On Equivalence of Data Informativity for Identification and Data-Driven Control of Partially Observable Systems

Tomonori Sadamoto, *Member, IEEE*

***Abstract*— This study shows that the informativity for the identification of partially observable systems is equivalent to that for designing dynamical measurement-feedback stabilizers. This finding is entirely different from the input-state case, where the direct data-driven design of state-feedback stabilizers requires less informativity than system identification. We derive the equivalence between the two types of informativity based on a newly introduced vector autoregressive with exogenous input (VARX) framework, which is suitable for time-domain analyses such as state-space models while directly representing input–output characteristics such as transfer functions. Moreover, we show a duality between the characterization of all VARX models explaining data and that of all VARX controllers stabilizing such VARX models.**

***Index Terms*— Data Informativity, Data-Driven Control, System Identification, Partially Observable Systems**

I. Introduction

Data-driven methodologies for control can be classified into indirect and direct approaches. Indirect approaches have a two-step design consisting of system identification [1] from data and the application of model-based controller design methods such as optimal control [2], whereas direct approaches [3], [4] aim to design controllers directly from data without the identification process.

Many studies have compared these two approaches from several perspectives. For example, [5] bridges them through a multi-criteria formulation with a trade-off between system identification and control objectives when sufficient data are available. From a different perspective, several studies have investigated the sample complexity [6] of system identification [7], [8] and data-driven linear quadratic regulator (LQR) design [9]. Furthermore, [10] compares direct and indirect model predictive control (MPC) in terms of sample complexity and control performance. These sample complexity analyses derive the upper bounds of identification error and control performance; however, they do not reveal the necessary amount (i.e., the lower bounds) of data for achieving the desired performance.

Of relevant to this article, we note a paper [11] that reveals the *informativity* of input-state data for performing direct and indirect approaches. Their findings are two-fold: the informativity (i.e., the amount of data) for direct LQR designs must be equivalent to that for identifying the true system, however, the informativity for the direct design of stabilizers can be less. Based on these findings, extensions to suboptimal control [12] and multiple-dataset-driven design [13] have been proposed. However, the informativity framework is limited to input-state systems.

In this study, we investigate the informativity for the identification of partially observable systems and that for designing dynamical measurement-feedback stabilizers. We show that these two must be equivalent. This finding is completely different from the input-state case in [11] and provides a theoretical justification for imposing the strong persistency of excitation conditions [14] in both direct and indirect approaches to partially observable systems. In the derivation of the equivalence, a newly developed *vector autoregressive with exogenous input (VARX)* framework plays a central role. The VARX model, which is an alternative description of a discrete-time system as a linear combination of finite-length input–output histories, has been proposed in the context of system identification. Because the VARX model of multiple-input multiple-output (MIMO) systems must be redundant in general, it has been thought that the representation is not entirely suitable for identification. On the contrary, we show that any minimal state-space model can be equivalently described as a VARX model obtained by projecting the redundant history dynamics onto their reachable subspace. Moreover, we show that the problem of designing dynamical measurement-feedback controllers for partially observable systems is equivalent to that of designing state-feedback (in other words, input-output history-feedback) controllers for VARX models. Based on these findings, we provide a necessary and sufficient condition for data by which the true system can be uniquely identified. Furthermore, incorporating the informativity framework of [1] with the VARX framework shows that the above two types of informativity are equivalent. Interestingly, we discover a duality between the characterization of all VARX models explaining data and that of all VARX controllers that stabilize such VARX models.

The remainder of this paper is organized as follows. Section II describes the problem settings and our main claim as a theorem. Section III introduces the VARX framework. Section IV presents a characterization of informativity for system identification. Section V presents the proof of the theorem

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[1]Tomonori Sadamoto is with Department of Mechanical and Intelligent Systems Engineering, Graduate School of Informatics and Engineering, The University of Electro-Communications, 1-5-1, Chofugaoka, Chofu, Tokyo 182-8585, Japan. Email: sadamoto@uec.ac.jp

stated in Section II. Moreover, we show a dual relation between the characterization of all VARX models explaining data and that of all VARX controllers that stabilize such VARX models. Section VI concludes this paper.

Notation: We denote the set of n-dimensional real vectors as $\mathbb{R}^n$, the set of n-dimensional complex vectors as $\mathbb{C}^n$, the set of natural numbers as $\mathbb{N}$, the range space spanned by the column vectors of P as $\mathrm{im}P$, the rank of the matrix P as $\mathrm{rank}P$, the Moore–Penrose inverse of P as $P^\dagger$, the set of eigenvalues of P as $\lambda(P)$, the dimension of a vector space X as $\dim(X)$, the n-dimensional identity matrix as I_n, and the n-by-m zero matrix as $0_{n\times m}$. The subscript n (resp. $n\times m$) of I_n (resp. $0_{n\times m}$) is omitted if obvious. Given a matrix, entries having a value of zero are left blank, unless this causes confusion. For example, $M=\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22}\end{bmatrix}$, where $M_{11}\neq 0$, $M_{22}\neq 0$, $M_{12}\neq 0$, and $M_{21}=0$, is described as $M=\begin{bmatrix} M_{11} & M_{12} \\ & M_{22}\end{bmatrix}$. Given a full-column rank matrix $P\in\mathbb{R}^{n\times m}$ satisfying $P^\top P=I$, $\overline{P}\in\mathbb{R}^{n\times(n-m)}$ is defined such that $[P\ \overline{P}]$ is unitary. An n-dimensional state-space model $\boldsymbol{\Sigma}$: $x_{t+1}=Ax_t+Bu_t$ and $y_t=Cx_t+Du_t$ is said to be stable if all absolute values of the eigenvalues of A are less than 1. Further, given $L\in\mathbb{N}$, we define $R_L(\boldsymbol{\Sigma}):=[A^{L-1}B,\ldots,AB,B]$, $O_L(\boldsymbol{\Sigma}):=[C^\top,(CA)^\top,\ldots,(CA^{L-1})^\top]^\top$, and

$$H_L(\boldsymbol{\Sigma}):=\begin{bmatrix} D & & & \\ CB & \ddots & & \\ \vdots & \ddots & \ddots & \\ CA^{L-2}B & \cdots & CB & D \end{bmatrix}.$$

In particular, we denote the reachability matrix of $\boldsymbol{\Sigma}$ as $R(A,B):=R_n(\boldsymbol{\Sigma})$. Given two systems $\boldsymbol{\Sigma}$ and $\boldsymbol{K}$, the closed-loop system composed of these is denoted as $(\boldsymbol{\Sigma},\boldsymbol{K})$. The set of given data $\{x_{t_1},\ldots,x_{t_2}\}$ is denoted by $\{x\}_{t_2}^{t_1}$, and the stack of x_k for $k\in[t_1,t_2]$ is given by $[x]_{t_2}^{t_1}:=[x_{t_1}^\top,\ldots,x_{t_2}^\top]^\top$. The set of the infinite sequence $x_T,x_{T+1},\cdots$ is denoted by $\{x\}_{t\geq T}$.

II. Problem Settings and Main Result

Let $\mathcal{D}:=\{u^\star,y_{\rm s}^\star\}_N^0$ be a given dataset, where $y_{\rm s}^\star$ is the output of

$$\boldsymbol{\Sigma}_{\rm s}:\begin{cases} x_{{\rm s},t+1}=A_{\rm s}x_{{\rm s},t}+B_{\rm s}u_t \\ y_{{\rm s},t}\ =C_{\rm s}x_{{\rm s},t}\end{cases},\ x_{\rm s}\in\mathbb{R}^n, y_{\rm s}\in\mathbb{R}^r, u\in\mathbb{R}^m \tag{1}$$

when $u=u^\star$. We assume that $\boldsymbol{\Sigma}_{\rm s}$ is minimal. Further, we assume $N\geq n$. The case where a feedthrough term exists, i.e., $y_{{\rm s},t}=C_{\rm s}x_{{\rm s},t}+D_{\rm s}u_t$, will be discussed in Remark 4. In this paper, the system $\boldsymbol{\Sigma}_{\rm s}$ is referred to as the *true* system.

Data-driven designs of a controller stabilizing $\boldsymbol{\Sigma}_{\rm s}$ can be classified into two types: the *indirect* method, in which a model explaining the dataset $\mathcal{D}$ is identified and model-based designs are then applied, and the *direct* method, in which a controller is designed directly from the dataset. However, the necessary and sufficient conditions of $\mathcal{D}$ for identifying the true system and for designing a stabilizing controller have not yet been studied. Therefore, the present study aims to compare the conditions.

