof. We can write $g(x^2) = (-1)^d f(x+2\mu)f(-x+2\mu) \equiv (x+1)^{2d} \pmod{2\mathbb{Z}[x]}$. Since $(x+1)^{2d} \equiv \sum_{i=0}^d \binom{2d}{2i} x^{2i} \pmod{2\mathbb{Z}[x]}$, we see that $g(x) \equiv (x+1)^d \pmod{2\mathbb{Z}[x]}$, as required. \square

By Corollary 3.5, the polynomial $F(x)$ is congruent to $(x+1)^d$ modulo $2\mathbb{Z}[x]$. Hence, by Lemma 4.3, the polynomial $G(x)$ is also congruent to $(x+1)^d$ modulo $2\mathbb{Z}[x]$, and since $\mathbb{Z}[x]/2\mathbb{Z}[x]$ is a UFD, each irreducible factor of $G(x)$ of degree e is congruent to $(x+1)^e$ modulo $2\mathbb{Z}[x]$. Next we consider all possible candidates for the irreducible factors of $G(x)$.

Define the set $\mathcal{T}(d, t) := \{p(x) \in T(d, t) \mid p(x) \equiv (x+1)^d \pmod{2\mathbb{Z}[x]}\}$. We distinguish between two kinds of irreducible factors $\eta(x)$ of $G(x)$ depending on whether or not $\eta(x^2)$ is irreducible. Define the sets

$$\begin{aligned} \mathcal{T}_{\text{irr}}(d, k) &:= \{p(x) \in \mathcal{T}(d, k+d) \mid p(x^2) \text{ is irreducible}\} \\ \mathcal{T}_{\text{red}}(d, k) &:= \{p(x) \in \mathcal{T}(d, k+d) \mid p(x^2) \text{ is reducible}\}. \end{aligned}$$

Lemma 4.4. *Suppose that $g(x) \in \mathcal{T}_{\text{irr}}(d, k)$ is a factor of $G(x)$. Then $g(x)^2$ divides $G(x)$.*

Proof. We have $G(x^2) = (-1)^{17}F(10+x)F(10-x)$. Therefore $(-1)^{17}F(10+x)F(10-x) = g(x^2)h(x^2)$, for some monic polynomial $h(x) \in \mathbb{Z}[x]$. Since $g(x^2)$ is irreducible, it must divide $F(10+x)$ or $F(10-x)$. Furthermore, since $g(x^2)$ is invariant under substituting $-x$ for x , it must divide both $F(10+x)$ and $F(10-x)$. Hence $g(x^2)$ is also a factor of $h(x^2)$, as required. \square

We will use the following consequence of Lemma 4.4.

Corollary 4.5. *Let $g(x)$ be an irreducible factor of $G(x)$ of degree d . Then either $g(x) \in \mathcal{T}_{\text{irr}}(d, k)$ with $k \leq 4$ or $g(x) \in \mathcal{T}_{\text{red}}(d, k)$ with $k \leq 8$.*

The next theorem is a result of McKee [22].

Theorem 4.6. *Let $f(x) \in \mathbb{Z}[x]$ be a monic totally-positive irreducible polynomial of degree $d \geq 5$. Then $\text{tr } f(x) \geq \lceil 1.78839d \rceil$.*

It follows that we can further improve the bounds on the degree of the putative irreducible factors of $G(x)$.

Corollary 4.7. *Let $g(x)$ be an irreducible factor of $G(x)$ of degree d . Suppose that $g(x) \in \mathcal{T}_{\text{irr}}(d, k)$ (resp. $g(x) \in \mathcal{T}_{\text{red}}(d, k)$) for some d and k . Then $d \leq 5$ (resp. $d \leq 10$).*

By Corollary 4.5 and Corollary 4.7, each irreducible factor of $G(x)$ belongs to either $\mathcal{T}_{\text{irr}}(d, k)$ with $d \leq 5$ and $k \leq 4$ or $\mathcal{T}_{\text{red}}(d, k)$ with $d \leq 10$ and $k \leq 8$.

4.3. Polynomial enumeration algorithm. We use a method due to Robinson to find all possible irreducible factors of $G(x)$. This method has been detailed by Smyth [27] and McKee and Smyth [23]. For completeness, following McKee and Smyth [23, Section 3.2], we describe the algorithm below. We will also use this algorithm in Proposition 5.8.

First we state a result about the interlacing of the zeros of a totally real polynomial and its derivative. This result is a straightforward consequence of Rolle's theorem.

Proposition 4.8. *Let $d \geq 2$ and let $p(x)$ be a monic degree- d polynomial having zeros $\alpha_1 < \dots < \alpha_d$. Denote by $\beta_1 < \dots < \beta_{d-1}$ the zeros of its derivative $p'(x)$. Then*

$$\alpha_1 < \beta_1 < \alpha_2 < \dots < \alpha_{d-1} < \beta_{d-1} < \alpha_d.$$

Fix an integer t and real numbers l and u . Let $\mathfrak{P} \subset \mathbb{Z}[x]$ be the set of monic polynomials having trace t and all zeros inside the interval $[l, u]$. We would like to find all polynomials in \mathfrak{P} . Suppose $p(x) \in \mathfrak{P}$. Then

$$p(x) = x^d - tx^{d-1} + a_2x^{d-2} + \dots + a_{d-1}x + a_d,$$

for some integers a_2, \dots, a_d . For $r = d, d-1, \dots, 1$, define

$$p_r(x) = \frac{r!}{d!} \frac{d^{d-r}}{dx^{d-r}} p(x) = x^r - t \frac{r}{d} x^{r-1} + \dots + a_r \frac{r!(d-r)!}{d!}.$$

