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On equivariant holomorphic imbeddings of Siegel domains to compact complex homogeneous spaces

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Introduction

Let D be a Siegel domain of the second kind. If $Aut(D)$ is so "small" that $Aut(D) = Aff(D)$, the domain D can be equivariantly imbedded as an open set of complex projective space. In the case where $Aut(D)$ is the "largest", i.e., the domain D is symmetric, D can be also equivariantly imbedded as an open set of the hermitian symmetric space of compact type dual to D. Therefore it is natural to ask whether there exists an equivariant open imbedding of a Siegel domain D to a compact complex homogeneous space M. In this paper, we shall prove the following:

(*a*) *If there exists an open equivariant imbedding of D to M, then M must be a hermitian symmetric space of compact type* (*Theorem* 8).

(*b*) *D can be equivariantly imbedded as an open set of $P^n(\boldsymbol{C})$ if and only if $Aut(D) = Aff(D)$ or D is holomorphically equivalent to a disk* (*Theorem* 9).

(*c*) *There exists a Siegel domain which does not admit open equivariant imbeddings to compact complex homogeneous spaces* (§ 6).

Throughout this paper, we use the following notations: $Aut(M)$ means the group of all holomorphic transformations of a complex manifold M. For a real vector space or a real Lie algebra V, V^c denotes its complexification. We denote by $Gr(W, r)$ the complex grassmann manifold consisting of all r-dimensional subspaces of a complex vector space W.

§ 1. Homogeneous spaces associated with complex graded Lie algebras of certain type

Let $\mathcal{A}$ be a totally ordered abelian group which satisfies the following conditions;

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$$\alpha>\beta \text{ if and only if } \alpha-\beta>0.$$
$$\alpha>0 \text{ if and only if } -\alpha<0$$

Let $\mathfrak{g}$ be a finite dimensional complex Lie algebra and let $\{\mathfrak{g}_\alpha\}_{\alpha\in\mathcal{A}}$ be a family of subspaces of $\mathfrak{g}$ satisfying

$$\mathfrak{g}=\sum_{\alpha\in\mathcal{A}}\mathfrak{g}_\alpha \text{ (direct sum)}$$
$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta]\subset\mathfrak{g}_{\alpha+\beta}.$$

We further assume that there exists $\alpha_0<0$ such that

(1) For $X\in\mathfrak{g}_\alpha$ $\alpha\leqq\alpha_0$, the condition "$[X, \mathfrak{g}_0]=0$" implies $X=0$;
$\mathfrak{b}=\sum_{\alpha>\alpha_0}\mathfrak{g}_\alpha$ is a subalgebra of $\mathfrak{g}$.

Under these conditions we shall prove the following

Theorem 1. *Let G be a connected Lie group with the Lie algebra $\mathfrak{g}$ and let B be any Lie subgroup of G with the Lie algebra $\mathfrak{b}$. Then*
(i) *B is closed and connected.*
(ii) *The homogeneous space G/B is simply connected.*

Proof. We set $B^*=\{a\in G; Ad\,a\mathfrak{b}=\mathfrak{b}\}$. Then B^* is a closed subgroup of G containing B. From the condition (1), we can show that the Lie algebra of B^* coincides with $\mathfrak{b}$. Therefore the group B is closed in B^* and hence closed in G. We put $\mathfrak{n}=\sum_{\alpha\leqq\alpha_0}\mathfrak{g}_\alpha$. Clearly $\mathfrak{n}$ is a nilpotent subalgebra. We assert that the mapping $\pi\circ\exp$ is an imbedding of $\mathfrak{n}$ into G/B, where π denotes the natural projection of G onto G/B. Let N be a connected subgroup of G corresponding to the subalgebra $\mathfrak{n}$. It is sufficiant to prove that $N\cap B=\{e\}$, e denoting the unit element in G. Let $a\in N\cap B$. Since N is nilpotent, there exists $X\in\mathfrak{n}$ such that $\exp X=a$. We can write $X=\sum_{\alpha\leqq\alpha_0}X_\alpha$. For any $Y\in\mathfrak{g}_0$, $Ad(\exp X)Y\in\mathfrak{b}$. Suppose that $X_\alpha=0$ for $\alpha>\alpha'$ and $X_{\alpha'}\neq0$. Then the $\mathfrak{g}_{\alpha'}$-part of $Ad(\exp X)Y$ is equal to $[X_{\alpha'}, Y]$. Hence $[X_{\alpha'}, Y]=0$ for any $Y\in\mathfrak{g}_0$. It follows that $X_{\alpha'}=0$ by (1), contradicting the assumption $X_{\alpha'}\neq0$. Thus we get $X=0$, proving our assertion.