As a model explaining the dataset, let us consider

$$\boldsymbol{\Sigma}:\begin{cases} x_{t+1}=Ax_t+Bu_t \\ y_t\ =Cx_t\end{cases},\ x\in\mathbb{R}^n,\ y\in\mathbb{R}^r,\ u\in\mathbb{R}^m, \tag{2}$$

which is assumed to be minimal without loss of generality. The set of $\boldsymbol{\Sigma}$, i.e., the set of minimal n-dimensional systems explaining $\mathcal{D}$, can be defined as

$$\boldsymbol{\Sigma}_{\rm i/o}:=\left\{\boldsymbol{\Sigma}\text{ in (2)}\mid {}^\exists\{x\}_N^0\text{ s.t. }\{u,y\}_N^0=\mathcal{D}\right\}. \tag{3}$$

Under this setting, we define the *informativity* for system identification as follows.

Definition 1. We state that $\mathcal{D}$ is *informative for system identification* if $\boldsymbol{\Sigma}_{\rm i/o}=\{\boldsymbol{\Sigma}_{\rm s}\}$ up to the similarity transformation.

If $\mathcal{D}$ is informative for system identification, we can construct a model $\boldsymbol{\Sigma}$ with input–output behavior that exactly matches $\boldsymbol{\Sigma}_{\rm s}$, based on which we can design a measurement-feedback minimally realized controller,

$$\boldsymbol{K}:\begin{cases} \xi_{t+1}=A_{\boldsymbol{K}}\xi_t+B_{\boldsymbol{K}}y_t \\ u_t\ =C_{\boldsymbol{K}}\xi_t\end{cases},\ \xi\in\mathbb{R}^\kappa \tag{4}$$

that stabilizes $\boldsymbol{\Sigma}_{\rm s}$. We assume $\kappa\leq n$. However, it may be possible to design a stabilizing controller $\boldsymbol{K}$ from $\mathcal{D}$ without identifying the true system. To investigate this possibility, we introduce the following informativity.

Definition 2. We state that $\mathcal{D}$ is *informative for stabilization by measurement feedback* if there exists $\boldsymbol{K}$ in (4) such that $\boldsymbol{\Sigma}_{\rm i/o}\subseteq\boldsymbol{\Sigma}_{\boldsymbol{K}}$, where

$$\boldsymbol{\Sigma}_{\boldsymbol{K}}:=\{\boldsymbol{\Sigma}\text{ in (2)}\mid(\boldsymbol{\Sigma},\boldsymbol{K})\text{ is stable}\}. \tag{5}$$

If $\mathcal{D}$ is informative for stabilization by measurement feedback, the controller $\boldsymbol{K}$ can stabilize any minimal n-dimensional system explaining $\mathcal{D}$. Hence, if there exists a situation where $\mathcal{D}$ is informative for measurement feedback but not for system identification, the amount of data required for the aforementioned direct approach can be less than that required for the indirect approach. However, the following theorem shows that such situations do not exist.

Theorem 1. A given $\mathcal{D}$ is informative for stabilization by measurement feedback if and only if it is informative for system identification.

This theorem implies that the direct and indirect approaches are equivalent in terms of the amount of data. This result is different from the case for state-feedback control [11], where the informativity for stabilization by *state feedback* was shown to be less than that for system identification by using the input-state data. To prove Theorem 1, in the next section, we introduce a new mathematical tool called the *VARX framework*.

Remark 1. Any state-feedback controller cannot be represented as $\boldsymbol{K}$ in (4) because it does not have a feedthrough term. Thus, even if $C_{\rm s}=I$ in (1), Theorem 1 does not include the result in [11] as a special case.

Remark 2. The work [11] shows the necessity of Theorem 1: if $\mathcal{D}$ is informative for system identification, then it is informative for stabilization by measurement feedback. In contrast, our major contribution is to demonstrate its sufficiency.

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III. VARX Framework

VARX models [15], especially for ARX models of single-input single-output (SISO) systems [16], are alternative representations of dynamical systems. We newly define the VARX models with a slight modification as follows.

Definition 3. Given $L \in \mathbb{N}$, consider

$$\hat{\mathbf{\Sigma}}: \ y_t = \Sigma v_t, \quad v_t := \begin{bmatrix} [u]_{t-1}^{t-L} \\ [y]_{t-1}^{t-L} \end{bmatrix}, \quad t \geq L \tag{6}$$

such that

$$v_L \in \mathbb{V}, \quad \mathbb{V} := \{v_t, t > L|^{\forall}\{u\}_{t>L}, \ v_L = 0\}, \tag{7}$$

where $u \in \mathbb{R}^m$ is the input, $y \in \mathbb{R}^r$ is the output, and $\Sigma \in \mathbb{R}^{r \times L(m+r)}$ is a parameter. We call $\hat{\mathbf{\Sigma}}$ and v an *L-length VARX model* and *L-length input-output history*, respectively, or simply a *VARX model* and *history*[1], respectively.

In general, the pair $\{u, y\}$ in (6) is not related to the input–output pair of $\mathbf{\Sigma}$ in (2). However, these pairs can be identical in a certain situation, as will be shown later. The set $\mathbb{V}$ in (7) represents a reachable subspace of v for u under the condition that the initial history is zero. Later we will show this as (14). The condition (7) means that the initial history lies in the reachable subspace. This condition is needed to relate VARX and state-space models.

We note that $\hat{\mathbf{\Sigma}}$ is a dynamical system in the sense that the output y_t in (6) is used to define v_{t+1}, and subsequently, v_{t+1} is used to generate y_{t+1}. To express this recursive relation explicitly, we show an equivalent state-space realization of (6) as follows.

Lemma 1. The VARX model (6) is equivalent to

$$\hat{\mathbf{\Sigma}}: \ \begin{cases} v_{t+1} &= \Phi v_t + \Gamma u_t \\ y_t &= \Sigma v_t \end{cases} \tag{8}$$

where $\Phi := \Psi + \Theta\Sigma$, $\Theta = [0_{r\times(Lm+Lr-r)}, I_r]^{\mathsf{T}}$,

$$\Psi = \begin{bmatrix} \begin{matrix} 0_{m\times m} & I_{(L-1)m} \end{matrix} & \\ & \begin{matrix} & I_{(L-1)r} \\ 0_{r\times r} & \end{matrix} \end{bmatrix}, \ \Gamma = \begin{bmatrix} I_m \\ \\ \end{bmatrix}. \tag{9}$$

By abuse of notation, we call (8) as a *VARX model*.

In addition, we introduce the notion of the *essential dimension* into the VARX model from the perspective of reachability. Owing to the structure of Φ and Γ, it follows from a simple calculation that

$$Lm \leq \operatorname{rank} R(\Phi, \Gamma) \leq Lm + Lr. \tag{10}$$

Based on this relation, we define the *essential dimension* of the VARX model (8) as

$$\operatorname{ess.dim}(\hat{\mathbf{\Sigma}}) := \operatorname{rank} R(\Phi, \Gamma) - Lm. \tag{11}$$

It can be seen that the VARX model given by (8) can be reduced to a lower-dimensional system, unless the essential dimension is Lr. The following lemma summarizes this fact.

[1]Because the parameter and history depend on L, their symbols should be $\hat{\mathbf{\Sigma}}_L$ and $v_{t,L}$, respectively. However, to simplify the notation, we omit the subscript L.

Lemma 2. Given a VARX model $\hat{\mathbf{\Sigma}}$ in (8), let $n := \operatorname{ess.dim}(\hat{\mathbf{\Sigma}})$, where $\operatorname{ess.dim}(\cdot)$ is defined as (11). Then, v follows

$$\begin{cases} w_{t+1} &= P^{\mathsf{T}}\Phi P w_t + P^{\mathsf{T}}\Gamma u_t \\ v_t &= P w_t \end{cases}, \quad w_L := P^{\mathsf{T}} v_L \tag{12}$$

for any u, $t \geq L$, and v_L satisfying (7), where $P \in \mathbb{R}^{(Lm+Lr)\times(Lm+n)}$ satisfies

$$\operatorname{im} P = \operatorname{im} R(\Phi, \Gamma), \quad P^{\mathsf{T}} P = I. \tag{13}$$

Moreover, $\mathbb{V}$ in (7) is described as

$$\mathbb{V} = \operatorname{im} R(\Phi, \Gamma). \tag{14}$$

Proof: From the definition of the essential dimension, P satisfies (13). Let

$$w := P^{\mathsf{T}} v, \quad \overline{w} := \overline{P}^{\mathsf{T}} v. \tag{15}$$

Then, (8) can be written as

$$\begin{bmatrix} w_{t+1} \\ \overline{w}_{t+1} \end{bmatrix} = \begin{bmatrix} P^{\mathsf{T}}\Phi P & P^{\mathsf{T}}\Phi\overline{P} \\ \overline{P}^{\mathsf{T}}\Phi P & \overline{P}^{\mathsf{T}}\Phi\overline{P} \end{bmatrix} \begin{bmatrix} w_t \\ \overline{w}_t \end{bmatrix} + \begin{bmatrix} P^{\mathsf{T}}\Gamma \\ \overline{P}^{\mathsf{T}}\Gamma \end{bmatrix} u_t.$$

The equivalence between (7) and (14) is clear. From (13) and (14), we have $\overline{P}^{\mathsf{T}}\Phi P = 0$, $\overline{P}^{\mathsf{T}}\Gamma = 0$, and $\overline{w}_L = 0$. Hence, $\overline{w}_t \equiv 0$ for $t \geq L$. Thus, the lemma follows. ☐

Under these settings, we introduce the notion of *consistency* as follows.