Let $\gamma_1^{(r)} \leq \gamma_2^{(r)} \leq \dots \leq \gamma_r^{(r)}$ be the zeros of $p_r(x)$. Then, by Proposition 4.8, we have that $l < \gamma_1^{(r)}$ and $\gamma_r^{(r)} < u$. Given a candidate for $p_i(x)$, having i distinct zeros,

that is, given values for a_2, \dots, a_i , we seek the (possibly empty) range of values for a_{i+1} such that $p_{i+1}(x)$ has $i + 1$ distinct zeros in the interval $[l, u]$. To find this range of values for a_{i+1} we need to find the middle (with a left bias for discriminant of odd degree) two zeros of the discriminant of $p_{i+1}(x)$, which is a polynomial in a_{i+1} . We refer to [23, Section 3.2] for the details.

This gives us a tree with root (or first generation) $a_1 = -(d + k)$. The r th generation consisting of nodes v such that each path $a_1 = v_1, v_2, \dots, v_r = v$ from the root a_1 to v corresponds to the polynomial p_r , where $a_i = v_i$ for all $1 \leq i \leq r$. Each path of length d in the tree corresponds to a candidate for $p(x)$. This way we obtain all elements of \mathfrak{P} .

Now we remark on the applications of this algorithm.

When finding polynomials in $\mathcal{T}(d, t)$, we can speed up the algorithm by taking advantage of the fact that each polynomial in $\mathcal{T}(d, t)$ is congruent to $(x+1)^d$ modulo $2\mathbb{Z}[x]$. That is, for each $i \in \{2, \dots, d\}$, we restrict each a_i so that $a_i \equiv \binom{d}{i} \pmod{2}$.

When applying the algorithm to the proofs of Theorems 5.6, 5.7, 5.9, and Proposition 5.8, we not only fix the trace, we also fix the coefficient a_2 .

4.4. Candidates for irreducible factors of $G(x)$. Now we detail the results of the computation of all polynomials in $\mathcal{T}_{\text{irr}}(d, k)$ for $d \in \{1, \dots, 5\}$ and $k \in \{0, 2, 4\}$ and all polynomials in $\mathcal{T}_{\text{red}}(d, k)$ for $d \in \{1, \dots, 10\}$ and $k \in \{0, 2, 4, 6, 8\}$. We list all the polynomials in these sets in Tables 2 and 3.

$\mathcal{T}_{\text{irr}}(1, 2)$	$x - 3$
$\mathcal{T}_{\text{irr}}(1, 4)$	$x - 5$
$\mathcal{T}_{\text{irr}}(2, 2)$	$x^2 - 4x + 1$
$\mathcal{T}_{\text{irr}}(2, 4)$	$x^2 - 6x + 3$
	$x^2 - 6x + 7$
$\mathcal{T}_{\text{irr}}(3, 4)$	$x^3 - 7x^2 + 9x - 1$
	$x^3 - 7x^2 + 11x - 3$
	$x^3 - 7x^2 + 13x - 5$
$\mathcal{T}_{\text{irr}}(4, 4)$	$x^4 - 8x^3 + 16x^2 - 8x + 1$
	$x^4 - 8x^3 + 18x^2 - 10x + 1$
	$x^4 - 8x^3 + 20x^2 - 16x + 1$

TABLE 2. The elements of $\mathcal{T}_{\text{irr}}(d, k)$ for $d \in \{1, \dots, 5\}$ and $k \in \{0, 2, 4\}$

Note that we need only to consider even values of k . Indeed, by Lemma 4.3, each degree- d , irreducible factor $g(x)$ of $G(x)$ must be congruent to $(x+1)^d \pmod{2\mathbb{Z}[x]}$. Hence the trace of $g(x)$ is congruent to d modulo 2. Furthermore, by Theorem 4.6, for $t < \lceil 1.78839d \rceil$ we know that $\mathcal{T}(d, t)$ is empty.

To simplify our task, we take advantage of related computations that have already been performed.

Trace minus degree at most 6. Smyth [27] has enumerated the elements of the sets $T(d, t)$ for all integers $d \geq 1$ and $t \geq d$ such that $t - d \leq 6$. We can use the lists of Smyth to find the sets $\mathcal{T}_{\text{irr}}(d, k)$ and $\mathcal{T}_{\text{red}}(d, k)$ for $k \leq 6$.

Degree 10. For degree-10 candidates, using the above restrictions, we need only compute the set $\mathcal{T}_{\text{red}}(10, 8)$. The set $T(10, 18)$ was computed in [23]. It consists of three polynomials, from which it is straightforward to check that the set $\mathcal{T}(10, 18)$ is empty.

