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# On equidistant sets in normed linear spaces

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In this note some results concerning the equidistant set $E(-x, x)$ and the kernel $M^{\theta}$ of the metric projection $P_M$, where $M$ is a Chebyshev subspace of a normed linear space $X$, have been obtained. In particular, when $X = l^p$ $(1 < p < \infty)$, it has been proved that every equidistant set is closed in the $bw$-topology of the space. In $c_0$ no equidistant set has this property.

## 0. Introduction

Let $X$ be a real normed linear space. For any two distinct points $x$ and $y$ of $X$, let $E(x, y)$ denote the equidistant set from $x$ and $y$; that is, the set of points $p$ in $X$ for which $\|p-x\| = \|p-y\|$. Such sets were introduced by Kalisch and Straus in [6] in connection with their study of "determining" sets in Banach spaces. In an inner-product space every set $E(x, y)$ is a closed hyperplane, but in general it may not be even weakly closed. Not much is known about spaces other than inner-product and finite dimensional spaces in which sets $E(x, y)$ are weakly or weakly sequentially closed. The purpose of this paper is to make an attempt in that direction.

In the first section we shall study a few geometrical and topological properties of the set $E(x, y)$. For example, in Theorem 1.2 we prove that, if $E(x, y)$ is convex, then it is a hyperplane and as a consequence,

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the convexity of all sets $E(x, y)$ implies that the space $X$ is an inner-product space. The connection between the structural properties of the set $E(-x, x)$ and those of the kernel $M^{\theta}$ of the metric projection $P_M$ where $M$ is the linear span of the point $x$, is then exhibited in Theorem 1.4 and Lemma 1.5. These results are closely related to the recent works of Holmes and Kripke [4], Kottman and Lin [8], and Holmes [3].

In the second section of this paper we show that $l^p$ spaces $(1 < p < \infty)$ have the property that all sets $E(x, y)$ are closed in the bounded weak topology. Thus these spaces satisfy the $P_2$-property (see Klee [7], p. 298). In contrast to $l^p$-spaces, we find that in $c_0$, $E(x, y)$ is not even weakly sequentially closed for any $x$ and $y \in c_0$.

## 1. Some properties of the equidistant set $E(-x, x)$

We begin by recalling some notations and definitions. Let $X$ be a normed linear space over the real numbers $R$, with $\theta$ as its zero element. Let $x \in X$ and $K \subset X$. A point $y \in K$ is called a *nearest point* of $x$ in $K$, if $\|x-y\| \le \|x-z\|$ for every $z \in K$. A set $K \subset X$ is said to be *proximal* (respectively, *Chebyshev*) if for each point $x \in X$, there exists a (respectively, *a unique*) nearest point of $x$ in $K$. Let $M$ be a Chebyshev linear subspace of $X$. The metric projection supported by $M$ will be denoted by $P_M$. It is known [3, p. 160] that $P_M$ induces a direct sum decomposition of $X$. Namely, every $x \in X$ can be written uniquely as $x = m + y$ where $m \in M$ and $y \in M^{\theta}$, where $M^{\theta} = \{x \in X : P_M(x) = \theta\}$. $M^{\theta}$ is called the kernel of $P_M$.

For $x \ne \theta$ in $X$, let $E(-x, x)$ denote the *equidistant set* from $x$ and $-x$; that is, the set of points $y \in X$ such that $\|y-x\| = \|y+x\|$. Observe that each equidistant set is closed. If $x$ and $y \in X$ and $\|x-y\| = \|x+y\|$ we say that $x$ is *orthogonal* to $y$ and write $x \perp y$. Thus $E(-x, x)$ is then the set of all vectors in $X$ which are orthogonal to $x$. This concept of orthogonality is named the isosceles orthogonality and has been studied by James in [5]. We shall need the following result from [5]. For each pair of linearly independent vectors $x$ and $y$ in

$X$, there exists a number $t \in R$ such that $tx + y \perp x$. By a *cone* in $X$, we shall mean a set $K$ such that $x \in K \Rightarrow tx \in K$, for every non-negative number $t$. With these preliminaries we pass on to the study of some geometrical and topological properties of the set $E(-x, x)$.