Definition 4. We state that $\mathbf{\Sigma}$ in (2) (resp. $\hat{\mathbf{\Sigma}}$ in (6)) is *consistent* with $\hat{\mathbf{\Sigma}}$ (resp. $\mathbf{\Sigma}$) if

$$^{\exists}\{x\}_{t\geq 0}, \ ^{\forall}\{u, y\}_{t\geq 0} \text{ s.t. (6) and (7) hold.} \tag{16}$$

A pair $\{\mathbf{\Sigma}, \hat{\mathbf{\Sigma}}\}$ satisfying (16) is called a *consistent pair*. Moreover, a consistent pair $\{\mathbf{\Sigma}, \hat{\mathbf{\Sigma}}\}$ such that $\dim(\mathbf{\Sigma}) = \operatorname{ess.dim}(\hat{\mathbf{\Sigma}}) = n$ is an *n-dimensional* consistent pair.

The relation in (16) implies that the output of $\hat{\mathbf{\Sigma}}$ is identical to that of $\mathbf{\Sigma}$ for any common input u. In other words, a consistent pair $\{\mathbf{\Sigma}, \hat{\mathbf{\Sigma}}\}$ represents the same dynamical system. Next, we present characterizations of consistency.

Lemma 3. Given a L-length VARX model $\hat{\mathbf{\Sigma}}$ in (8), let $n := \operatorname{ess.dim}(\hat{\mathbf{\Sigma}})$, where $\operatorname{ess.dim}(\cdot)$ is defined as (11). Then, any minimal $\mathbf{\Sigma}$ that is consistent with $\hat{\mathbf{\Sigma}}$ can be written in the form of (2) with

$$A = P'^{\dagger} A' P', \ B = P'^{\dagger} B', \ C = C' P', \ x_0 = P'^{\dagger} x_0'. \tag{17}$$

Here, $P' \in \mathbb{R}^{Lr\times n}$ is a full-column rank matrix such that $\operatorname{im} P' = \operatorname{im} R(A', B')$,

$$A' := \begin{bmatrix} & & & A_L \\ I & & & A_{L-1} \\ & \ddots & & \vdots \\ & & I & A_1 \end{bmatrix}, \ B' := \begin{bmatrix} B_L \\ \vdots \\ B_2 \\ B_1 \end{bmatrix}, \ C' := \begin{bmatrix} 0_{r\times r} \\ \vdots \\ 0_{r\times r} \\ I \end{bmatrix}^{\mathsf{T}}$$
$$x_0' := V'^{\dagger}[y]_{L-1}^0 - V'^{\dagger} H' [u]_{L-1}^0, \tag{18}$$

where $A_i \in \mathbb{R}^{r\times r}$ and $B_i \in \mathbb{R}^{r\times m}$ are partitions of Σ such that $\Sigma = [B_L, \cdots, B_1, A_L \cdots, A_1]$. Additionally, $V' := O_L(\mathbf{\Sigma}')$

and $H' := H_L(\mathbf{\Sigma}')$ with $\mathbf{\Sigma}' : \ x'_{t+1} = A'x'_t + B'u'_t,\ y'_t = C'x'_t$. Moreover, any such $\mathbf{\Sigma}$ satisfies

$$\mathrm{rank} O_L(\mathbf{\Sigma}) = n. \tag{19}$$

Proof: Note that V' is invertible. Hence, it follows from the dynamics of $\mathbf{\Sigma}'$ that

$$[y']^0_{L-1} = V'x'_0 + H'[u']^0_{L-1} = [y]^0_{L-1} + H'[u'-u]^0_{L-1}$$

and

$$y'_t = \Sigma v'_t, \quad v'_t := \begin{bmatrix} [u']^{t-L}_{t-1} \\ [y']^{t-L}_{t-1} \end{bmatrix}, \quad t \geq L.$$

Hence, $y'_t \equiv y_t$ if $u'_t \equiv u_t$ for any $t \geq 0$. This implies that there exists $\{x'\}_{t\geq 0}$ for any $\{u,y\}_{t\geq 0}$ following (6). In the following, we assume that $u' \equiv u$, without loss of generality.

Let $n' := \dim(\mathrm{im} R(A', [B', x'_0]))$. Then, there exists a full-column rank matrix $P' \in \mathbb{R}^{Lr\times n'}$ and $\zeta_t \in \mathbb{R}^{n'}$ such that $x'_t = P'\zeta_t$ for any u and t. Because x'_0 depends on $\{u,y\}^0_{L-1}$, the value of n' can vary depending on $\{u,y\}^0_{L-1}$. We show that $n' = n$ for any $\{u,y\}^0_{L-1}$ satisfying (7). It follows from the dynamics of $\mathbf{\Sigma}'$ that

$$v_t = v'_t = \Xi' \begin{bmatrix} [u]^{t-L}_{t-1} \\ \zeta_{t-L} \end{bmatrix}, \quad \Xi' := \begin{bmatrix} I & \\ H' & V'P' \end{bmatrix} \tag{20}$$

for any u, ζ, and $t \geq L$. Because of the reachability of ζ, the history v_t can be any element of $\mathrm{im}\Xi'$ for any $t \geq L$. On the other hand, from (8), v_t can be any element of $\mathrm{im} R(\Phi, [\Gamma, v_L])$ for any $t \geq L$. Thus, we have

$$\mathrm{im}\Xi' = \mathrm{im} R(\Phi, [\Gamma, v_L]) = \mathrm{im} R(\Phi, \Gamma)$$

because $v_L \in \mathrm{im} R(\Phi, \Gamma)$. This relation implies that the reachable subspace of the VARX model $\hat{\mathbf{\Sigma}}$ in (8) is $\mathrm{im}\Xi'$. Additionally, $\mathrm{rank}\Xi' = Lm + n'$. Hence, $\mathrm{rank} R(\Phi, \Gamma) = Lm + n'$. Therefore, it follows from (11) that $n' = n$. Note that this holds even for $x'_0 = 0$. Hence, from the definition, P' satisfies $\mathrm{im} P' = \mathrm{im} R(A', B')$.

Moreover, $\mathbf{\Sigma}'$ is observable because V' is invertible. Therefore, a minimal realization of $\mathbf{\Sigma}'$ can be written in the form of a (2). Because any n-dimensional minimal realizations of a given system are equivalent up to the similarity transformation [17], the lemma follows. □

Lemma 3 shows that $\mathbf{\Sigma}$ is consistent with a given VARX model $\hat{\mathbf{\Sigma}}$. The next lemma shows the opposite characteristic.