We set $N'=\pi\circ\exp\mathfrak{n}$. N' is an open orbite of N through the origin o of G/B. Let C be the union of all singular orbites of N and let $(X_1^*, \cdots, X_n^*)$ be a family of holomorphic vector fields on G/B corresponding to a base $(X_1, \cdots, X_n)$ of $\mathfrak{n}$. Then the subset C of G/B is defined by the equation $X_1^*\wedge\cdots\wedge X_n^*=0$. Therefore $G/B-C$ is connected and hence coincides with N'. Let B_0 be the identity component of B. Then G/B_0 is a covering space of G/B. We denote by ρ the covering map of G/B_0 to G/B. It is clear that the open (resp. a singular) orbite of N in G/B_0 is mapped by ρ to the open (resp. to a singular) orbite in G/B. On the other hand ρ is a homomorphism on the open orbite because $\pi\circ\exp$ is an imbedding on $\mathfrak{n}$. Therefore $G/B_0=G/B$ and hence $B_0=B$.

Let $\widetilde{G}$ be the universal covering group of G and let ϖ be the covering map of $\widetilde{G}$ to G. We now know that $\varpi^{-1}(B)$ is closed and connected. Therefore $G/B=\widetilde{G}/\varpi^{-1}(B)$ is simply connected. q.e.d.

§ 2. Siegel domains and equivariant holomorphic mappings

Let D be a Siegel domain of the second kind due to Pyatetski-Shapiro [5] and let $G(D)$ be the identity component of $Aut(D)$. Denote by $\mathfrak{g}(D)$ the Lie algebra of $G(D)$. For each $X\in\mathfrak{g}(D)$, X^* means the vector field on D generated by $\{\exp tX\}_{t\in\boldsymbol{R}}$. Then the correspondence: $X\rightarrow X^*$ can be extended to an injective linear mapping of $\mathfrak{g}(D)^c$ to the space of all vector fields on D by putting $(\sqrt{-1}X)^*=JX^*$ for $X\in\mathfrak{g}(D)$, where J denotes the complex structure on D. It is easy to see that for any point $z\in D$, $T_z(D)=\{X_z^*; X\in\mathfrak{g}(D)^c\}$.

We set

(2) $$\mathfrak{b}_z=\{X\in\mathfrak{g}(D)^c; X_z^*=0\}.$$

Then $\mathfrak{b}_z$ is a complex subalgebra of $\mathfrak{g}(D)^c$ and $\dim\mathfrak{b}_z$ is constant for any $z\in D$. Therefore the assignment: $z\rightarrow\mathfrak{b}_z$ gives a holomorphic mapping Φ of D into $Gr(\mathfrak{g}(D)^c, r)$, where $r=\dim\mathfrak{b}_z$.**) The group $G(D)$ acts on $Gr(\mathfrak{g}(D)^c, r)$ by its adjoint representation. Clearly

(3) $$\Phi(az)=Ad\,a\,\Phi(z) \quad \text{for } a\in G(D) \text{ and } z\in D.$$

Let M be a complex manifold such that $Aut(M)$ is a Lie group. A holomorphic mapping f of D to M will be called *equivariant* if there exists a homomorphism τ of $G(D)$ to $Aut(M)$ such that

(4) $$f(ap)=\tau(a)f(p) \quad \text{for } a\in G(D) \text{ and } p\in D.$$

By (3) the mapping Φ is equivariant.

Let f be a equivariant holomorphic mapping of D into M with a homomorphism τ: $G(D)\rightarrow Aut(M)$. We now assume that $Aut(M)$ is a complex Lie group and denote by $\mathfrak{g}(M)$ the Lie algebra of $Aut(M)$. Let τ_* be the homomorphism of $\mathfrak{g}(D)$ to $\mathfrak{g}(M)$ induced by τ. The mapping τ_* can be extended to a homomorphism of $\mathfrak{g}(D)^c$ to $\mathfrak{g}(M)$ complex linearly, which is denoted by the same letter τ_*. It follows that

(5) $$f_*X^*=(\tau_*X)^* \quad \text{for } X\in\mathfrak{g}(D),$$

where $(\tau_*X)^*$ denotes the vector field on M corresponding to $\tau_*X\in\mathfrak{g}(M)$.

**) J. Hano [1] constructed the mapping Φ for an effective homogeneous space G/K with an invariant complex structure and the non-degenerate canonical hermitian form.

Lemma 2. *The equation* (5) *holds for* $X\in\mathfrak{g}(D)^c$.

Proof. We set $X=X_1+\sqrt{-1}X_2\,(X_1, X_2\in\mathfrak{g}(D))$. Then $f_*X^*=f_*(X_1^*+JX_2^*)=f_*X_1^*+J'f_*X_2^*$, where J' denotes the complex structure on M. On the other hand $(\tau_*X)^*=(\tau_*X_1)^*+(\sqrt{-1}\tau_*X_2)^*=(\tau_*X_1)^*+J'(\tau_*X_2)^*$. q.e.d.