$\mathcal{T}_{\text{red}}(1, 0)$	$x - 1$
$\mathcal{T}_{\text{red}}(1, 8)$	$x - 9$
$\mathcal{T}_{\text{red}}(2, 4)$	$x^2 - 6x + 1$
$\mathcal{T}_{\text{red}}(3, 4)$	$x^3 - 7x^2 + 11x - 1$
$\mathcal{T}_{\text{red}}(3, 8)$	$x^3 - 11x^2 + 7x - 1$ $x^3 - 11x^2 + 23x - 1$ $x^3 - 11x^2 + 27x - 1$ $x^3 - 11x^2 + 31x - 25$ $x^3 - 11x^2 + 31x - 9$
$\mathcal{T}_{\text{red}}(4, 8)$	$x^4 - 12x^3 + 26x^2 - 12x + 1$ $x^4 - 12x^3 + 34x^2 - 20x + 1$ $x^4 - 12x^3 + 38x^2 - 40x + 9$ $x^4 - 12x^3 + 38x^2 - 16x + 1$ $x^4 - 12x^3 + 42x^2 - 44x + 1$ $x^4 - 12x^3 + 46x^2 - 64x + 25$ $x^4 - 12x^3 + 46x^2 - 56x + 1$
$\mathcal{T}_{\text{red}}(5, 8)$	$x^5 - 13x^4 + 42x^3 - 46x^2 + 13x - 1$ $x^5 - 13x^4 + 46x^3 - 42x^2 + 13x - 1$ $x^5 - 13x^4 + 50x^3 - 66x^2 + 17x - 1$ $x^5 - 13x^4 + 50x^3 - 62x^2 + 21x - 1$ $x^5 - 13x^4 + 54x^3 - 90x^2 + 53x - 1$ $x^5 - 13x^4 + 54x^3 - 86x^2 + 49x - 9$ $x^5 - 13x^4 + 54x^3 - 78x^2 + 33x - 1$ $x^5 - 13x^4 + 54x^3 - 74x^2 + 21x - 1$ $x^5 - 13x^4 + 58x^3 - 106x^2 + 73x - 9$ $x^5 - 13x^4 + 58x^3 - 102x^2 + 61x - 9$ $x^5 - 13x^4 + 58x^3 - 98x^2 + 41x - 1$
$\mathcal{T}_{\text{red}}(6, 8)$	$x^6 - 14x^5 + 59x^4 - 96x^3 + 59x^2 - 14x + 1$ $x^6 - 14x^5 + 63x^4 - 104x^3 + 63x^2 - 14x + 1$ $x^6 - 14x^5 + 67x^4 - 136x^3 + 111x^2 - 26x + 1$ $x^6 - 14x^5 + 67x^4 - 132x^3 + 99x^2 - 26x + 1$ $x^6 - 14x^5 + 67x^4 - 132x^3 + 103x^2 - 22x + 1$ $x^6 - 14x^5 + 71x^4 - 160x^3 + 151x^2 - 38x + 1$ $x^6 - 14x^5 + 71x^4 - 160x^3 + 155x^2 - 50x + 1$ $x^6 - 14x^5 + 71x^4 - 156x^3 + 135x^2 - 26x + 1$
$\mathcal{T}_{\text{red}}(7, 8)$	$x^7 - 15x^6 + 81x^5 - 203x^4 + 243x^3 - 125x^2 + 23x - 1$ $x^7 - 15x^6 + 81x^5 - 195x^4 + 215x^3 - 101x^2 + 19x - 1$ $x^7 - 15x^6 + 85x^5 - 227x^4 + 287x^3 - 149x^2 + 23x - 1$ $x^7 - 15x^6 + 85x^5 - 227x^4 + 291x^3 - 165x^2 + 35x - 1$

TABLE 3. The elements of $\mathcal{T}_{\text{red}}(d, k)$ for $d \in \{1, \dots, 10\}$ and $k \in \{0, 2, 4, 6, 8\}$

Degree 9. For degree-9 candidates, using the above restrictions, we need only compute the set $\mathcal{T}_{\text{red}}(9, 8)$. It was shown in [1] that the cardinality of the set $T(9, 17)$ is 686. We recomputed this list of 686 polynomials and found that just two of these polynomials $p_1(x)$ and $p_2(x)$ (say) are congruent to $(x+1)^9 \pmod{2\mathbb{Z}[x]}$. However, both $p_1(x^2)$ and $p_2(x^2)$ are irreducible. Hence, $\mathcal{T}_{\text{red}}(9, 8)$ is empty.

Trace minus degree 8. It remains for us to compute the sets $\mathcal{T}_{\text{red}}(d, 8)$ for $d \in \{1, \dots, 8\}$. We used the algorithm described in Section 4.3. The computations ran in SageMath [28] on a single core of an Intel Core i7 at 2.9 GHz. Computing the sets $\mathcal{T}_{\text{red}}(d, 8)$ for $d \in \{1, \dots, 8\}$ took 8 hours and 24 minutes in total. We have listed the elements of these sets in Table 3. We found that, although the set $\mathcal{T}(8, 16)$ has cardinality 48, the set $\mathcal{T}_{\text{red}}(8, 8)$ is empty.