Lemma 4. Given $\mathbf{\Sigma}$ in (2), let L be given such that (19) holds. Then, any L-length VARX model $\hat{\mathbf{\Sigma}}$ that is consistent with $\mathbf{\Sigma}$ can be written in the form of (6) with

$$\Sigma = \Sigma_0 + \bar{\Sigma}, \tag{21}$$

where

$$\Sigma_0 := \left[C(R_L(\mathbf{\Sigma}) - A^L O^\dagger_L(\mathbf{\Sigma})H_L(\mathbf{\Sigma})),\ CA^L O^\dagger_L(\mathbf{\Sigma})\right] \tag{22}$$

and $\bar{\Sigma} \in \mathbb{R}^{r\times L(m+r)}$ satisfies $\ker\bar{\Sigma} = \mathrm{im} R(\Psi + \Theta\Sigma_0, \Gamma)$ with Ψ, Θ, and Γ in (9). Moreover, any $\hat{\mathbf{\Sigma}}$ satisfies

$$n = \mathrm{ess.\,dim}(\hat{\mathbf{\Sigma}}). \tag{23}$$

Proof: We first show that $\hat{\mathbf{\Sigma}}$ with $\Sigma = \Sigma_0$ satisfies (16). It follows from (2) that

$$x_t = A^L x_{t-L} + R_L(\mathbf{\Sigma})[u]^{t-L}_{t-1}, \tag{24}$$

$$[y]^{t-L}_{t-1} = O_L(\mathbf{\Sigma})x_{t-L} + H_L(\mathbf{\Sigma})[u]^{t-L}_{t-1}. \tag{25}$$

From (19), note that there exists $O^\dagger_L(\mathbf{\Sigma})$ such that $O^\dagger_L(\mathbf{\Sigma})O_L(\mathbf{\Sigma}) = I$. Hence, from a simple calculation, any $\{u,y\}_{t\geq 0}$ satisfying (2) follows (6) with $\Sigma = \Sigma_0$. Using the same procedure as in the proof of Lemma 3, we can see that v_t in (8) satisfies

$$v_t \in \mathrm{im}\Xi,\ t \geq L, \quad \Xi := \begin{bmatrix} I & \\ H_L(\mathbf{\Sigma}) & O_L(\mathbf{\Sigma}) \end{bmatrix}. \tag{26}$$

In addition, v_t can be any element of $\mathrm{im}\Xi$ because of the reachability of x. This implies that

$$\mathrm{im}\Xi = \mathrm{im} R(\Psi + \Theta\Sigma_0, \Gamma).$$

Thus, $\hat{\mathbf{\Sigma}}$ with $\Sigma = \Sigma_0$ satisfies (16). Because $\bar{\Sigma}v_t \equiv 0$ for any v_t, any $\hat{\mathbf{\Sigma}}$ with Σ in (21) satisfies (16). Conversely, if there exists a Σ that cannot be written as (21), the corresponding $\hat{\mathbf{\Sigma}}$ does not satisfy (6). Therefore, the parameter Σ of any $\hat{\mathbf{\Sigma}}$ that is consistent with $\mathbf{\Sigma}$ has the form of (21). Finally, because $\mathrm{rank}\Xi = Lm + n$, the relation (23) holds. This completes the proof. □

Because the outputs of $\mathbf{\Sigma}$ and $\hat{\mathbf{\Sigma}}$ are identical if they are consistent, the eigenvalues of A in (2) and of Φ in (8) should be related. The following lemma demonstrates this fact.

Lemma 5. Consider an n-dimensional consistent pair $\{\mathbf{\Sigma}, \hat{\mathbf{\Sigma}}\}$ in Definition 4. Then,

$$\lambda(P^\mathsf{T}\Phi P) = \lambda(A) \cup \{0, \ldots, 0\}, \tag{27}$$

where P is defined in (13).

Proof: For $i \in \{1, \ldots, n\}$, let $\lambda_i \in \mathbb{C}$ and $\nu_i \in \mathbb{C}^n$ be the i-th eigenvalue and eigenvector of A. Suppose $u_t \equiv 0$ for $t \geq 0$, and $x_0 = \nu_i$. Then, $y_t = \lambda^t_i C\nu_i$. Thus, it follows that

$$v_t = \begin{bmatrix} [u]^{t-L}_{t-1} \\ [y]^{t-L}_{t-1} \end{bmatrix} = \lambda^{t-L}_i \pi_i, \quad t \geq L$$

where $\pi_i := [0_{1\times Lm}, (C\nu_i)^\mathsf{T}, \cdots, \lambda^{L-1}_i (C\nu_i)^\mathsf{T}]^\mathsf{T}$. Then, w in (15) can be described as $w_t = \lambda^{t-L}_i P^\mathsf{T}\pi_i$ for $t \geq L$. Because (12) holds, we have

$$\lambda_i P^\mathsf{T}\pi_i = w_{L+1} = P^\mathsf{T}\Phi P w_L = (P^\mathsf{T}\Phi P)P^\mathsf{T}\pi_i,$$

which implies that λ_i is an eigenvalue of $P^\mathsf{T}\Phi P$.

Next, we show that the eigenvalues of $P^\mathsf{T}\Phi P$ other than $\lambda(A)$ are all zero. Let μ_i be such an eigenvalue, i.e., $\mu_i \in \lambda(P^\mathsf{T}\Phi P)\setminus\lambda(A)$, and let ω_i be the corresponding eigenvector. When $w_L = \omega_i$ and $u_t \equiv 0$, it follows that

$$w_t = \mu^{t-L}_i \omega_i, \quad t \geq L. \tag{28}$$

Because $\{\mathbf{\Sigma}, \hat{\mathbf{\Sigma}}\}$ is an n-dimensional consistent pair, (19) follows. Hence, $x_t = \Upsilon v_t$ holds, where

$$\Upsilon := [R_L(\mathbf{\Sigma}) - A^L O^\dagger_L(\mathbf{\Sigma})H_L(\mathbf{\Sigma}), A^L O^\dagger_L(\mathbf{\Sigma})].$$

Therefore,

$$x_t = \mu^{t-L}_i \Upsilon P\omega_i, \quad t \geq L \tag{29}$$

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holds. If $\Upsilon P\omega_i \neq 0$, the relation (29) implies that μ_i and $\Upsilon P\omega_i$ are the eigenvalue and the corresponding right eigenvector of A. This is a contradiction to $\mu_i \in \lambda(P^\mathsf{T}\Phi P)\setminus\lambda(A)$. Thus, by reductio ad absurdum, $\Upsilon P\omega_i = 0$. Because $\Phi = \Psi + (\Theta C\Upsilon + \bar{\Sigma})$, where Ψ, Γ, and Θ are defined in (8) and $\bar{\Sigma}$ is given by (21), we have

$$\mu_i\omega_i = P^\mathsf{T}\Phi P\omega_i = P^\mathsf{T}(\Psi + \Theta(C\Upsilon + \bar{\Sigma}))P\omega_i = P^\mathsf{T}\Psi P\omega_i,$$

where the property $\ker\bar{\Sigma} = \operatorname{im}P$ is used to derive the last equation. Hence, μ_i is an eigenvalue of Ψ. Because Ψ is nilpotent, $\mu_i = 0$. This completes the proof. □

A relation similar to Lemma 5 also holds for closed-loop systems. Consider a κ-dimensional consistent pair $\{\boldsymbol{K}, \hat{\boldsymbol{K}}\}$, where $\boldsymbol{K}$ has the form of (4), while

$$\hat{\boldsymbol{K}}:\ u_t = Kv_t, \quad t \geq L. \tag{30}$$

We call the control (30) an *input–output–history (IOH)* feedback. Note that (12) holds for n-dimensional consistent pairs $\{\boldsymbol{\Sigma}, \hat{\boldsymbol{\Sigma}}\}$. Thus, the closed-loop $(\hat{\boldsymbol{\Sigma}}, \hat{\boldsymbol{K}})$ is described as

$$(\hat{\boldsymbol{\Sigma}}, \hat{\boldsymbol{K}}): \begin{cases} w_{t+1} &= P^\mathsf{T}(\Phi + \Gamma K)Pw_t \\ v_t &= Pw_t \end{cases}, \quad w_L := P^\mathsf{T}v_L \tag{31}$$

for $t \geq L$. On the other hand, the closed loop $(\boldsymbol{\Sigma}, \boldsymbol{K})$ is described as:

$$(\boldsymbol{\Sigma}, \boldsymbol{K}): x_{\mathrm{cl},t+1} = A_{\mathrm{cl}}x_{\mathrm{cl},t}, \quad y_t = C_{\mathrm{cl}}^{\mathrm{y}}x_{\mathrm{cl},t}, \quad u_t = C_{\mathrm{cl}}^{\mathrm{u}}x_{\mathrm{cl},t},$$

where

$$A_{\mathrm{cl}} := \begin{bmatrix} A & BC_{\boldsymbol{K}} \\ B_{\boldsymbol{K}}C & A_{\boldsymbol{K}} \end{bmatrix}, \quad C_{\mathrm{cl}}^{\mathrm{y}} := [C\ 0], \quad C_{\mathrm{cl}}^{\mathrm{u}} := [0\ C_{\boldsymbol{K}}]. \tag{32}$$

From the definition of consistency, L satisfies $\operatorname{rank}O_L(\boldsymbol{\Sigma}) = n$ and $\operatorname{rank}O_L(\boldsymbol{K}) = \kappa$. Because of the consistency of $\{\boldsymbol{\Sigma}, \hat{\boldsymbol{\Sigma}}\}$ and $\{\boldsymbol{K}, \hat{\boldsymbol{K}}\}$, the history v_t defined by (6), where u and y follow (32), is identical to the output of (31). Therefore, we have the following corollary of Lemma 5.