Proposition 3. *Let* $z\in D$ *and let* B' *be the isotropy subgroup of* $Aut(M)$ *at* $f(z)$. *Then* $f(D)\subset Aut(M)/B'$.

Proof. For any $z'\in D$, there exist $p_0, p_1, \cdots, p_m\in D$, $t_1, \cdots, t_m\in\boldsymbol{R}$ and $X_1, \cdots, X_m\in\mathfrak{g}(D)^c$ such that

$$p_0=z,\ p_m=z',$$
$$p_i=\exp t_iX_i^*(p_{i-1})\quad 1\leqq i\leqq m\,,$$

$\{\exp tX_i^*\}$ denoting the one parameter group of local transformations of D generated by the vector field X_i^*. It follows by Lemma 2

$$\begin{aligned} f(z')&=f(\exp t_mX_m^*(p_{m-1}))\\ &=\exp t_m\tau_*X_m\circ f(p_{m-1})\\ &=\exp t_m\tau_*X_m\circ\cdots\circ\exp t_1\tau_*X_1\circ f(z) \end{aligned}$$ q.e.d.

§ 3. The mapping Φ and Tanaka's imbeddings

The Lie algebra $\mathfrak{g}(D)$ has a graded structure such that (cf. [2])

$$\mathfrak{g}(D)=\mathfrak{g}_{-2}+\mathfrak{g}_{-1}+\mathfrak{g}_0+\mathfrak{g}_1+\mathfrak{g}_2\,.^{***)}$$

Let E and I are elements in $\mathfrak{g}_0$ as in § 1 of [3]. We set

$$\begin{aligned} &\theta_{-1}=\mathfrak{g}_{-2}^c+\{X-\sqrt{-1}[I,X]\,;X\in\mathfrak{g}_{-1}\}\\ &\theta_0=\{X+\sqrt{-1}[I,X]\,;X\in\mathfrak{g}_{-1}\}+\mathfrak{g}_0^c+\{Y-\sqrt{-1}[I,Y]\,;\,Y\in\mathfrak{g}_1\}\\ &\theta_1=\{Y+\sqrt{-1}[I,Y]\,;\,Y\in\mathfrak{g}_1\}+\mathfrak{g}_2^c\\ &H_0=\frac{1}{2}(E+\sqrt{-1}I)\,. \end{aligned}\tag{6}$$

Then

$$\begin{aligned} &\mathfrak{g}(D)^c=\theta_{-1}+\theta_0+\theta_1\ \text{(direct sum)}\\ &[\theta_\lambda,\theta_\mu]\subset\theta_{\lambda+\mu}\\ &\theta_\lambda=\{X\in\mathfrak{g}(D)^c;\,[H_0,X]=\lambda X\}\quad\lambda=-1,0,1\,. \end{aligned}\tag{7}$$

Let G^c be the adjoint group of $\mathfrak{g}(D)^c$. Since $G(D)$ is centerless ([2]),

***) Our suspace $\mathfrak{g}_\lambda$ corresponds to $\mathfrak{g}_{\lambda/2}$ in [2].

$G(D)$ is identified with a subgroup of G^c. Define a closed subgroup B of G^c by

(8) $$B=\{a\in G^c; a(\theta_0+\theta_1)=\theta_0+\theta_1\}.$$

The Lie algebra of B coincides with $\theta_0+\theta_1$ as is easily observed (cf. [8]). Hence B is connected by Theorem 1. According to [3], we identify the domain D with an open subset of θ_{-1} and define a holomorphic mapping h of D onto an open set of G^c/B by putting

(9) $$h(z)=\pi\circ\exp z \qquad z\in D,$$

where π denotes the projection of G^c onto G^c/B. The map h is equivariant (Lemma 2.4 in [3]) and called Tanaka's imbedding. We now fix a point $z\in D$. Let B_z be the isotropy subgroup of G^c at $h(z)$. Then from (9) we obtain

(10) $$B_z=\exp z\cdot B\cdot(\exp z)^{-1}.$$

For any $X\in\mathfrak{g}(D)^c$, the vector field on G^c/B generated by $\{\exp tX\}_{t\in\boldsymbol{R}}$ is equal to X^* on $D\subset G^c/B$. Therefore from (2), the Lie algebra of B_z coincides with $\mathfrak{b}_z$. Thus we get from (10),

(11) $$\mathfrak{b}_z=Ad\exp z(\theta_0+\theta_1).$$

The group G^c acts on $Gr(\mathfrak{g}(D)^c, r)$ in a natural manner. By (8), (10) and (11), the isotropy subgroup of G^c at $\Phi(z)$ is B_z. It follows from Proposition 3 that $\Phi(D)$ is contained in the homogeneous space G^c/B_z. We define a holomorphic diffeomorphism φ of G^c/B onto G^c/B_z by

$$\varphi(aB)=a(\exp z)^{-1}B_z \quad a\in G^c.$$

Then the following equality holds;

(12) $$\varphi(aq)=a\varphi(q) \qquad \text{for } a\in G^c,\ q\in G^c/B.$$

Lemma 4. $\Phi=\varphi\circ h$.