4.5. From $G(x)$ to $F(x)$. Now we have computed all possible candidates (see Tables 2 and 3) for the irreducible factors of $G(x)$, we can find all candidates for the polynomial $G(x)$. By Lemma 4.4, each factor $g(x) \in \mathcal{T}_{\text{irr}}(d, k)$ of $G(x)$ for some d and k must be a factor with even multiplicity. Hence we form the set

$$\mathcal{U} := \bigcup_{d=1}^5 \bigcup_{k=1}^4 \{g(x)^2 \mid g(x) \in \mathcal{T}_{\text{irr}}(d, k)\} \cup \bigcup_{d=1}^{10} \bigcup_{k=1}^8 \mathcal{T}_{\text{red}}(d, k).$$

Then define $\overline{\mathcal{U}} := \bigcup_{i=1}^{17} \mathcal{U}^i$ to be a union of cartesian powers of \mathcal{U} . For a vector of l polynomials, $\mathbf{v} = (v_1(x), v_2(x), \dots, v_l(x))$, define $p_{\mathbf{v}}(x) := \prod_{i=1}^l v_i(x)$. Now form the set

$$\mathcal{G} = \{p_{\mathbf{v}}(x) \mid \mathbf{v} \in \overline{\mathcal{U}}, \deg p_{\mathbf{v}}(x) = 17, \text{ and } \text{tr } p_{\mathbf{v}}(x) = 25\},$$

which contains all candidates for the polynomial $G(x)$. The cardinality of \mathcal{G} is 55.

Now we can construct all possible candidates for the polynomial $F(x)$. Each polynomial $g(x) \in \mathcal{T}_{\text{irr}}(d, k)$ corresponds to the polynomial $f(x) = g((x - 10)^2)$, which is a potential factor of $F(x)$. Each polynomial $g(x) \in \mathcal{T}_{\text{red}}(d, k)$ corresponds to the polynomial $g((x - 10)^2) = f_1(x)f_2(x)$, where both $f_1(x)$ and $f_2(x)$ are potential factors of $F(x)$. Therefore, for each $g(x) \in \mathcal{T}_{\text{irr}}(d, k)$, we define the set $\mathcal{H}(g(x)) := \{g((x - 10)^2)^{1/2}\}$ and, for each $g(x) \in \mathcal{T}_{\text{red}}(d, k)$, we define the set $\mathcal{H}(g(x)) := \{f(x) \mid f(x) \text{ is a factor of } g((x - 10)^2)\}$.

For a polynomial $G(x) \in \mathcal{G}$, denote by $\text{irr}(G(x))$ the multiset of irreducible factors of $G(x)$ and define the set

$$P(G(x)) := \prod_{\nu(x) \in \text{irr}(G(x))} \mathcal{H}(\nu(x)).$$

Now define the set of polynomials

$$\mathcal{R}(G(x)) := \{p_{\mathbf{v}}(x) \mid \mathbf{v} \in P(G(x))\}.$$

The set $\mathcal{F} = \bigcup_{G(x) \in \mathcal{G}} \mathcal{R}(G(x))$ contains all candidates for the polynomial $F(x)$. For each $p(x) \in \mathcal{F}$, the polynomial $c(x) := (x + 5)^{33}p(x)$ is a candidate for the characteristic polynomial of S . We now check that the polynomial $c(x)$ satisfies the necessary conditions for $c(x)$ being the characteristic polynomial of a Seidel matrix of order 50. By Remark 4.2, the top three coefficients of $c(x)$ must be 1, 0, and $-50 \cdot 49/2 = -1225$ respectively. Hence the top three coefficients of $p(x)$ must be 1, -165 , and 12800, respectively. Define

$$\mathcal{C} := \{(x + 5)^{33}p(x) \mid p(x) \in \mathcal{F} \text{ and } p(x) = x^{17} - 165x^{16} + 12800x^{15} + \dots\}.$$

Recall that S is the Seidel matrix for \mathcal{L} , a putative set of 50 equiangular lines in \mathbb{R}^{17} . The set \mathcal{C} , which consists of 102 polynomials, contains all candidates for the characteristic polynomial $\chi_S(x)$ of S . Write $c(x - 1) = \sum_{i=0}^{50} a_i x^{50-i}$. By Corollary 3.2, for each $i \in \{0, \dots, 50\}$, the coefficient a_i must be divisible by 2^i .

This condition weeds out all but three possibilities from \mathcal{C} , and we thus find that $\chi_S(x)$ must be one of

$$\begin{aligned} &(x+5)^{33}(x-9)^{10}(x-11)^5(x^2-20x+95), \\ &(x+5)^{33}(x-9)^{12}(x-11)^4(x-13), \\ &\text{or } (x+5)^{33}(x-7)(x-9)^9(x-11)^7. \end{aligned}$$

This proves Proposition 4.1.

5. NONEXISTENCE RESULTS FOR SEIDEL MATRICES WITH CERTAIN CHARACTERISTIC POLYNOMIALS

In this section we show that there does not exist a Seidel matrix having any of the characteristic polynomials of Proposition 4.1, thereby showing that there cannot exist 50 equiangular lines in \mathbb{R}^{17} .

Let M be a real symmetric matrix of order n having m distinct eigenvalues $\lambda_1 > \lambda_2 > \dots > \lambda_m$. We write $\Lambda(M) = \{\lambda_1, \dots, \lambda_m\}$ for the set of distinct eigenvalues of M . For each $i \in \{1, \dots, m\}$, denote by $\mathcal{E}(\lambda_i)$ the eigenspace of λ_i and let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be the standard basis of \mathbb{R}^n . Denote by P_i the orthogonal projection of \mathbb{R}^n onto $\mathcal{E}(\lambda_i)$. We write $\alpha_{ij} = \|P_i \mathbf{e}_j\|$ for all $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$. As is customary, we refer to the numbers α_{ij} as the **angles** of M .