Corollary 1. Consider an n-dimensional consistent pair $\{\boldsymbol{\Sigma}, \hat{\boldsymbol{\Sigma}}\}$ in Definition 4 and an κ-dimensional consistent pair $\{\boldsymbol{K}, \hat{\boldsymbol{K}}\}$, where $\boldsymbol{K}$ and $\hat{\boldsymbol{K}}$ are defined by in (4) and (30), respectively, and A_{cl} is defined by (32). Then, it follows that

$$\lambda(P^\mathsf{T}(\Phi + \Gamma K)P) = \lambda(A_{\mathrm{cl}}) \cup \{0, \ldots, 0\}. \tag{33}$$

This corollary implies that the problem of designing measurement-feedback dynamical controllers for state-space models is equivalent to the problem of designing state-feedback controllers for VARX models. An interesting example is the case where $\kappa \neq n$, which is described below.

Example 1. Consider a three-dimensional, single-input, two-output system $\boldsymbol{\Sigma}$ in (2) with

$$A = \begin{bmatrix} 0.5 & 0.3 & 0.3 \\ 0.1 & 0.6 & -0.4 \\ 0.5 & -0.2 & 0.2 \end{bmatrix}, \ B = \begin{bmatrix} -1.2 \\ 0 \\ -1.4 \end{bmatrix}, \ C = \begin{bmatrix} 0 & -0.6 \\ 2.2 & 1.1 \\ -2.5 & 0.4 \end{bmatrix}^\mathsf{T}.$$

Let $L = 2$. Then, the condition (19) holds. Based on Lemma 4, we choose an L-length VARX model that is consistent with the $\boldsymbol{\Sigma}$ in (6) with $\Sigma = \Sigma_0$, where Σ_0 is defined as (22). Based on Lemma 1, we construct $\Phi \in \mathbb{R}^{6\times 6}$ and $\Gamma \in \mathbb{R}^6$; subsequently, we choose a matrix $P \in \mathbb{R}^{6\times 5}$ satisfying (13) with the Gram–Schmidt process. Then, we design $K \in \mathbb{R}^{1\times 6}$ in (30) such that $P^\mathsf{T}(\Phi+\Gamma K)P$ has desirable poles as follows. Let the poles be chosen as $\{0.2, 0.4, 0.6, 0.7, 0.9\}$. Using the pole placement method, we can find a matrix $K_P \in \mathbb{R}^{1\times 5}$ such that $P^\mathsf{T}\Phi P + P^\mathsf{T}\Gamma K_P$ has the chosen poles. When $K = K_P P^\mathsf{T}$, which in our case is

$$K = [0.1727 \quad -0.4115 \quad -0.0989 \ \ 0.0276 \quad -0.4383 \ \ 1.5],$$

it follows that

$$\lambda(P^\mathsf{T}(\Phi + \Gamma K)P) = \lambda(P^\mathsf{T}\Phi + P^\mathsf{T}\Gamma K_P) = \{0.2, 0.4, 0.6, 0.7, 0.9\}.$$

A state-space model $\boldsymbol{K}$ in (4) that is consistent with $\hat{\boldsymbol{K}}$ having this K can be given as (17). In this example, the system matrices are

$$A_{\boldsymbol{K}} = \begin{bmatrix} 0 & -0.4383 \\ 1 & 1.5 \end{bmatrix}, \ B_{\boldsymbol{K}} = \begin{bmatrix} 0.1727 & -0.4115 \\ 0.09889 & 0.0276 \end{bmatrix}, \ C_{\boldsymbol{K}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}^\mathsf{T},$$

where $P' = I_2$ in (17). Then, we have

$$\lambda(A_{\mathrm{cl}}) = \{0.2, 0.4, 0.6, 0.7, 0.9\},$$

where A_{cl} is defined in (32). This example shows that for an n-dimensional dynamical system, we can construct a lower-dimensional measurement-feedback dynamical controller based on the standard pole placement method via the VARX framework.

IV. Informativity for System Identification

The VARX framework is useful for system identification because the VARX model $\hat{\boldsymbol{\Sigma}}$ in (6) has

- only one parameter Σ to be identified and
- a direct relation with input–output data.

Furthermore, we can see from Lemma 3 that identifying a VARX model is equivalent to specifying a consistent state-space model. Based on these observations, we characterize the informativity for system identification in Definition 1. In the following, we assume that $\mathcal{D}$, i.e., the input–output data of the true system $\boldsymbol{\Sigma}_{\mathrm{s}}$ in (1), is given. Let n denote the dimensionality of $\boldsymbol{\Sigma}_{\mathrm{s}}$. For $L \in \mathbb{N}$, we define

$$\mathcal{V} := \left[v_{\mathrm{s},L}^\star, \ldots, v_{\mathrm{s},N}^\star\right], \quad v_{\mathrm{s},t}^\star := \begin{bmatrix} [u^\star]_{t-1}^{t-L} \\ [y_{\mathrm{s}}^\star]_{t-1}^{t-L} \end{bmatrix}. \tag{34}$$

In addition, we define $\mathcal{P}$ as a full-column rank matrix such that

$$\operatorname{im}\mathcal{P} = \operatorname{im}\mathcal{V}, \quad \mathcal{P}^\mathsf{T}\mathcal{P} = I. \tag{35}$$

Under these settings, we first introduce the following lemma.

Lemma 6. Consider $\mathcal{V}$ in (34) and $\boldsymbol{\Sigma}_{\mathrm{i/o}}$ in (3). Then,

$$\operatorname{rank}\mathcal{V} \leq Lm + n, \quad {}^\forall N \in \mathbb{N}, \quad {}^\forall u_t \in \mathbb{R}^m. \tag{36}$$

Moreover, if there exists $L \in \mathbb{N}$ such that

$$\operatorname{rank}\mathcal{V} = Lm + n, \tag{37}$$

then it follows that

$$\operatorname{rank}O_L(\boldsymbol{\Sigma}) = n, \quad {}^\forall\boldsymbol{\Sigma} \in \boldsymbol{\Sigma}_{\mathrm{i/o}}. \tag{38}$$

Proof: It follows from (1) that

$$v^\star_{\mathrm{s},t} = \Xi_\mathrm{s}\begin{bmatrix}[u^\star]^{t-L}_{t-1}\\ x^\star_{\mathrm{s},t-L}\end{bmatrix},\quad \Xi_\mathrm{s} := \begin{bmatrix} I & \\ H_L(\mathbf{\Sigma}_\mathrm{s}) & O_L(\mathbf{\Sigma}_\mathrm{s})\end{bmatrix} \tag{39}$$

holds for $t \geq L$, where $x^\star_\mathrm{s}$ denotes the state of $\mathbf{\Sigma}_\mathrm{s}$ when $u^\star$ is applied. Because $\mathrm{rank}\,\Xi_\mathrm{s} \leq Lm+n$, (36) follows. Next, we show (38). Similarly to (39), we have

$$v_t = \Xi\begin{bmatrix}[u]^{t-L}_{t-1}\\ x_{t-L}\end{bmatrix}, \tag{40}$$

where Ξ is defined in (26). Because $\mathbf{\Sigma} \in \mathbf{\Sigma}_{\mathrm{i/o}}$, it follows that

$$v_t = v^\star_{\mathrm{s},t},\quad t \in \{L,\dots,N\}$$

when $\{u\}^0_{N-1} = \{u^\star\}^0_{N-1}$. Thus, it follows from (40) that

$$\mathcal{V} = \Xi\mathcal{W},\quad \mathcal{W} := \begin{bmatrix}[u^\star]^0_{L-1} & \cdots & [u^\star]^{N-L}_{N-1}\\ x^\star_{\mathrm{s},0} & \cdots & x^\star_{\mathrm{s},N-L},\end{bmatrix}. \tag{41}$$

Because (37) holds, we have $\mathrm{rank}\,\Xi = Lm+n$. This implies (38) and completes the proof. □

We can see from (38) and Lemma 4 that for any $\mathbf{\Sigma}$ explaining the data (as well as for $\mathbf{\Sigma}_\mathrm{s}$ because $\mathbf{\Sigma}_\mathrm{s} \in \mathbf{\Sigma}_{\mathrm{i/o}}$), there exists a consistent L-length VARX model. Moreover, (36) implies that $\mathrm{rank}\,\mathcal{V}$ cannot be larger than $Lm+n$, even for sufficiently large data samples N. Hence, the condition (37) implies that all information of the true system's behavior is included in the dataset $\mathcal{D}$. Hence, it can be seen that (37) is a necessary and sufficient condition of informativity for system identification. The following theorem shows that this conjecture is true.