LEMMA 1.1. *Let $x$ be any point of a two dimensional normed linear space $X$. If $E(-x, x)$ is convex then it must be a line through the origin.*

Proof. Let $E(-x, x)$ be convex and $z \neq \theta \in E(-x, x)$. By the result of James for isosceles orthogonality such a point $z$ exists. We shall show that $E(-x, x) = [z]$, the one-dimensional subspace spanned by $z$. First, since $E(-x, x)$ is symmetric about the origin, the convexity implies that $\{tz : |t| \leq 1\} \subset E(-x, x)$. If $y \in E(-x, x)$ is linearly independent from $z$, then either $y$ and $z$ or $-y$ and $z$ are separated by the line $[x]$. Since $y \in E(-x, x)$ implies $-y \in E(-x, x)$, we assume that the former holds. Then the line segment joining $y$ and $z$ is contained in $E(-x, x)$; but since this line intersects $[x]$ at a point other than the origin it cannot be a point of $E(-x, x)$. Hence there is a contradiction.

Now we show that $E(-x, x)$ is unbounded. Let $z \in E(-x, x)$ and $\lambda > 1$ be arbitrary. Then again by James' result there exists a $t$ in $R$ such that $\lambda z + tx \in E(-x, x)$. This implies that $z$ and $\lambda z + tx$ must be linearly dependent. This is possible only if $t = 0$. Hence the result is proved.

THEOREM 1.2. *Let $x \neq \theta$ be any point of a normed linear space $X$. If $E(-x, x)$ is convex then it must be a proximal subspace of codimension one.*

Proof. Let $E(-x, x)$ be convex and $z$ be any point of $X$ outside $[x]$. Then $E(-x, x) \cap [x, z]$ is convex and by Lemma 1.1, it must be a line. Thus if $z \in E(-x, x)$, then $[z] \subset E(-x, x)$. Consequently, $E(-x, x)$ is a convex cone symmetric about the origin. Hence it is a subspace. Now let $u \in X$. Then either $u = \lambda x$ or, by James' result, $u + \lambda x = z \in E(-x, x)$ for some $\lambda$. Thus $E(-x, x)$ and $x$ together span $X$. Therefore $E(-x, x)$ is of codimension $1$. Since every equidistant set is closed it follows that $E(-x, x)$ is a closed subspace.

Now let $h \in E(-x, x)$. Then $\|x-h\| = \|x+h\|$ and hence $\theta$ is a nearest point of $x$ in $E(-x, x)$ . If $\alpha \in R$ , then

$$\|\alpha x-h\| = |\alpha|\|x-\alpha^{-1}h\| = |\alpha|\|x+\alpha^{-1}h\| = \|\alpha x+h\|$$

for all $h$ in $E(-x, x)$ and hence $\theta$ is also a nearest point in $E(-x, x)$ to $\alpha x$ . As any $w \in X$ has a representation $w = \alpha x + h$ , where $h \in E(-x, x)$ , we have

$$\|w-h\| = \|\alpha x\| \leq \|\alpha x-z\| , \quad z \in E(-x, x) ,$$

which implies

$$\|w-h\| \leq \|w-z-h\| , \quad z \in E(-x, x) .$$

But $E(-x, x)$ being a subspace, $z + h \in E(-x, x)$ and hence $\|w-h\| \leq \|w-v\|$ for every $v$ in $E(-x, x)$ . Thus every $w$ in $X$ has a nearest point in $E(-x, x)$ .

As a consequence of the above theorem we have, under weaker assumptions, the following characterization of inner-product spaces [1, Theorem 5.4].

COROLLARY 1.3. *Let $X$ be a normed linear space. If $E(-x, x)$ is convex for each $x \in X$ , then $X$ must be an inner-product space.*

Proof. Immediate from the above theorem and Day's result.

In the sequel, $M^\theta$ denotes the kernel of the metric projection $P_M$ , where $M$ is a Chebyshev subspace. We then have the following theorem.

THEOREM 1.4. *Let $M$ be the one dimensional span $[x]$ of $x$ in a normed linear space $X$ . Let $M$ be Chebyshev. Then the following hold:*

(1°) $$M^\theta \subset E(-x, x) \Rightarrow M^\theta = E(-x, x) ;$$

(2°) $$E(-x, x) \text{ is a cone} \Rightarrow M^\theta = E(-x, x) .$$

We need the following result in its proof.