Proposition 5.1. [5, Prop. 4.2.1] *Let M be a real symmetric matrix of order n with $\Lambda(M) = \{\lambda_1, \dots, \lambda_m\}$ and angles α_{ij} . Then $\sum_{j=1}^n \alpha_{ij}^2 = \dim \mathcal{E}(\lambda_i)$ for each $i \in \{1, \dots, m\}$, and $\sum_{i=1}^m \alpha_{ij}^2 = 1$ for each $j \in \{1, \dots, n\}$.*

Now we define the **angle matrix** of M to be the n -by- m matrix with (j, i) -entry equal to α_{ij}^2 . Note that the angle matrix defined here is the transpose of the angle matrix defined in [5, Chapter 4]. Let $M[r]$ denote the principal submatrix of M obtained by deleting the r -th row and column of M . For a proof of the next result see [5, (4.2.8)] or [9].

Proposition 5.2. *Let M be a real symmetric matrix of order n with $\Lambda(M) = \{\lambda_1, \dots, \lambda_m\}$ and angles α_{ij} . Then, for each $j \in \{1, \dots, n\}$, we have*

$$\chi_{M[j]}(x) = \chi_M(x) \sum_{i=1}^m \frac{\alpha_{ij}^2}{x - \lambda_i}.$$

By Proposition 5.2, each characteristic polynomial of an $(n-1)$ -by- $(n-1)$ principal submatrix of M corresponds to a row of the angle matrix of M . We take advantage of this fact in the main results of this section.

In this section we will also make use of Cauchy's interlacing theorem [4, 8, 16].

Theorem 5.3. *Let M be a real symmetric matrix having eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ and suppose $M[j]$, for some $j \in \{1, \dots, n\}$, has eigenvalues $\mu_1 \leq \mu_2 \leq \dots \leq \mu_{n-1}$. Then*

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \dots \leq \lambda_{n-1} \leq \mu_{n-1} \leq \lambda_n.$$

Next we prove a result about the top three coefficients of the characteristic polynomial of a principal submatrix of a Seidel matrix.

Lemma 5.4. *Let S be a Seidel matrix of order n and let $T = S[j]$ for some $j \in \{1, \dots, n\}$. Suppose S has minimal polynomial $m_S(x) = \sum_{i=0}^d a_i x^{d-i}$. Then*

$$\chi_T(x) = \frac{\chi_S(x)}{m_S(x)} \sum_{i=0}^{d-1} b_i x^{d-1-i},$$

where $b_0 = 1$, $b_1 = a_1$, $b_2 = a_2 + n - 1$, and $b_i \in \mathbb{Z}$ for $i \in \{3, \dots, d-1\}$.

Proof. The proof is a straightforward consequence of Proposition 5.2, using the fact that $\text{tr}(S) = \text{tr}(T) = 0$, $\text{tr}(S^2) = n(n-1)$, and $\text{tr}(T^2) = (n-1)(n-2)$. \square

The next result will be used repeatedly in this section.

Lemma 5.5. *Let \mathcal{P} denote the set of congruence classes of characteristic polynomials of Seidel matrices of order 49 modulo $32\mathbb{Z}[x]$. Then \mathcal{P} has 16 elements.*

Proof. By Corollary 3.13, the set \mathcal{P} has cardinality at most 16. Using a computer, it is straightforward to construct the set \mathcal{P} , and confirm that it has precisely 16 elements. \square

Now we can prove the first main result of this section.

Theorem 5.6. *There does not exist a Seidel matrix S with the characteristic polynomial $\chi_S(x) = (x+5)^{33}(x-9)^{10}(x-11)^5(x^2-20x+95)$.*

Proof. Suppose for a contradiction that S is a Seidel matrix with the proposed spectrum, and let $\chi_S(x)$ be its characteristic polynomial. Delete a row and (its corresponding) column of S to form the Seidel matrix S' . By Lemma 5.4, we have

$$\chi_{S'}(x) = \frac{(x+5)^{33}(x-9)^{10}(x-11)^5(x^2-20x+95)}{(x+5)(x-9)(x-11)(x^2-20x+95)}(x^4-35x^3+443x^2-r_1x+r_0),$$

for some integers r_1 and r_0 . By Theorem 5.3, the zeros of $\chi_{S'}(x)$ must interlace those of $\chi_S(x)$. Hence the four zeros $\gamma_1 \leq \gamma_2 \leq \gamma_3 \leq \gamma_4$ of $f(x) = x^4 - 35x^3 + 443x^2 - r_1x + r_0$ must satisfy $\gamma_1 \in [-5, 10 - \sqrt{5}]$, $\gamma_2 \in [10 - \sqrt{5}, 9]$, $\gamma_3 \in [9, 11]$, and $\gamma_4 \in [11, 10 + \sqrt{5}]$. Therefore, using the ideas in Section 4.3, we find 286 possibilities for $f(x)$.

Let \mathcal{P} denote the set of congruence classes of characteristic polynomials of Seidel matrices of order 49 modulo $32\mathbb{Z}[x]$. Using Lemma 5.5, we can produce all 16 elements of \mathcal{P} . By checking against the congruence classes in \mathcal{P} , we reduce the number of possibilities for $f(x)$ down to three, that is, $f(x) = x^4 - 35x^3 + 443x^2 - 2381x + 4516$, $f(x) = x^4 - 35x^3 + 443x^2 - 2369x + 4392$, or $f(x) = x^4 - 35x^3 + 443x^2 - 2365x + 4356$. Each possibility for $f(x)$ corresponds to a putative characteristic polynomial for S' .