Proof. Let $a\in G(D)$. By (12), $\Phi(az)=a\Phi(z)=a\varphi\circ h(z)=\varphi\circ h(az)$. Therefore $\Phi=\varphi\circ h$ on the orbite of $G(D)$ through z. Since both Φ and $\varphi\circ h$ are holomorphic, we can conclude that $\Phi=\varphi\circ h$ on D (cf. Lemma 2.5 in [3]). q.e.d.

Corollary 5. *Φ is an imbedding of D.*

We can now verify the following

Theorem 6. *Let f be an equivariant holomorphic mapping of D into* *a complex manifold M. Assume that Aut(M) is a complex Lie group. Then there exists a unique holomorphic mapping* ψ: $G^c/B_z \rightarrow M$ *such that* $f=\psi\circ\Phi$.

Proof. The uniqueness follows from the fact that $\Phi(D)$ is open in G^c/B_z. Let $\tilde{G}^c$ be the universal covering group of G^c and let τ be the homomorphism of $G(D)$ to $Aut(M)$ attaches to f. We denote by $\tilde{\tau}$ the homomorphism of $\tilde{G}^c$ to $Aut(M)$ corresponding to the homomorphism τ_* of $\mathfrak{g}(D)^c$ to $\mathfrak{g}(M)$. Let $\tilde{B}_z$ be the connected subgroup of $\tilde{G}^c$ with the Lie algebra $\mathfrak{b}_z$. By Lemma 2, we see that for any $X\in\mathfrak{b}_z$ $(\tau_* X)^*_{f(z)}=f_* X^*_z=0$. Therefore $\tilde{\tau}(\tilde{B}_z)$ is contained in the isotropy subgroup of $Aut(M)$ at $f(z)$. Hence $\tilde{\tau}$ induces a holomorphic mapping ψ of $\tilde{G}^c/\tilde{B}_z$ to M. We now set

$$H=Ad(\exp z)H_0$$
$$\mathfrak{a}_\lambda=Ad(\exp z)\theta_\lambda \quad \lambda=-1,0,1\,. \tag{13}$$

It follows from (7)

$$\mathfrak{g}(D)^c=\mathfrak{a}_{-1}+\mathfrak{a}_0+\mathfrak{a}_1 \quad \text{(direct sum)}$$
$$[\mathfrak{a}_\lambda, a_\mu]\subset\mathfrak{a}_{\lambda+\mu} \tag{14}$$
$$\mathfrak{a}_\lambda=\{X\in\mathfrak{g}(D)^c;\ [H,X]=\lambda X\}.$$

By (14) the graded Lie algebra $\mathfrak{g}(D)^c=\mathfrak{a}_{-1}+\mathfrak{a}_0+\mathfrak{a}_1$ and the subalgebra $\mathfrak{b}_z=\mathfrak{a}_0+\mathfrak{a}_1$ satisfy the condition (1). Hence applying Theorem 1, we have $G^c/B_z=\tilde{G}^c/\tilde{B}_z$. The equation $f=\psi\circ\Phi$ follows from Proposition 3. q.e.d.

§ 4. Equivariant open imbeddings of Siegel domains to compact complex homogeneous spaces

Let f be an equivariant holomorphic immersion of a Siegel domain D to a compact complex homogeneous space M with a homomorphism τ: $G(D)\rightarrow Aut(M)$. Note that $Aut(M)$ is a complex Lie group. We further assume that $f(D)$ is open in M. Let τ_* be the homomorphism of $\mathfrak{g}(D)^c$ to $\mathfrak{g}(M)$ defined in § 2. Suppose that $\tau_* X=0$, $X\in\mathfrak{g}(D)^c$. By Lemma 2, $f_* X^*=(\tau_* X)^*=0$. Since f is an immersion, we have $X^*=0$. Therefore τ_* is injective. In what follows, we consider $\mathfrak{g}(D)^c$ as a complex subalgebra of $\mathfrak{g}(M)$. Let ψ be the holomorphic mapping of $G^c/B_z(=\tilde{G}^c/\tilde{B}_z)$ to M given by Theorem 6 and let B' be the isotropy subgroup of $Aut(M)$ at $f(z)$. Since f is an immersion, the Lie algebra of $\tau^{-1}(B')$ is $\mathfrak{b}_z$. Therefore by Theorem 1, $\tilde{B}_z=\tilde{\tau}^{-1}(B')$ and hence ψ is an imbedding.