Theorem 2. Let $v_\mathrm{s} := [([u]^{t-L}_{t-1})^\mathsf{T}, ([y_\mathrm{s}]^{t-L}_{t-1})^\mathsf{T}]^\mathsf{T}$, where y_s is defined in (1). Consider $\mathcal{V}$ in (34). The following are equivalent:

i) There exists $L \in \mathbb{N}$ such that (37) holds.
ii) For any v_s satisfying $\mathbf{\Sigma}_\mathrm{s}$, it follows that $v_\mathrm{s} \in \mathrm{im}\,\mathcal{V}$ for L such that $\mathrm{rank}\,O_L(\mathbf{\Sigma}_\mathrm{s}) = n$.
iii) $\mathcal{D}$ is informative for system identification.

Proof: We first show i) ⇒ ii). Let L be chosen such that (37) holds. From Lemma 6, L satisfies $\mathrm{rank}\,O_L(\mathbf{\Sigma}_\mathrm{s}) = n$. It follows from (39) that $\mathcal{V} = \Xi_\mathrm{s}\mathcal{W}$, where $\mathcal{W}$ is defined in (41). Thus, $\mathrm{im}\,\mathcal{V} \subseteq \mathrm{im}\,\Xi_\mathrm{s}$. On the other hand, $\mathrm{rank}\,\Xi_\mathrm{s} = \mathrm{rank}\,\mathcal{V} = Lm+n$. Thus, $\mathrm{im}\,\mathcal{V} = \mathrm{im}\,\Xi_\mathrm{s}$. Because $v_\mathrm{s} \in \mathrm{im}\,\Xi_\mathrm{s}$ follows from (39), Property ii) follows.

Next, we show ii) ⇒ i). Let L be chosen such that $\mathrm{rank}\,O_L(\mathbf{\Sigma}_\mathrm{s}) = n$. Property ii) is equivalent to stating that there always exists θ_t such that $v_{\mathrm{s},t} = \mathcal{V}\theta_t$ for any $v_{\mathrm{s},t}$ satisfying $\mathbf{\Sigma}_\mathrm{s}$. As $\mathbf{\Sigma}_\mathrm{s}$ is reachable, there exists $\{u\}^0_{N'-1}$ such that the response of $\mathbf{\Sigma}_\mathrm{s}$ satisfies $\mathrm{rank}\,\Xi_\mathrm{s}\mathcal{W}' = Lm+n$, where

$$\mathcal{W}' := \begin{bmatrix}[u']^0_{L-1} & \cdots & [u']^{N-L}_{N-1}\\ x_{\mathrm{s},0} & \cdots & x_{\mathrm{s},N-L}\end{bmatrix}. \tag{42}$$

Then, we have

$$\Xi_\mathrm{s}\mathcal{W}' = [v_{\mathrm{s},L},\dots,v_{\mathrm{s},N'-1}] = \mathcal{V}[\theta_L,\dots,\theta_{N'-1}].$$

As $\mathrm{rank}\,\Xi_\mathrm{s}\mathcal{W}' = Lm+n$, Property i) follows.

In the remainder of this proof, we show the equivalence between ii) and iii). To this end, we first show that if (38) holds, then the error $\tilde{y} := y - y_\mathrm{s}$ follows

$$[\tilde{y}]^{t-L+1}_t = \begin{bmatrix}[0, I_{(L-1)r}]\\ \Sigma_\mathrm{y}\end{bmatrix}[\tilde{y}]^{t-L}_{t-1} + \begin{bmatrix} \\ (\Sigma-\Sigma_\mathrm{s})\overline{\mathcal{P}\mathcal{P}}^\mathsf{T}\end{bmatrix} v_{\mathrm{s},t} \tag{43}$$

for $t \geq L$, where $\mathcal{P}$ is defined in (35) and $\Sigma_\mathrm{y} \in \mathbb{R}^{r\times Lr}$ is defined such that $\Sigma = [\Sigma_\mathrm{u}, \Sigma_\mathrm{y}]$. It follows from (38) and Lemma 4 that there exists a consistent L-length VARX model $\hat{\mathbf{\Sigma}}$ in (6) for any $\mathbf{\Sigma} \in \mathbf{\Sigma}_{\mathrm{i/o}}$. In the following, we denote an L-length VARX model that is consistent with $\mathbf{\Sigma}_\mathrm{s}$ as

$$\hat{\mathbf{\Sigma}}_\mathrm{s}:\ y_\mathrm{s} = \Sigma_\mathrm{s}v_\mathrm{s},$$

the output of $\mathbf{\Sigma}$ when $u = u^\star$ as $y^\star$, and

$$v^\star_t := \begin{bmatrix}[u^\star]^{t-L}_{t-1}\\ [y^\star]^{t-L}_{t-1}\end{bmatrix},\quad \tilde{v}_t := \begin{bmatrix}0_{Lm\times 1}\\ [\tilde{y}]^{t-L}_{t-1}\end{bmatrix}.$$

Because $\Sigma_\mathrm{s}v^\star_{\mathrm{s},t} = y^\star_{\mathrm{s},t} = y^\star_t = \Sigma v^\star_t$ holds for $t \in \{L,\dots,N\}$, we have

$$\Sigma_\mathrm{s}\mathcal{V} = \Sigma\mathcal{V}, \tag{44}$$

or equivalently,

$$\Sigma = \Sigma_\mathrm{s} + (\Sigma-\Sigma_\mathrm{s})\overline{\mathcal{P}\mathcal{P}}^\mathsf{T}. \tag{45}$$

Using the relation $v_t = v_{\mathrm{s},t} + \tilde{v}_t$, we have

$$\tilde{y}_t = \Sigma(v_{\mathrm{s},t}+\tilde{v}_t) - \Sigma_\mathrm{s}v_{\mathrm{s},t} = (\Sigma-\Sigma_\mathrm{s})\overline{\mathcal{P}\mathcal{P}}^\mathsf{T}v_{\mathrm{s},t} + \Sigma\tilde{v}_t.$$

Then, we have $\Sigma\tilde{v}_t = \Sigma_\mathrm{y}[\tilde{y}]^{t-L}_{t-1}$. Using this representation, the dynamics of $\tilde{y}$ can be written as (43).

We now show ii) ⇒ iii). Let L be chosen such that $\mathrm{rank}\,O_L(\mathbf{\Sigma}_\mathrm{s}) = n$. Because ii) ⇔ i), (38) holds. Thus, $\tilde{y}$ satisfies (43). Moreover, Property ii) is equivalent to

$$\overline{\mathcal{P}}^\mathsf{T}v_{\mathrm{s},t} = 0,\quad {}^\forall u_t \in \mathbb{R}^m,\ {}^\forall t \geq L. \tag{46}$$

Thus, (43) is an autonomous system driven by $[\tilde{y}]^0_{L-1}$. If $[\tilde{y}]^0_{L-1} = 0$ holds, then it follows from (43) that

$$\tilde{y}_t \equiv 0,\quad {}^\forall u_t \in \mathbb{R}^m,\ {}^\forall t \geq 0, \tag{47}$$

which shows Property iii). Hence, we show that $[\tilde{y}]^0_{L-1} = 0$. Note that (25) holds for $t = L$ and $u = u^\star$. Thus, we have $O_L(\mathbf{\Sigma})x_0 = [y^\star_\mathrm{s}]^0_{L-1} - H_L(\mathbf{\Sigma})[u^\star]^0_{L-1}$. Therefore,

$$\begin{aligned}[y]^0_{L-1} &= O_L(\mathbf{\Sigma})x_0 + H_L(\mathbf{\Sigma})[u]^0_{L-1}\\ &= [y^\star_\mathrm{s}]^0_{L-1} + H_L(\mathbf{\Sigma})([u]^0_{L-1} - [u^\star]^0_{L-1}).\end{aligned}$$

Similarly, it follows from (1) that $[y^\star_\mathrm{s}]^0_{L-1} = O_L(\mathbf{\Sigma}_\mathrm{s})x_{\mathrm{s},0} + H_L(\mathbf{\Sigma}_\mathrm{s})[u^\star]^0_{L-1}$. Thus, we have

$$\begin{aligned}[y_\mathrm{s}]^0_{L-1} &= O_L(\mathbf{\Sigma}_\mathrm{s})x_{\mathrm{s},0} + H_L(\mathbf{\Sigma}_\mathrm{s})[u]^0_{L-1}\\ &= [y^\star_\mathrm{s}]^0_{L-1} + H_L(\mathbf{\Sigma}_\mathrm{s})([u]^0_{L-1} - [u^\star]^0_{L-1}).\end{aligned}$$

Hence, $[\tilde{y}]^0_{L-1} = (H_L(\mathbf{\Sigma}) - H_L(\mathbf{\Sigma}_\mathrm{s}))([u]^0_{L-1} - [u^\star]^0_{L-1})$. The relation (43) with (46) implies that the transfer function from u to $\tilde{y}$ is exactly zero. Hence, $H_L(\mathbf{\Sigma}) = H_L(\mathbf{\Sigma}_\mathrm{s})$. Thus, $[\tilde{y}]^0_{L-1} = 0$, and Property iii) follows.