Using Proposition 5.2, we find the row of the angle matrix of S corresponding to each of the three putative characteristic polynomials for S' . Using the ordering $\lambda_1 < \lambda_2 < \dots < \lambda_5$, where $\Lambda(S) = \{\lambda_1, \dots, \lambda_5\}$, we write these rows below.

$$\begin{aligned} \mathbf{r}_1 &= (2031/3080 \quad 2/55 + \sqrt{5}/110 \quad 1/7 \quad 1/8 \quad 2/55 - \sqrt{5}/110) \\ \mathbf{r}_2 &= (577/880 \quad 31/220 + \sqrt{5}/44 \quad 0 \quad 1/16 \quad 31/220 - \sqrt{5}/44) \\ \mathbf{r}_3 &= (36/55 \quad 19/110 + \sqrt{5}/55 \quad 0 \quad 0 \quad 19/110 - \sqrt{5}/55) \end{aligned}$$

If S exists then, by Proposition 5.1, there must exist nonnegative integers n_1, n_2, n_3 such that $\sum_{i=1}^3 n_i = 50$ and $\sum_{i=1}^3 n_i \mathbf{r}_i = (33, 1, 10, 5, 1)$. Since \mathbf{r}_1 is the only row

to have a nonzero in the third column, we must have $n_1 = 70$, which is clearly impossible. Thus such a Seidel matrix S cannot exist. \square

The proof of the next theorem is similar to that of Theorem 5.6.

Theorem 5.7. *There does not exist a Seidel matrix S with the characteristic polynomial $\chi_S(x) = (x + 5)^{33}(x - 9)^{12}(x - 11)^4(x - 13)$.*

Proof. Suppose for a contradiction that S is a Seidel matrix with the proposed spectrum, and let $\chi_S(x)$ be its characteristic polynomial. Delete a row and (its corresponding) column of S to form the Seidel matrix S' . By Lemma 5.4, we have

$$\chi_{S'}(x) = \frac{(x + 5)^{33}(x - 9)^{12}(x - 11)^4(x - 13)}{(x + 5)(x - 9)(x - 11)(x - 13)}(x^3 - 28x^2 + 243x - r),$$

for some integer r . By Theorem 5.3, the zeros of $\chi_{S'}(x)$ must interlace those of $\chi_S(x)$. Hence the three zeros $\gamma_1 \leq \gamma_2 \leq \gamma_3$ of $x^3 - 28x^2 + 243x - r$ must satisfy $\gamma_1 \in [-5, 9]$, $\gamma_2 \in [9, 11]$, and $\gamma_3 \in [11, 13]$. Therefore, we find that we must have $r \in \{616, \dots, 624\}$ (see Section 4.3).

Let \mathcal{P} denote the set of congruence classes of characteristic polynomials of Seidel matrices of order 49 modulo $32\mathbb{Z}[x]$. Using Lemma 5.5, we can produce all 16 elements of \mathcal{P} . By checking against the congruence classes in \mathcal{P} , we find that $r = 616$. Hence we must have $\chi_{S'}(x) = (x + 5)^{32}(x - 9)^{11}(x - 11)^4(x^2 - 17x + 56)$.

By Proposition 5.2, we find that, for all $j \in \{1, \dots, 50\}$, the j th row of the angle matrix of S is $(\alpha_{1j}^2, \dots, \alpha_{4j}^2) = (83/126, 2/7, 0, 1/18)$. However,

$$\sum_{j=1}^{50} \alpha_{1j}^2 = 50 \cdot 83/126 \neq \dim \mathcal{E}(\lambda_1) = 33.$$

This contradicts Proposition 5.1. Thus such a Seidel matrix S cannot exist. \square

It remains to deal with the third possibility for the characteristic polynomial in Proposition 4.1. We first establish a preparatory nonexistence result whose proof uses the same ideas as the previous two proofs.

Proposition 5.8. *There does not exist a Seidel matrix S with characteristic polynomial $\chi_S(x) = (x + 5)^{32}(x - 7)(x - 9)^8(x - 11)^6(x^2 - 15x + 48)$.*

Proof. Assume that S is a Seidel matrix with the proposed spectrum, and let $\chi_S(x)$ be its characteristic polynomial. Delete a row and (its corresponding) column of S to form the Seidel matrix S' . By Lemma 5.4, we have

$$\chi_{S'}(x) = (x + 5)^{31}(x - 9)^7(x - 11)^5 f(x),$$

where $f(x) = x^5 - 37x^4 + 530x^3 + r_2x^2 + r_1x + r_0$ for some integers r_2 , r_1 , and r_0 . Let $\alpha_1 \leq \alpha_2$ be the two zeros of $x^2 - 15x + 48$. Since the zeros of $\chi_{S'}(x)$ must interlace those of $\chi_S(x)$, the five zeros $\gamma_1 \leq \dots \leq \gamma_5$ of $f(x)$ must satisfy $\gamma_1 \in [-5, \alpha_1]$, $\gamma_2 \in [\alpha_1, 7]$, $\gamma_3 \in [7, 9]$, $\gamma_4 \in [9, \alpha_2]$, and $\gamma_5 \in [\alpha_2, 11]$.