Lemma 7. *Let* λ *be an eigenvalue of ad* H *on* $\mathfrak{g}(M)$, *where* H *is an element of* $\mathfrak{g}(D)^c$ *defined by* (13). *Then* λ *is an integer and* $\lambda\geqq-1$.

Proof. Let $\mathfrak{b}'$ be the Lie algebra of B'. Since $f(D)$ is open, we know $\dim \mathfrak{g}(D)^c/\mathfrak{b}_z = \dim \mathfrak{g}(M)/\mathfrak{b}'$. Define subspaces $\{\mathfrak{b}'_m\}_{m\geq -1}$ of $\mathfrak{g}(M)$ by setting

$$\mathfrak{b}'_{-1} = \mathfrak{g}(M), \quad \mathfrak{b}'_0 = \mathfrak{b}'.$$
$$\mathfrak{b}'_m = \{X \in \mathfrak{b}'_{m-1};\ [X, \mathfrak{g}(M)] \subset \mathfrak{b}'_{m-1}\} \quad (m \geqq 1).$$

Let $X \in \mathfrak{b}'_m$. Then all m'-th derivatives of X^* at $f(z)$ must be zeros where $m' < m$. Therefore $\bigcap_{m=-1}^{\infty} \mathfrak{b}'_m = 0$. As a consequence, there exists $m_0 \geqq 0$ such that $\mathfrak{b}'_m = 0$ for $m > m_0$ and $\mathfrak{b}'_{m_0} \neq 0$. Thus we get a sequence $\mathfrak{b}'_{-1} \supsetneqq \mathfrak{b}'_0 \supsetneqq \cdots \supsetneqq \mathfrak{b}'_{m_0} \supsetneqq \mathfrak{b}'_{m_0+1} = 0$. It is easy to see the following equality holds

$$ad H X \equiv mX \pmod{\mathfrak{b}'_{m+1}} \quad \text{for } X \in \mathfrak{b}'_m. \qquad \text{q.e.d.}$$

We now set

$$\mathfrak{a}'_\lambda = \{X \in \mathfrak{g}(M);\ (adH - \lambda)^m = 0 \text{ for some } m\} \tag{15}$$

Then

$$\mathfrak{g}(M) = \sum_{\lambda \geq -1} \mathfrak{a}'_\lambda, \quad \mathfrak{b}'_\mu = \sum_{\lambda \geq \mu} \mathfrak{a}'_\lambda$$
$$[\mathfrak{a}'_\lambda, \mathfrak{a}'_\mu] \subset \mathfrak{a}'_{\lambda+\mu}.$$

By (14) and (15) we have $\mathfrak{a}_\lambda \subset \mathfrak{a}'_\lambda$. Therefore $\mathfrak{a}'_{-1} = \mathfrak{a}_{-1}$ because $\dim \mathfrak{a}'_{-1} = \dim \mathfrak{a}_{-1}$. From (15), we see that the pair $(\mathfrak{g}(M), \mathfrak{b}')$ satisfies (1) and hence the homogeneous space M is simply connected. As a result M is a C-space due to Wang [9] and hence $\mathfrak{g}(M)$ is reductive ([9]). Let X be an element in the center of $\mathfrak{g}(M)$. We can write $X = \sum_{\lambda=-1}^{m_0} X_\lambda$ $(X_\lambda \in \mathfrak{a}'_\lambda)$. Since $[H, X] = 0$, we have $X_\lambda = 0$ for $\lambda \neq 0$. Then $[\mathfrak{g}(M), X_0] = [\mathfrak{g}(M), X] = 0$. Thus we know $X_0 \in \mathfrak{b}'_1$ and hence $X_0 = 0$. Consequently the Lie algebra $\mathfrak{g}(M)$ is semi-simple and $\mathfrak{a}'_\lambda = 0$ for $\lambda > 1$. We now know from [7] that M is a hermitian symmetric space of compact type. We have thereby prove the following

Theorem 8. *If there exists an open immersion f of a Siegel domain D to a compact complex homogeneous space M, then M must be a hermitian symmetric space of compact type and f is an imbedding.*

Remark. Note that the following equility hold: $\mathfrak{a}'_\lambda = \{X \in \mathfrak{g}(M);\ [H, X] = \lambda X\}$ $\lambda = -1, 0, 1$. Indeed, an endomorphism η defined by $\eta(X) = \lambda X$ for $X \in \mathfrak{a}'_\lambda$ is a derivation of the semi-simple Lie algebra $\mathfrak{g}(M)$. Therefore there exists $H' \in \mathfrak{g}(M)$ such that $adH' = \eta$. It is easy to see that $H' \in \mathfrak{a}'_0$. Both H and H' are in $\mathfrak{a}'_0$ and $adH = adH'$ on $\mathfrak{a}'_{-1}$. Hence we get $H = H'$.