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Finally, we show iii) ⇒ ii). Let L satisfy $\mathrm{rank} O_L(\mathbf{\Sigma}_\mathrm{s}) = n$. Then, Property iii) implies (38). Hence, $\tilde{y}$ satisfies (43). Moreover, because (47) follows from Property iii), we have

$$(\Sigma - \Sigma_\mathrm{s})\overline{\mathcal{P}\mathcal{P}}^\mathsf{T} v_{\mathrm{s},t} \equiv 0, \quad {}^\forall u_t \in \mathbb{R}^m, \ {}^\forall t \geq L,$$

which is equivalent to

$$(\Sigma - \Sigma_\mathrm{s})\overline{\mathcal{P}\mathcal{P}}^\mathsf{T} = 0, \quad \text{or} \quad (46).$$

Because (45) holds, the first condition becomes $\Sigma = \Sigma_\mathrm{s}$. However, because of Lemma 4, this does not hold in general. Hence, (46) holds, implying Property ii). This completes the proof. □

Theorem 2 shows the necessary and sufficient condition for the data by which the true system is identifiable. One may consider that Property i) is the same as the well-known sufficient condition for subspace identification [1], i.e., the existence of an even number L satisfying (37) and

$$\mathrm{rank} O_{L/2}(\mathbf{\Sigma}_\mathrm{s}) = n. \tag{48}$$

However, Property i) is more relaxed than this condition. For comparison, we consider a situation where $\mathbf{\Sigma}_\mathrm{s}$ is a SISO system. Let $L_\mathrm{SYSID} := 2n$, which is the minimum L that satisfies (48). Because $\mathcal{V} \in \mathbb{R}^{2L\times(N-L+1)}$ and $m = 1$, to satisfy (37), we need $N - L_\mathrm{SYSID} + 1 \geq L_\mathrm{SYSID} + n$, or equivalently, $N \geq 5n + 1$. On the other hand, let $L_\mathrm{s} := n$, which is the smallest L that satisfies $\mathrm{rank} O_L(\mathbf{\Sigma}_\mathrm{s}) = n$. In this case, $N - L_\mathrm{s} + 1 \geq L_\mathrm{s} + n$, or equivalently, $N \geq 3n + 1$ is required to satisfy (37). Therefore, the number of data samples for Property i) is less than that required to satisfy the well-known sufficient condition.

Remark 3. Property ii) has been shown in some studies on data-driven control such as [11], [14]. In contrast to these studies, we connect it to the informativity for system identification.

Remark 4. A generalization of Theorem 2 to the case where $\mathbf{\Sigma}_\mathrm{s}$ has a feedthrough term (i.e., $y_{\mathrm{s},t} = C_\mathrm{s} x_{\mathrm{s},t} + D_\mathrm{s} u_t$) can be described as follows. Let $v_\mathrm{s} := [([u]_t^{t-L})^\mathsf{T}, ([y_\mathrm{s}]_{t-1}^{t-L})^\mathsf{T}]^\mathsf{T}$. Then, Properties ii)–iii) presented above and the existence of $L \in \mathbb{N}$ satisfying $\mathrm{rank}\mathcal{V} = (L+1)m + n$ are equivalent. This claim follows by replacing v in Theorem 2 with $[([u]_t^{t-L})^\mathsf{T}, ([y]_{t-1}^{t-L})^\mathsf{T}]^\mathsf{T}$.

Remark 5. The characterization shown in Remark 4 is the same as the result of [18], where the derivation is described in the language of behavioral theory. In contrast, we present an alternative proof based on state-space realizations. Consequently, we will find a duality between the characterization of all VARX models explaining data and that of all VARX controllers that stabilize such VARX models; please see the end of Section V.

V. Proof of Theorem 1

This section proves Theorem 1 through the VARX framework introduced in Section III. Similar to the previous section, we assume that $\mathcal{D}$ is given, and we let $n := \dim(\mathbf{\Sigma}_\mathrm{s})$. For $L \in \mathbb{N}$, we consider a VARX model $\hat{\boldsymbol{K}}$ in (30), the essential dimensionality of which is denoted by κ. From Lemma 3, there exists a consistent κ-dimensional model $\boldsymbol{K}$ in the form of (4). Let

$$\mathcal{V}_+ := [v^\star_{\mathrm{s},L+1}, \ldots, v^\star_{\mathrm{s},N+1}], \quad \mathcal{U} := [u^\star_L, \ldots, u^\star_N]. \tag{49}$$

Following [11], we define

$$\begin{aligned}\hat{\mathbf{\Sigma}}_{\mathrm{u/v}} &:= \{(\Phi,\Gamma) \mid \mathcal{V}_+ = \Phi\mathcal{V} + \Gamma\mathcal{U},\ \mathrm{rank}R(\Phi,\Gamma) - Lm = n\} \\ \hat{\mathbf{\Sigma}}_{\hat{K}} &:= \{(\Phi,\Gamma) \mid P^\mathsf{T}(\Phi + \Gamma K)P \text{ is stable}, \\ &\qquad \text{where } P \text{ is in (13)}, \mathrm{rank}R(\Phi,\Gamma) - Lm = n\}. \end{aligned} \tag{50}$$

Note that $\hat{\mathbf{\Sigma}}_{\hat{K}}$ is independent of P because the eigenvalues of $P^\mathsf{T}(\Phi+\Gamma K)P$ are invariant with respect to P satisfying (13). In addition, because $\Phi \in \mathbb{R}^{(Lm+Lr)\times(Lm+Lr)}$, when $L \leq n/r$, the sets $\hat{\mathbf{\Sigma}}_{\mathrm{u/v}}$ and $\hat{\mathbf{\Sigma}}_{\hat{K}}$ are empty. Using these notations, we define the following type of informativity.

Definition 5. Given $L \in \mathbb{N}$, we state that $\mathcal{D}$ is *informative for stabilization by L-length IOH feedback* if there exists $\hat{\boldsymbol{K}}$ in (30) such that $\hat{\mathbf{\Sigma}}_{\mathrm{u/v}} \subseteq \hat{\mathbf{\Sigma}}_{\hat{K}}$.

Similar to the discussion in [11], a necessary condition of the informativity for stabilization by L-length IOH feedback can be summarized as follows.

Lemma 7. Given $L \in \mathbb{N}$, we assume that $\mathcal{D}$ is informative for stabilization by L-length IOH feedback. Let $\hat{\boldsymbol{K}}$ be a controller such that $\hat{\mathbf{\Sigma}}_{\mathrm{u/v}} \subseteq \hat{\mathbf{\Sigma}}_{\hat{K}}$. Then, (37) holds.