LEMMA 1.5 [8, Lemma 1]. *If $x \in X$ and $M = [x]$ is Chebyshev, then*

$$P_M\big(E(-x, x)\big) \subset \{tx : -1 \leq t \leq 1\} .$$

Proof of Theorem 1.4 (1°). If $u \in E(-x, x)$ then $P_M(u) = \alpha x$ with $|\alpha| \leq 1$ . Since $\|u-x\| = \|u+x\|$ and $u$ has a unique nearest point in

$[x]$ , $|\alpha| \neq 1$ . We can write $u = u_\theta + \alpha x$ where $u_\theta \in M^\theta$ and, since $\lambda u_\theta \in M^\theta \subset E(-x, x)$ for all $\lambda \in R$ , we have $u_\theta \perp \mu x$ for all $\mu \in R$ ; that is,

$$\|u_\theta - \mu x\| = \|u_\theta + \mu x\| , \quad \mu \in R .$$

In particular, with $\mu = 1 - \alpha$ , we have

$$\|u+(1-2\alpha)x\| = \|u_\theta + \alpha x + (\mu - \alpha)x\| = \|u_\theta + \mu x\|$$
$$= \|u_\theta - \mu x\| = \|u - x\| = \|u + x\| = r , \text{ (say)}.$$

So the sphere centred at $u$ and radius $r$ meets $M$ in at least three points: $-x$, $+x$ , and $(-1+2\alpha)x$ , which is impossible unless $1 - 2\alpha = \pm 1$ . Thus $\alpha = 0$ or $\alpha = +1$ and the latter, we saw above, is also impossible. Therefore $\alpha = 0$ and $u \in M^\theta$ .

(2°). Let $u \in M^\theta$ . Then there exists a number $t$ such that $u - tx \in E(-x, x)$ . Since $P_M\big(E(-x, x)\big) \subset \{ax : |a| \leq 1\}$ , we have $|t| \leq 1$ . Because $E(-x, x)$ is a cone, $u - tx \in E(-x, x)$ implies $\lambda u - \lambda tx \in E(-x, x)$ for arbitrary $\lambda$ in $R$ . But $\lambda u \in M^\theta$ , $\forall \lambda \in R$ , and hence we must have $|\lambda t| \leq 1$ . This is possible only when $t = 0$ . Hence $M^\theta \subset E(-x, x)$ and, by (1°) above, $M^\theta = E(-x, x)$ .

However, $M^\theta$ is a subspace does not imply that $E(-x, x)$ is also a subspace. In fact Kottman and Lin [8] have given an example where $M^\theta$ is a closed hyperplane, but $E(-x, x)$ is not even weakly sequentially closed.

In the following we see the relation between $M^\theta$ and $E(-x, x)$ as regards weak topology, where $M = [x]$ is given to be Chebyshev. We give a simple proof of a result in [8].

**THEOREM 1.6.** *Let $M = [x]$ be a Chebyshev subspace of a normed linear space $X$ . Then $M^\theta$ is weakly (bounded weakly, or weakly sequentially) closed if $E(-x, x)$ is weakly (bounded weakly, or weakly sequentially) closed.*

Proof. We consider the case when $E(-x, x)$ is weakly closed, the

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other cases being similar. Let $\{u_\alpha\} \subset M^\theta$ be a net which converges weakly to $u \in X$. Suppose that $u \notin M^\theta$; then taking $2z = P_M(u)$ we can find a net $\{t_\alpha\}$ of real numbers such that $u_\alpha - t_\alpha z \in E(-z, z)$ and $|t_\alpha| \le 1$. If $t_0$ is a cluster point of the net $\{t_\alpha\}$, then $|t_0| \le 1$, and since $E(-z, z)$ is weakly closed, being a scalar multiple of $E(-x, x)$, $u - t_0 z \in E(-z, z)$. Therefore, $P_M(u-t_0 z) = 2z - t_0 z \in \{tz : |t| \le 1\}$. This means $1 \le t_0 \le 3$ and hence $t_0 = 1$. It follows then that $u - z \in E(-z, z)$; that is, $\|u-\theta\| = \|u-2z\| = \|u-P_M(u)\|$, and this contradicts the Chebyshev property of $M$. This proves the result.

In the following we consider a structural property of the set $E(-x, x)$.