§ 5. Siegel domains which can be equivariantly imbedded as an open subset of $P^n(C)$

In this section, we shall determine Siegel domains which admit equivariant open imbeddings to the complex projective space $P^n(\boldsymbol{C})$. We prove the following

Theorem 9. *Let D be a Siegel domain of the second kind. Then D can be equivariantly imbedded as an open set of $P^n(\boldsymbol{C})$ if and only if D is one of the following two type*;

(i) *D is holomorphically equivalent to the disk, i.e.,*

$$D\cong\{(z_1, \cdots, z_n)\in\boldsymbol{C}^n; \sum_{i=1}^{n}|z_i|^2<1\},$$

(ii) *$Aut(D)=Aut(D)$, where $Aff(D)$ denotes the affine transformation group of D.*

It is well known that if D satisfies (i) or (ii), then D can be imbedded equivariantly as an open set of $P^n(\boldsymbol{C})$. We verify the converse.

Let f be an open equivariant imbedding of D to $P^n(\boldsymbol{C})$ with a homomorphism $\tau: G(D)\rightarrow SL(n+1, \boldsymbol{C})$. Let $z\in D$ and let $(w_1, \cdots, w_{n+1})$ be a homogeneous coordinate system of $P^n(\boldsymbol{C})$. We may assume $f(z)=(1, 0, \cdots, 0)$. As in § 4, we identify $\mathfrak{g}(D)^c$ with a subalgebra of $\mathfrak{sl}(n+1, \boldsymbol{C})$ by τ_*. Let $\mathfrak{a}'_\lambda$ $(\lambda=-1, 0, 1)$ be the subspaces of $\mathfrak{sl}(n+1, \boldsymbol{C})$ defined by (15).

Lemma 10. *Let $Y\in\mathfrak{a}'_1$. If $Y\neq 0$, the linear mapping adY is injective on $\mathfrak{a}'_{-1}$.*

Proof. Let $\mathfrak{b}'$ be the isotropy subalgebra at $f(z)$. Then $\mathfrak{b}'=\mathfrak{a}'_0+\mathfrak{a}'_1$ and

$$\mathfrak{b}'=\left\{X\in\mathfrak{sl}(n+1, \boldsymbol{C});\ X=\begin{pmatrix}-\mathrm{Tr}\,A & \xi\\ 0 & A\end{pmatrix};\ \xi\in\boldsymbol{C}^n \text{ and } A\in\mathfrak{gl}(n, \boldsymbol{C})\right\}.$$

We set

$$\mathfrak{a}''_{-1}=\left\{X\in\mathfrak{sl}(n+1, \boldsymbol{C});\ X=\begin{pmatrix}0 & 0\\ {}^t\eta & 0\end{pmatrix};\ \eta\in\boldsymbol{C}^n\right\}$$

$$\mathfrak{a}''_0=\left\{X\in\mathfrak{sl}(n+1, \boldsymbol{C});\ X=\begin{pmatrix}-\mathrm{Tr}\,A & 0\\ 0 & A\end{pmatrix};\ A\in\mathfrak{gl}(n, \boldsymbol{C})\right\}$$

$$\mathfrak{a}''_1=\left\{X\in\mathfrak{sl}(n+1, \boldsymbol{C});\ X=\begin{pmatrix}0 & \xi\\ 0 & 0\end{pmatrix};\ \xi\in\boldsymbol{C}^n\right\}.$$

Since $\mathfrak{a}'_1=\{X\in\mathfrak{b}'; [\mathfrak{sl}(n+1, \boldsymbol{C}), X]\subset\mathfrak{b}'\}$. We can show $\mathfrak{a}''_1=\mathfrak{a}'_1$. There exists a unique element $H^\#$ of $\mathfrak{a}''_0$ such that $adH^\# X=\lambda X$ for $X\in\mathfrak{a}''_1$ $(\lambda=-1, 0, 1)$, because $\mathfrak{sl}(n+1, \boldsymbol{C})$ is semi-simple. Both H and $H^\#$ are in $\mathfrak{b}'$ and $adHX\equiv adH^\# X\equiv -X(\mathrm{mod}\ \mathfrak{b}')$ for any $X\in\mathfrak{sl}(n+1, \boldsymbol{C})$. Therefore we can write $H^\#-H=Z$, $Z\in\mathfrak{a}'_1$. Then $H=Ad(\exp Z)H^\#$ and hence $\mathfrak{a}'_\lambda=Ad(\exp Z)\mathfrak{a}''_\lambda$.

By direct calculations, we know that for any $Y \in \mathfrak{a}_1''$ $(Y \neq 0)$, adY is injective on $\mathfrak{a}_{-1}''$. q.e.d.