Proof: Let (Φ,Γ) be an entry of $\hat{\mathbf{\Sigma}}_{\mathrm{u/v}}$, and let P be chosen such that (13) is satisfied. From this assumption, $F := P^\mathsf{T}(\Phi + \Gamma K)P$ is stable. We define

$$\hat{\mathbf{\Pi}}_{\mathrm{u/v}} := \{(\Phi_0,\Gamma_0) \mid 0 = \Phi_0\mathcal{V} + \Gamma_0\mathcal{U}\} \tag{51}$$

and $F_0 := P^\mathsf{T}(\Phi_0 + \Gamma_0 K)P$. For any $\alpha \geq 0$, we have $F + \alpha F_0 = P^\mathsf{T}(\tilde{\Phi} + \tilde{\Gamma}K)P$, where $\tilde{\Phi} := \Phi + \alpha\Phi_0$ and $\tilde{\Gamma} := \Gamma + \alpha\Gamma_0$. Because $\tilde{\Phi}\mathcal{V} + \tilde{\Gamma}\mathcal{U} = \mathcal{V}_+$, it follows that $(\tilde{\Phi},\tilde{\Gamma}) \in \hat{\mathbf{\Sigma}}_{\mathrm{u/v}}$. Hence, from the definition of $\hat{\boldsymbol{K}}$, $F + \alpha F_0$ is stable for any $\alpha \geq 0$. Dividing by α, it follows that for all $\alpha \geq 1$, the spectral radius of the matrix $M_\alpha := F/\alpha + F_0$ is smaller than $1/\alpha$. From the continuity of the spectral radius by taking the limit as $\alpha \to \infty$, we see that F_0 is nilpotent for any $(\Phi_0,\Gamma_0) \in \hat{\mathbf{\Pi}}_{\mathrm{u/v}}$. Note that

$$(Q\Phi_0, Q\Gamma_0) \in \hat{\mathbf{\Pi}}_{\mathrm{u/v}} \tag{52}$$

for any $(\Phi_0,\Gamma_0) \in \hat{\mathbf{\Pi}}_{\mathrm{u/v}}$ and Q. It follows that $(Q\Phi_0, Q\Gamma_0) \in \hat{\mathbf{\Pi}}_{\mathrm{u/v}}$, and thus, $P^\mathsf{T}(Q\Phi_0 + Q\Gamma_0 K)P$ is also nilpotent. Let $Q = (\Phi_0 + \Gamma_0 K)^\mathsf{T} P P^\mathsf{T}$. Then, it follows that $P^\mathsf{T}(Q\Phi_0 + Q\Gamma_0 K)P = F_0^\mathsf{T} F_0$ is nilpotent. Because the only symmetric nilpotent matrix is the zero matrix, $F_0 = 0$, or equivalently,

$$P^\mathsf{T}[\Phi_0 \ \Gamma_0]\begin{bmatrix} P \\ KP \end{bmatrix} = 0, \quad {}^\forall(\Phi_0,\Gamma_0) \in \hat{\mathbf{\Pi}}_{\mathrm{u/v}}. \tag{53}$$

We will show that this is equivalent to

$$\mathrm{im}\begin{bmatrix} P \\ KP \end{bmatrix} \subseteq \mathrm{im}\begin{bmatrix} \mathcal{V} \\ \mathcal{U} \end{bmatrix}. \tag{54}$$

The necessity is clear. To demonstrate the sufficiency, we assume (53). Because (52) holds even when $Q = P$, (53) yields

$$[\check{\Phi}_0 \ \check{\Gamma}_0]\begin{bmatrix} P \\ KP \end{bmatrix} = 0, \quad {}^\forall(\check{\Phi}_0,\check{\Gamma}_0) \in \hat{\mathbf{\Pi}}_{\mathrm{u/v}}$$

where $\check{\Phi}_0 := P^\mathsf{T}\Phi_0$ and $\check{\Gamma}_0 := P^\mathsf{T}\Gamma_0$. This yields (54). Finally, (54) yields

$$\operatorname{im} P \subseteq \operatorname{im} \mathcal{V}. \tag{55}$$

Because $\operatorname{rank} P = Lm + n$ and (36) holds, (55) is equivalent to (37). This completes the proof. □

Based on this lemma, we show the proof of Theorem 1 as follows.

Proof of Theorem 1: Suppose that $\mathcal{D}$ is informative for system identification. Because $\mathbf{\Sigma}_\mathrm{s}$ is reachable, there exists $\boldsymbol{K}$ such that $(\mathbf{\Sigma}_\mathrm{s}, \boldsymbol{K})$ is stable. From Definition 1, $\mathcal{D}$ is informative for measurement feedback.

In the following, we show the sufficiency. Suppose that $\mathcal{D}$ is informative for measurement feedback. Let L be chosen as the smallest value such that

$$\operatorname{rank} O_L(\mathbf{\Sigma}) = n \ ^\forall \mathbf{\Sigma} \in \mathbf{\Sigma}_{\mathrm{i/o}} \cup \mathbf{\Sigma}_{\boldsymbol{K}}, \quad \wedge \quad \operatorname{rank} O_L(\boldsymbol{K}) = \kappa. \tag{56}$$

Since $\kappa \le n$, L satisfies $L \le n$. Hence, $\mathcal{V}$, $\mathcal{V}_+$ and $\mathcal{U}$ in (50) are not empty because $N \ge n$. Then, from Lemma 4, there exists $\hat{\mathbf{\Sigma}}$ in (8) (resp. $\hat{\boldsymbol{K}}$ in (30)) that is consistent with $\mathbf{\Sigma}$ for any $\mathbf{\Sigma} \in \mathbf{\Sigma}_{\mathrm{i/o}} \cup \mathbf{\Sigma}_{\boldsymbol{K}}$ (resp. $\boldsymbol{K}$). Thus, from Corollary 1,

$$\bigcup_{\mathbf{\Sigma} \in \mathbf{\Sigma}_{\boldsymbol{K}}} \{n\text{-dim. } \hat{\mathbf{\Sigma}} \text{ being consistent with } \mathbf{\Sigma}\} = \hat{\mathbf{\Sigma}}_{\hat{\boldsymbol{K}}} \tag{57}$$

holds. Similarly, we have

$$\bigcup_{\mathbf{\Sigma} \in \mathbf{\Sigma}_{\mathrm{i/o}}} \{n\text{-dim. } \hat{\mathbf{\Sigma}} \text{ being consistent with } \mathbf{\Sigma}\} = \hat{\mathbf{\Sigma}}_{\mathrm{u/v}}. \tag{58}$$

Therefore, $\mathcal{D}$ is informative for stabilization by L-length IOH feedback, and $\hat{\boldsymbol{K}}$ is such a controller. Thus, from Lemma 7, we have (37). This implies that Property i) in Theorem 2 holds. Hence, $\mathcal{D}$ is informative for system identification. This completes the proof. □

Theorem 1 implies that finding a system that explains $\mathcal{D}$ is equivalent to finding a controller that stabilizes all systems explaining $\mathcal{D}$ in terms of data informativity.

We conclude this section by showing a *duality* between the former model and the latter controller in the VARX framework. Suppose that $\mathcal{D}$ is informative for stabilization by L-length IOH feedback. From Lemma 7, (37) and (54) hold. Because (35) holds, we have $\operatorname{im} P \subseteq \operatorname{im} \mathcal{P}$. Because $\operatorname{rank} P = Lm + n$ and (36) holds, the equality holds, i.e.,

$$\operatorname{im} P = \operatorname{im} \mathcal{P}. \tag{59}$$

Therefore, we assume that $\mathcal{P} = P$ without loss of generality. Then, (54) can be written as

$$\operatorname{im} \begin{bmatrix} P \\ KP \end{bmatrix} = \operatorname{im} \begin{bmatrix} PP^\mathsf{T} \\ K\mathcal{P}\mathcal{P}^\mathsf{T} \end{bmatrix} \subseteq \operatorname{im} \begin{bmatrix} \mathcal{V} \\ \mathcal{U} \end{bmatrix}. \tag{60}$$

Therefore, any controller $\hat{\boldsymbol{K}}$ in (30) that stabilizes all systems explaining $\mathcal{D}$ satisfies

$$K\mathcal{P}\mathcal{P}^\mathsf{T} = \mathcal{U}\mathcal{V}^\mathrm{R}, \tag{61}$$

where $\mathcal{V}^\mathrm{R}$ is a full-column rank matrix that satisfies $\mathcal{V}\mathcal{V}^\mathrm{R} = PP^\mathsf{T}$. On the other hand, because $\mathcal{D}$ is informative for system identification, any VARX model $\hat{\mathbf{\Sigma}}$ in (6) satisfies (44). By postmultiplying (44) with $\mathcal{V}^\mathrm{R}$, we obtain

$$\Sigma\mathcal{P}\mathcal{P}^\mathsf{T} = \mathcal{Y}\mathcal{V}^\mathrm{R}, \tag{62}$$

where $\mathcal{Y} := [y_L^\star, \dots, y_N^\star]$. Equations (61) and (62) show a dual relationship between models that explain $\mathcal{D}$ and controllers that stabilize all systems explaining $\mathcal{D}$.

VI. Conclusion

In this study, we showed that the informativity for the identification of partially observable systems must be equivalent to that for designing dynamical measurement-feedback stabilizers. This finding is entirely different from the input-state case in [11] and provides theoretical justification for imposing the strong persistency of excitation conditions [14] in both direct and indirect approaches to partially observable systems. Moreover, we showed a duality between the characterization of all VARX models explaining data and that of all VARX controllers that stabilize such VARX models. Future works include the extension of the results to nonlinear (and/or stochastic) systems based on the developed VARX framework.

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