THEOREM 1.7. *Let $E(-x, x)$ be a convex subset of a normed linear space $X$ with $\|x\| = 1$. Then $E(-x, x)$ is Chebyshev if and only if $x$ is an extreme point of the unit ball of $X$.*

Proof. Let $E(-x, x)$ be a Chebyshev set. It will be actually a subspace because of Theorem 1.2. If $x$ is not an extreme point of the unit ball of $X$, then there exists a pair of points $x_1$ and $x_2$ in the unit sphere $S = \{z \in X : \|z\| = 1\}$ such that $x = \frac{1}{2}(x_1+x_2)$ and $I = \{tx_1+(1-t)x_2 : 0 \le t \le 1\}$ is contained in $S$. Now

$$\|x_1-x-x\| = \|x_2\| = 1 = \|x_1\| = \|x_1-x+x\|$$

and hence $x_1 - x \in E(-x, x)$. Similarly $x_2 - x \in E(-x, x)$. Thus $x_1, x_2 \in E(\theta, 2x)$ and since $E(-x, x)$ is a subspace, $I \subset E(\theta, 2x)$. As $E(-x, x)$ is Chebyshev and $h \in E(-x, x)$ implies that $\|x-h\| = \|x+h\|$, we must have

$$1 = \|x\| = \inf\{\|x-h\| : h \in E(-x, x)\} .$$

Hence the origin is the nearest point of $x$ in $E(-x, x)$. This in turn implies that the origin has the nearest point $x$ in $E(\theta, 2x)$. But $x \in I$ and every point of $I$ has norm $1$. This contradicts the Chebyshev property of $E(\theta, 2x)$.

Conversely, it is easy to see that if $x$ is an extreme point of the unit ball, then $\theta$ is the unique nearest point in $E(-x, x)$ to $\lambda x$, $\lambda \in R$. Hence if $u = z + \lambda x$, and $z \in E(-x, x)$, then $z$ is the unique nearest point to $u$. Therefore $E(-x, x)$ is Chebyshev.

In the following we illustrate Theorem 1.7 by two examples.

EXAMPLE 1.8. Take $X = R^2$ with the sup norm, $x = (1, 1)$ and $z = (-1, 1)$. It is easy to see that $E(-x, x) = [z]$ and $E(-z, z) = [x]$ are Chebyshev subspaces, and $x$ and $z$ are extreme points of the unit ball of $X$.

EXAMPLE 1.9. Let $X = l^1$ and let $e_i$ be the vector with 1 at the $i$th place and zero otherwise. Then $E(-e_i, e_i) = \{z \in l^1 : z(i) = 0\}$ is a closed hyperplane. If $u \in l^1$, then the unique nearest point to $u$ in $E(-e_i, e_i)$ is $z$, where $z(j) = (1-\delta_{ij})u(j)$, $\delta_{ij}$ being the Kronecker delta. Thus the set $E(-e_i, e_i)$ is Chebyshev. Clearly $e_i$ is an extreme point of the unit ball of $l^1$. Also, if we write $M_i = |e_i|$, then $M_i^\theta = E(-e_i, e_i)$.

## 2. Nature of equidistant sets in $l^p$ spaces

Let $X$ be a normed linear space and let $E(x, y)$ be the equidistant set from $x$ and $y \in X$. The space $X$ is said to have

(1) property $P_1$ if for all $x, y \in X$, $E(x, y)$ is weakly closed,

(2) property $P_2$ if for each $x \in X$ with $\|x\| = 1$, there exists $\varepsilon_x > 0$ such that whenever $y$ and $z$ are distinct points of the set $x + \varepsilon_x U$, then the intersection $E(y, x) \cap (\varepsilon_x U)$ is weakly closed, $U$ denoting the unit cell of $X$.

That there is a connection between properties $P_1$ and $P_2$ and the continuity behaviour of metric projections onto Chebyshev sets is indicated by a result of Klee [7, Proposition 2.5]. Not much is known about spaces having the property $P_1$. Apart from the finite dimensional and inner-

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product spaces, no other example of spaces possessing the property $P_1$ has appeared in the literature. In the following we shall show that each equidistant set in an $l^p$ space $(1 < p < \infty)$ is closed in the bounded weak topology. It is easy to see that we need only consider equidistant sets of the form $E(-x, x)$. We start by proving a simple inequality.

LEMMA 2.1. *Let* $p \geq 1$ *and* $y$ *and* $z$ *be any two complex numbers. Then the following inequality holds:*

$$\left| |y+z|^p - |y-z|^p \right| \leq 2^p p\left( |y^{p-1} z| + |z|^p \right) . \tag{2.1}$$

Proof. Using the triangle inequality we see that we need only prove

$$(|y|+|z|)^p - \left| |y|-|z| \right|^p \leq 2^p p\left( |y^{p-1} z| + |z|^p \right) . \tag{2.2}$$

The result then follows from the following simple inequality, which can be proved by using elementary methods of differential calculus:

$$(1+x)^p - (1-x)^p \leq 2^p p\left(x+x^p\right) , \quad 0 \leq x \leq 1 . \tag{2.3}$$

We next prove a variant of Lebesgue's Dominated Convergence Theorem for $l^1$. This will be used to prove the main result of this section.