We assume that $Aut(D) \neq Aff(D)$, which is equivalent to $\mathfrak{g}_2 \neq 0$ ([3]). Suppose that $\mathfrak{g}(D)$ is not semi-simple. Then there exists a non-zero element X in $\mathfrak{a}_{-2}$ such that $[X, \mathfrak{g}_2] = 0$ (Remark 1 in [4]), which contradicts to Lemma 10 because $Ad(\exp z)X$ is in $\mathfrak{a}_{-1}'$ and $Ad(\exp z)\mathfrak{g}_2 \subset \mathfrak{a}_1'$. Therefore $\mathfrak{g}(D)$ is semi-simple and the domain D is symmetric. When D is symmetric, it is well known that G^c/B is the compact hermitian symmetric space dual to non-compact D and that h is the Borel imbedding (cf. [6]). Since $\psi(G^c/B_z)$ is a compact open subset of $P^n(\boldsymbol{C})$, we get $G^c/B \cong G^c/B_z \cong P^n(\boldsymbol{C})$, completing the proof of Theorem 9.

§ 6. An example

Let D be a 4-dimensional homogeneous Siegel domain defined by

$$D = \{(z_1, z_2, z_3, z_4) \in \boldsymbol{C}^4 ; \operatorname{Im} z_3 > 0 \quad \text{and} \quad \operatorname{Im} z_3 (\operatorname{Im} z_1 - |z_4|^2) - (\operatorname{Im} z_2)^2 > 0\}.$$

The domain D is not symmetric. In this section we prove that D does not admit open equivariant imbeddings to compact complex homogeneous spaces. We first calculate $\mathfrak{g}(D)$. Let $\mathfrak{r}$ denote the radical of $\mathfrak{g}(D)$. Then $\mathfrak{r}$ is of the form: $\mathfrak{r} = \mathfrak{r}_{-2} + \mathfrak{r}_{-1} + \mathfrak{r}_0$, $\mathfrak{r}_\lambda \subset \mathfrak{g}_\lambda$ ([2]). By using results in [4], we have

$$\dim \mathfrak{g}_{-2} = 3, \quad \dim \mathfrak{g}_{-1} = 2, \quad \dim \mathfrak{g}_0 = 4,$$
$$\dim \mathfrak{g}_1 = 0, \quad \dim \mathfrak{g}_2 = 1.$$

By (6) we can decompose $\mathfrak{r}^c$ into the form: $\mathfrak{r}^c = \mathfrak{r}_{-1}' + \mathfrak{r}_0'$, where $\mathfrak{r}_\lambda' = \mathfrak{r}^c \cap \theta_\lambda$. By Theorem 1. 1 in [4], there exists a semi-simple subalgebra $\mathfrak{s}$ of $\mathfrak{g}(D)^c$ such that

$$\mathfrak{s} = \mathfrak{s}_{-1} + \mathfrak{s}_0 + \mathfrak{s}_1, \quad \mathfrak{s}_\lambda = \mathfrak{s} \cap \theta_\lambda,$$
$$\mathfrak{g}(D)^c = \mathfrak{r}^c + \mathfrak{s} \quad \text{(direct sum)}.$$

Since $\mathfrak{s}_1 = \theta_1 = \mathfrak{g}_2^c$, $\dim_{\boldsymbol{C}} \mathfrak{s}_1 = 1$ and hence $\dim_{\boldsymbol{C}} \mathfrak{s}_{-1} = 1$. Because $\mathfrak{s}$ is semi-simple, there exists a unique $H_s \in \mathfrak{s}_0$ such that $[H_s, X] = \lambda X$ for $X \in \mathfrak{s}_\lambda$.

Suppose that there exists an equivariant imbedding of D onto an open set of a compact complex homogeneous space M. By Theorem 8, M must be a hermitian symmetric space of compact type. We also know that $\mathfrak{g}(D)^c$ is identified with a subalgebra of $\mathfrak{g}(M)$. We set $\theta_1' = Ad(\exp z)^{-1}\mathfrak{a}_\lambda'$ where $\mathfrak{a}_\lambda'$ is as in § 4. Then $\mathfrak{g}(M) = \theta_{-1}' + \theta_0' + \theta_1'$ and $\theta_\lambda \subset \theta_\lambda'$. We assert that M must be irreducible. If M is a product of two hetmitian symmetric spaces M_1 and M_2, then $\mathfrak{g}(M) = \mathfrak{g}(M_1) + \mathfrak{g}(M_2)$. Being an ideal of $\mathfrak{g}(M)$, $\mathfrak{g}(M_i)$ is decomposed as follows $(i = 1, 2)$;

$$\mathfrak{g}(M_i) = \theta_{-1}^i + \theta_0^i + \theta_1^i, \quad \theta_\lambda^i = \mathfrak{g}(M_i) \cap \theta_\lambda'.$$