To find the possible values of r_2 , r_1 , and r_0 , we apply the polynomial enumeration algorithm described in Section 4.3. We find 22023 possibilities for the vector (r_2, r_1, r_0) and therefore there are 22023 possibilities for $f(x)$. This computation ran in SageMath [28] on a single core of an Intel Core i7 at 2.9 GHz and took 134 seconds.

From this list of 22023 possibilities for $f(x)$, we produce the polynomial $g(x) = (x+5)^{31}(x-9)^7(x-11)^5f(x)$ and we sieve out those polynomials for which $g(x-1)$ does not satisfy Corollary 3.2. This leaves us with 12 possible polynomials for $f(x)$:

$$\begin{aligned} &x^5 - 37x^4 + 530x^3 - 3650x^2 + 11997x - 14985; & x^5 - 37x^4 + 530x^3 - 3650x^2 + 11949x - 14553; \\ &x^5 - 37x^4 + 530x^3 - 3666x^2 + 12237x - 15785; & x^5 - 37x^4 + 530x^3 - 3658x^2 + 12109x - 15345; \\ &x^5 - 37x^4 + 530x^3 - 3658x^2 + 12109x - 15313; & x^5 - 37x^4 + 530x^3 - 3658x^2 + 12109x - 15281; \\ &x^5 - 37x^4 + 530x^3 - 3658x^2 + 12093x - 15169; & x^5 - 37x^4 + 530x^3 - 3650x^2 + 11965x - 14665; \\ &x^5 - 37x^4 + 530x^3 - 3650x^2 + 11981x - 14841; & x^5 - 37x^4 + 530x^3 - 3650x^2 + 11965x - 14697; \\ &x^5 - 37x^4 + 530x^3 - 3642x^2 + 11837x - 14193; & x^5 - 37x^4 + 530x^3 - 3642x^2 + 11821x - 14049. \end{aligned}$$

Using Proposition 5.2, we find the row of the angle matrix of S corresponding to each of the 12 putative characteristic polynomials for S' . Using the ordering $\lambda_1 < \lambda_2 < \dots < \lambda_6$, where $\Lambda(S) = \{\lambda_1, \dots, \lambda_6\}$, we write these rows below.

$$\begin{array}{ccccccc} \left(\frac{385}{592} & -\frac{7\sqrt{33}}{6512} + \frac{11}{592} & \frac{1}{8} & 0 & \frac{7\sqrt{33}}{6512} + \frac{11}{592} & \frac{3}{16}\right) & \left(\frac{24}{37} & -\frac{15\sqrt{33}}{814} + \frac{13}{74} & 0 & 0 & \frac{15\sqrt{33}}{814} + \frac{13}{74} & 0\right) \\ \left(\frac{170}{259} & -\frac{31\sqrt{33}}{2442} + \frac{17}{222} & 0 & \frac{4}{21} & \frac{31\sqrt{33}}{2442} + \frac{17}{222} & 0\right) & \left(\frac{145}{222} & -\frac{19\sqrt{33}}{888} + \frac{39}{296} & \frac{1}{12} & 0 & \frac{19\sqrt{33}}{888} + \frac{39}{296} & 0\right) \\ \left(\frac{8119}{12432} & -\frac{191\sqrt{33}}{19536} + \frac{131}{1776} & \frac{1}{24} & \frac{2}{21} & \frac{191\sqrt{33}}{19536} + \frac{131}{1776} & \frac{1}{16}\right) & \left(\frac{1353}{2072} & \frac{3\sqrt{33}}{1628} + \frac{7}{444} & 0 & \frac{4}{21} & -\frac{3\sqrt{33}}{1628} + \frac{7}{444} & \frac{1}{8}\right) \\ \left(\frac{169}{259} & -\frac{19\sqrt{33}}{1221} + \frac{14}{111} & 0 & \frac{2}{21} & \frac{19\sqrt{33}}{1221} + \frac{14}{111} & 0\right) & \left(\frac{1345}{2072} & -\frac{5\sqrt{33}}{4884} + \frac{29}{444} & 0 & \frac{2}{21} & \frac{5\sqrt{33}}{4884} + \frac{29}{444} & \frac{1}{8}\right) \\ \left(\frac{577}{888} & -\frac{67\sqrt{33}}{9768} + \frac{21}{296} & \frac{1}{12} & 0 & \frac{67\sqrt{33}}{9768} + \frac{21}{296} & \frac{1}{8}\right) & \left(\frac{1153}{1776} & -\frac{247\sqrt{33}}{19536} + \frac{73}{592} & \frac{1}{24} & 0 & \frac{247\sqrt{33}}{19536} + \frac{73}{592} & \frac{1}{16}\right) \\ \left(\frac{31}{48} & \frac{\sqrt{33}}{528} + \frac{1}{16} & \frac{1}{24} & 0 & -\frac{\sqrt{33}}{528} + \frac{1}{16} & \frac{3}{16}\right) & \left(\frac{191}{296} & -\frac{19\sqrt{33}}{4884} + \frac{17}{148} & 0 & 0 & \frac{19\sqrt{33}}{4884} + \frac{17}{148} & \frac{1}{8}\right) \end{array}$$

Denote by $\mathbf{r}_1, \dots, \mathbf{r}_{12}$ the above twelve putative rows of the angle matrix of S . If S exists then, by Proposition 5.1, there must exist nonnegative integers n_1, \dots, n_{12} such that $\sum_{i=1}^{12} n_i = 49$ and $\sum_{i=1}^{12} n_i \mathbf{r}_i = (32, 1, 1, 8, 1, 6)$. However, it is straightforward to verify (via linear programming) that there does not exist such n_1, \dots, n_{12} . \square

Finally, using Proposition 5.8, we establish the nonexistence of a Seidel matrix having the remaining spectrum from Proposition 4.1.