THEOREM 2.2. *Let* $\{\phi_\alpha, D\}$ *be a net in* $l^1$ *converging pointwise to* $\phi$. *If there exists a net* $\{f_\alpha, D\}$ *in* $l^1$ *which converges in norm to an element* $f$ *and if* $|\phi_\alpha| \leq f_\alpha$ *for every* $\alpha \in D$, *then* $\phi \in l^1$ *and*

$$\sum_{i=1}^{\infty} \phi_\alpha(i) \to \sum_{i=1}^{\infty} \phi(i) .$$

Proof. Clearly $\phi \in l^1$. The rest then follows from the following inequality:

$$\left| \sum_{i=1}^{\infty} \phi_\alpha(i) - \sum_{i=1}^{\infty} \phi(i) \right| \leq \left| \sum_{i=1}^{i_0} \phi_\alpha(i) - \sum_{i=1}^{i_0} \phi(i) \right| + \|f_\alpha - f\| + 2 \sum_{i_0+1}^{\infty} f(i) .$$

REMARK 2.3. Taking

$$\phi_n = f_n = e_n/n \quad \text{where} \quad e_i(j) = \delta_{ij} ,$$

and observing that $\{\phi_n\}$ is not dominated by a single $f \in l^1$, we see that Theorem 2.2.could be applied in situations in which Lebesgue's Dominated Convergence Theorem does not help.

THEOREM 2.4. *Let* $x$ *be any point of* $l^p$ $(1 < p < \infty)$. *Then* $E(-x, x)$ *is closed in the bounded weak topology of the space.*

Proof. Let $\{u_\alpha, D\}$ be a bounded net in $E(-x, x)$ converging weakly to $u$. Then

$$\|u_\alpha - x\| = \|u_\alpha + x\| \quad \text{for all} \quad \alpha \in D ;$$

that is,

$$\sum_{i=1}^{\infty} \left| |u_\alpha(i) - x(i)|^p - |u_\alpha(i) + x(i)|^p \right| = 0 . \tag{2.4}$$

Let

$$z_\alpha(i) = |u_\alpha(i) - x(i)|^p - |u_\alpha(i) + x(i)|^p ,$$

$$z(i) = |u(i) - x(i)|^p - |u(i) + x(i)|^p ,$$

$$w_\alpha(i) = 2^p p \left[ \left| u_\alpha^{p-1}(i) x(i) \right| + |x(i)|^p \right] ,$$

$$w(i) = 2^p p \left[ |u^{p-1}(i) x(i)| + |x(i)|^p \right] ,$$

$$g_\alpha(i) = \left| \left| u_\alpha^{p-1}(i) \right| - |u^{p-1}(i)| \right| ,$$

$$y(i) = |x(i)| .$$

Clearly, $z_\alpha, w_\alpha, z, w \in l^1$ and $z_\alpha \to z$ pointwise. By Lemma 2.1, we have

$$|z_\alpha(i)| \le w_\alpha(i) \quad \text{for all} \quad \alpha \in D . \tag{2.5}$$

Also $\{g_\alpha\}$ is a bounded net in $l^q$ converging pointwise to $\theta$, where $\frac{1}{p} + \frac{1}{q} = 1$. As $p > 1$, this implies that $g_\alpha \xrightarrow{\omega} \theta$. Moreover,

$$\|w_\alpha - w\| = 2^p p \sum_{i=1}^{\infty} g_\alpha(i) y(i) = 2^p p \langle g_\alpha, y \rangle , \tag{2.6}$$

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where $\langle g_\alpha, y\rangle$ represents the value of the bounded linear functional $y \in l^p$ at $g_\alpha \in l^q$. An easy application of Theorem 2.2 to (2.5) and (2.6) then gives the required result.

REMARK 2.5. Let $x$ be an element of $l^p$ $(1 \le p < \infty)$ with finitely many nonzero coordinates. That $E(-x, x)$ is weakly closed can be verified easily. We do not know whether in Theorem 2.4 the bounded weak topology can be replaced by the weak topology or not.