Note that $\theta^i_{-1}\neq 0$. Let $X\in\theta^i_{-1}$. Since $\theta_{-1}=\theta'_{-1}$, we can write $X=X_s+X_r$ ($X_s\in\mathfrak{s}_{-1}$ and $X_r\in\mathfrak{r}'_{-1}$). If $X_s\neq 0$, then $\mathfrak{s}_1$ is contained in θ^i_1 because $\mathfrak{s}_1=[\mathfrak{s}_1, [\mathfrak{s}_1, \mathfrak{s}_{-1}]]$ and $\dim_C \mathfrak{s}_1=1$. Hence $\mathfrak{s}_{-1}$ is contained in θ^i_{-1} because $\mathfrak{s}_{-1}=[\mathfrak{s}_{-1}, [\mathfrak{s}_{-1}, \mathfrak{s}_1]]$. As a result $\mathfrak{s}_{-1}$ is contained in θ^1_{-1} or θ^2_{-1}. We may assume that $\mathfrak{s}_{-1}\subset\theta^1_{-1}$. It follows that $\mathfrak{s}_1\subset\theta^1_1$. For the domain D, we know from [4] that $\mathfrak{g}_{-2}=[\mathfrak{g}_{-2}, [\mathfrak{g}_{-2}, \mathfrak{g}_2]]$. Therefore $\mathfrak{g}_{-2}\subset\theta^1_{-1}$. Now let $Y\in\theta^2_{-1}$. We can write $Y=Y_{-2}+Y_{-1}-\sqrt{-1}[I, Y_{-1}]$, where $Y_{-2}\in\mathfrak{g}^c_{-2}$ and $Y_{-1}\in\mathfrak{g}_{-1}$. Since $[E, Y_\lambda]=\lambda Y_\lambda$ ($\lambda=-2, -1$), we have $Y_{-2}\in\theta^2_{-1}$ and $Y_{-1}-\sqrt{-1}[I, Y_{-1}]\in\theta^2_{-1}$. Hence $Y_{-2}=0$ and $[Y_{-1}-\sqrt{-1}[I, Y_{-1}], Y_{-1}+\sqrt{-1}[I, Y_{-1}]]=2\sqrt{-1}[Y_{-1}, [I, Y_{-1}]]\in\mathfrak{g}^c_{-1}\cap\theta^2_{-1}=0$. Consequently $Y_{-1}=0$ and hence $\theta^2_{-1}=0$. This is a contradiction. We have thus proved that M is irreducible.

A 4-dimensional irreducible hermitian symmetric space of compact type is $SU(5)/S(U_1\times U_4)$ or $SU(4)/S(U_2\times U_2)$. Hence M must be $SU(4)/S(U_2\times U_2)$ by Theorem 9. Then $\dim_C \mathfrak{g}(M)=\dim_R SU(4)=15$. Let $\mathfrak{t}_\lambda=\{X\in\mathfrak{g}(M);(adE-\lambda)^m X=0$ for some $m\}$. Then $\mathfrak{g}(M)=\sum_{\lambda\in C}\mathfrak{t}_\lambda$. Since $\mathfrak{g}(M)$ is semi-simple, $\dim_C \mathfrak{t}_\lambda=\dim_C \mathfrak{t}_{-\lambda}$. By using the fact $\mathfrak{g}_\lambda\subset\mathfrak{t}_\lambda$, we have $\dim_C \mathfrak{t}_{-2}=\dim_C \mathfrak{t}_2\geqq 3$, $\dim_C \mathfrak{t}_{-1}=\dim_C \mathfrak{t}_1\geqq 2$ and $\dim_C \mathfrak{t}_0\geqq 4$. Since $\dim_C\mathfrak{g}(M)=15$, the only possible case is the following: $\mathfrak{g}(M)=\mathfrak{t}_{-2}+\mathfrak{t}_{-1}+\mathfrak{t}_0+\mathfrak{t}_1+\mathfrak{t}_2$, $\dim_C \mathfrak{t}_{-2}=\dim_C \mathfrak{t}_2=3$, $\dim_C \mathfrak{t}_{-1}=\dim_C \mathfrak{t}_1=2$ and $\dim_C \mathfrak{t}_0=5$. In this case $\mathfrak{t}_{-2}=\mathfrak{g}^c_{-2}$ and $\mathfrak{t}_{-1}=\mathfrak{g}^c_{-1}$. Therefore $\mathfrak{t}_{-2}\neq[\mathfrak{t}_{-1}, \mathfrak{t}_{-1}]$. On the other hand, $\mathfrak{t}_{-2}=[\mathfrak{t}_{-1}, \mathfrak{t}_{-1}]$ because $\mathfrak{g}(M)$ is a simple Lie algebra (Lemma 1. 3 in [4]). Thus we have a contradiction.

DEPARTMENT OF MATHEMATICS
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