Theorem 5.9. *There does not exist a Seidel matrix with the characteristic polynomial $\chi_S(x) = (x+5)^{33}(x-7)(x-9)^9(x-11)^7$.*

Proof. Assume that S is a Seidel matrix with the proposed spectrum, and let $\chi_S(x)$ be its characteristic polynomial. Delete a row and (its corresponding) column of S to form the Seidel matrix S' . By Lemma 5.4, we have

$$\chi_{S'}(x) = \frac{(x+5)^{33}(x-7)(x-9)^9(x-11)^7}{(x+5)(x-7)(x-9)(x-11)}(x^3 - 22x^2 + 153x - r),$$

for some integer r . By Theorem 5.3, the zeros of $\chi_{S'}(x)$ must interlace those of $\chi_S(x)$. Hence the three zeros $\gamma_1 \leq \gamma_2 \leq \gamma_3$ of $x^3 - 22x^2 + 153x - r$ must satisfy $\gamma_1 \in [-5, 7]$, $\gamma_2 \in [7, 9]$, and $\gamma_3 \in [9, 11]$. Therefore, we find that we must have $r \in \{324, \dots, 336\}$ (see Section 4.3).

Let \mathcal{P} denote the set of congruence classes of characteristic polynomials of Seidel matrices of order 49 modulo $32\mathbb{Z}[x]$. Using Lemma 5.5, we can produce all 16 elements of \mathcal{P} . By checking against the congruence classes in \mathcal{P} , we find that either $r = 324$ or $r = 336$. Hence we must have either $\chi_{S'}(x) = (x+5)^{32}(x-4)(x-9)^{10}(x-11)^6$ or $\chi_{S'}(x) = (x+5)^{32}(x-7)(x-9)^8(x-11)^6(x^2 - 15x + 48)$.

Using Proposition 5.2, we find that the angle matrix of S has 42 rows, each equal to $(37/56, 0, 3/14, 1/8)$ and 8 rows, each equal to $(21/32, 1/8, 0, 7/32)$. Therefore, to show the nonexistence of S , it suffices to show the nonexistence of Seidel matrices having characteristic polynomial equal to either $(x+5)^{32}(x-7)(x-9)^8(x-11)^6(x^2 -$

$15x + 48$) or $(x + 5)^{32}(x - 4)(x - 9)^{10}(x - 11)^6$. By Proposition 5.8, we have nonexistence of the former. \square

Remark 5.10. *Proposition 4.1, together with Theorem 5.6, Theorem 5.7, and Theorem 5.9 shows that there cannot exist a Seidel matrix with smallest eigenvalue -5 having multiplicity 33. Furthermore, $N(17) \leq 49$.*

6. CONCLUDING REMARKS

The main ingredients for the improvement on the upper bound for $N(17)$ were obtained using the new restrictions on the characteristic polynomial of a Seidel matrix from Section 3 and the computation of totally positive monic integer polynomials with trace minus degree at most 8 in Section 4. The same approach can theoretically be applied to improve upper bounds for $N(d)$ for other d . However, the computation of the associated totally positive monic integer polynomials may become impractical.

In our case, we found that the existence of 50 equiangular lines in \mathbb{R}^{17} was related to the existence of the polynomial $G(x)$, which had degree 17 and trace 25. Here it turned out that we could compute all the relevant totally positive monic integer polynomials with trace minus degree at most 8 in reasonable time. In general, for n equiangular lines in \mathbb{R}^d , the analogous polynomial $G(x)$ will have degree d , but the trace may be much larger than d . When the trace of $G(x)$ is much larger than d , the computation of all possibilities for $G(x)$ will become much more computationally expensive. Hence, to successfully apply this method to improve the bounds for other d , we may require a more efficient method for computing totally positive monic integer polynomials.

However, there are other small dimensions d for which the analogous $G(x)$ has small trace relative to its degree. As an example, we can consider the problem of finding the value of $N(19)$. The current upper bound for $N(19)$ is 75 (see Table 1). Following Section 4.1, a system of 75 equiangular lines in \mathbb{R}^{19} would correspond to a Seidel matrix S of order 75 having smallest eigenvalue -5 with multiplicity 56. Let $-5 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{19}$ be the other eigenvalues of S . Then the analogues of (7) and (8) are $\sum_{i=1}^{19} \lambda_i = 56 \cdot 5 = 280$ and $\sum_{i=1}^{19} \lambda_i^2 = 75 \cdot 74 - 56 \cdot 5^2 = 4150$. It follows that

$$\sum_{i=1}^{19} (\lambda_i - 15)^2 = 25.$$

In this case, we cannot rule out the possibility that 15 is an eigenvalue of S . This adds an extra complication to the problem that we will consider in a future paper.

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