COROLLARY 2.6. *Let $M$ be a closed linear subspace of $l^p$ $(1 < p < \infty)$, $P_M$ the metric projection onto $M$. Then $P_M$ is continuous both from the strong to strong topology, and from the bounded weak to bounded weak topology on $l^p$.*

Proof. The uniform convexity of $l^p$ $(1 < p < \infty)$ implies that $M$ is Chebyshev and $P_M$ is continuous from the strong to strong topology of $l^p$. To show that $P_M$ is continuous in the bounded weak topology of $l^p$, we first observe that for each $x \in X$, and for $M_x = [x]$, $M_x^\theta$ is bounded weakly closed on account of Theorems 1.6 and 2.4. By the kernel intersection theorem of [4] we have $M^\theta = \bigcap_{x\in M} M_x^\theta$. Thus $M^\theta$ is bounded weakly closed. The result then follows from the following result of Holmes [3, p. 170]. If $M$ is reflexive, then $P_M$ is $bw$-continuous if and only if $M^\theta$ is $bw$-closed.

REMARK 2.7. The above has been essentially observed by Holmes [2] by using the fact that $l^p$ spaces $(1 < p < \infty)$ have a weakly continuous duality mapping.

In the case of $l^1$, since strong and weak sequential convergence coincide, $E(-x, x)$ is weakly sequentially closed for each $x$. However, this property of $l^p$ spaces is not present in $L^p(\mu)$ spaces $(1 < p < \infty$, $p \ne 2)$ where $\mu$ is a separable nonatomic measure. Lambert [9] has shown that $M^\theta$ is weakly sequentially dense for any finite dimensional Chebyshev subspace $M$ and consequently $E(-x, x)$ cannot be weakly sequentially closed for any $x$ in such spaces. In the following we show that $c_0$ also does not have this property.

THEOREM 2.8. *Let $x$ be any point of $c_0$. Then $E(-x, x)$ is not weakly sequentially closed.*

Proof. Let $x = (x_1, x_2, x_3, \ldots) \in c_0$. Take

$$z_n(i) = \begin{cases} 0 & , \text{ if } i \neq n , \\ 2\|x\|\operatorname{sgn}x_n + x_n & , \text{ if } i = n \text{ and } x_n \neq 0 , \\ 2\|x\| & , \text{ if } i = n \text{ and } x_n = 0 . \end{cases}$$

Then $\|z_n-x\| = \|z_n-2x\| = 2\|x\|$ for sufficiently large $n$. Hence $z_n \in E(x, 2x)$ eventually. But $z_n$ converges weakly to $\theta$ and $\theta \notin E(x, 2x)$. Therefore $E(x, 2x)$ and consequently $E(-x, x)$ is not weakly sequentially closed.

COROLLARY 2.9. *No one-dimensional Chebyshev subspace of $c_0$ can have a weakly sequentially continuous metric projection.*

Proof. Let $M = [x]$ be Chebyshev and $z_n$ be the sequence described in Theorem 2.8. Then $P_M(z_n) \in \{tx : 1 \leq t \leq 2\}$ for sufficiently large $n$, and $P_M(\theta) = \theta$. So $P_M(z_n) \not\to \theta$. Hence $P_M$ is not weakly sequentially continuous.

## References

[1] Mahlon M. Day, "Some characterizations of inner-product spaces", *Trans. Amer. Math. Soc.* 62 (1947), 320-337.

[2] R.B. Holmes, "On the continuity of best approximation operators", *Symposium on infinite dimensional theory*, 137-157 (Annals of Mathematics Studies, 69. Princeton University Press and University of Tokyo Press, Princeton, New Jersey, 1972).

[3] Richard B. Holmes, *A course on optimization and best approximation* (Lecture Notes in Mathematics, 257. Springer-Verlag, Berlin, Heidelberg, New York, 1972).

[4] Richard Holmes and Bernard Kripke, "Smoothness of approximation", *Michigan Math. J.* 15 (1968), 225-248.

[5] R.C. James, "Orthogonality in normed linear spaces", *Duke Math. J.* 12 (1945), 291-302.

[6] G.K. Kalisch and E.G. Straus, "On the determination of points in a Banach space by their distances from the points of a given set", *An. Acad. Brasil. Ci.* 29 (1957), 501-519.

[7] Victor Klee, "Convexity of Chebyshev sets", *Math. Ann.* 142 (1961), 292-304.

[8] Clifford A. Kottman and Bor-Luh Lin, "The weak continuity of metric projections", *Michigan Math. J.* 17 (1970), 401-404.

[9] Joseph Michael Lambert, "The weak sequential continuity of the metric projection in $L_p$ spaces", (Dissertation, Purdue University, 1970).

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