

## ON ESTIMATES FOR THE BESSEL TRANSFORM

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**Abstract.** Using a Bessel translation operator, we obtain a generalization of Theorem 2.2 in [3] for the Bessel transform for functions satisfying the  $(\psi, \delta, \beta)$ -Bessel Lipschitz condition in the space  $L_{2,\alpha}(\mathbb{R}^+)$ .

### 1. Introduction and preliminaries

Integral transforms and their inverses (e.g., the Bessel transform) are widely used to solve various problems in calculus, mechanics, mathematical physics, and computational mathematics (see, e.g., [6, 7, 9, 10]).

Let

$$B = \frac{d^2}{dt^2} + \frac{(2\alpha + 1)}{t} \frac{d}{dt},$$

be the Bessel differential operator. For  $\alpha > -\frac{1}{2}$ , we introduce the Bessel normalized function of the first kind  $j_\alpha$  defined by

$$j_\alpha(x) = \Gamma(\alpha + 1) \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n}}{n! \Gamma(n + \alpha + 1)},$$

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where  $\Gamma(x)$  is the gamma-function (see [5]). The function  $y = j_\alpha(x)$  satisfies the differential equation

$$By + y = 0$$

with the initial conditions  $y(0) = 1$  and  $y'(0) = 0$ . The function  $j_\alpha(x)$  is infinitely differentiable and even.

LEMMA 1.1. *The following inequalities are valid for Bessel function  $j_\alpha$*

- (1)  $|j_\alpha(x)| \leq 1$
- (2)  $1 - j_\alpha(x) = O(x^2)$ ,  $0 \leq x \leq 1$ .

PROOF. (See [1]) □

LEMMA 1.2. *The following inequality is true*

$$|1 - j_\alpha(x)| \geq c,$$

with  $x \geq 1$ , where  $c > 0$  is a certain constant.

PROOF. The asymptotic formulas for the Bessel function imply that  $j_\alpha(x) \rightarrow 0$  as  $x \rightarrow \infty$ . Consequently, a number  $x_0 > 0$  exists such that with  $x \geq x_0$  the inequality  $|j_\alpha(x)| \leq \frac{1}{2}$  is true. Let  $m = \min_{x \in [1, x_0]} |1 - j_\alpha(x)|$ . With  $x \geq 1$  we get the inequality

$$|1 - j_\alpha(x)| \geq c,$$

where  $c = \min(\frac{1}{2}, m)$ . □

Assume that  $L_{2,\alpha}(\mathbb{R}^+)$ ,  $\alpha > -\frac{1}{2}$ , is the Hilbert space of measurable functions  $f(x)$  on  $\mathbb{R}^+$  with the finite norm

$$\|f\| = \|f\|_{2,\alpha} = \left( \int_0^\infty |f(t)|^2 t^{2\alpha+1} dt \right)^{1/2}.$$

It is well known that the Bessel transform of a function  $f \in L_{2,\alpha}(\mathbb{R}^+)$  is defined (see [4, 5, 8]) by the formula

$$\widehat{f}(\lambda) = \int_0^\infty f(t) j_\alpha(\lambda t) t^{2\alpha+1} dt, \quad \lambda \in \mathbb{R}^+.$$

The inverse Bessel transform is given by the formula

$$f(t) = (2^\alpha \Gamma(\alpha + 1))^{-2} \int_0^\infty \widehat{f}(\lambda) j_\alpha(\lambda t) \lambda^{2\alpha+1} d\lambda.$$

THEOREM 1.3 ([4]). *If  $f \in L_{2,\alpha}(\mathbb{R}^+)$  then we have the Parseval's equality*

$$\|\widehat{f}\| = (2^\alpha \Gamma(\alpha + 1)) \|f\|.$$

In  $L_{2,\alpha}(\mathbb{R}^+)$ , consider the Bessel translation operator  $T_h$

$$T_h f(t) = c_\alpha \int_0^\pi f(\sqrt{t^2 + h^2 - 2th \cos \varphi}) \sin^{2\alpha} \varphi d\varphi,$$

where

$$c_\alpha = \left( \int_0^\pi \sin^{2\alpha} \varphi d\varphi \right)^{-1} = \frac{\Gamma(\alpha + 1)}{\Gamma(1/2)\Gamma(\alpha + \frac{1}{2})}.$$

It is easy to see that

$$T_0 f(x) = f(x).$$

The operator  $T_h$  is linear, homogeneous, and continuous. Below are some properties of this operator (see [5]):

- (1)  $T_h j_\alpha(\lambda x) = j_\alpha(\lambda h) j_\alpha(\lambda x)$ .
- (2)  $T_h$  is self-adjoint: If  $f(x)$  is continuous function such that

$$\int_0^\infty x^{2\alpha+1} |f(x)| dx < \infty$$

and  $g(x)$  is continuous and bounded for all  $x \geq 0$ , then

$$\int_0^\infty (T_h f(x)) g(x) x^{2\alpha+1} dx = \int_0^\infty f(x) (T_h g(x)) x^{2\alpha+1} dx.$$

- (3)  $T_h f(x) = T_x f(h)$ .
- (4)  $\|T_h f - f\| \rightarrow 0$  as  $h \rightarrow 0$ .

The following relation connect the Bessel translation operator, in [2], we have

$$(1.1) \quad \widehat{(T_h f)}(\lambda) = j_\alpha(\lambda h) \widehat{f}(\lambda).$$

For any function  $f(x) \in L_{2,\alpha}(\mathbb{R}^+)$  we define differences of the order  $m$  such that  $m \in \{1, 2, \dots\}$  with a step  $h > 0$  by

$$(1.2) \quad \Delta_h^m f(x) = (\mathbb{T}_h - \mathbb{I})^m f(x),$$

where  $\mathbb{I}$  is the unit operator.

LEMMA 1.4. *Let  $f \in L_{2,\alpha}(\mathbb{R}^+)$ . Then*

$$\|\Delta_h^m f(x)\|^2 = \frac{1}{(2^\alpha \Gamma(\alpha + 1))^2} \int_0^\infty |1 - j_\alpha(\lambda h)|^{2m} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda.$$

PROOF. From formulas (1.1) and (1.2), we have

$$(\widehat{\Delta_h^m f})(\lambda) = (j_\alpha(\lambda h) - 1)^m \widehat{f}(\lambda).$$

By Parseval's identity, we obtain the result.  $\square$

In [3], we have

THEOREM 1.5. *Let  $f \in L_{2,\alpha}(\mathbb{R}^+)$ . Then the following are equivalents*

- (1)  $f \in \text{Lip}(\psi, \alpha, 2)$
  - (2)  $\int_r^\infty |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} s\lambda = O(\psi(r^{-2}))$  as  $h \rightarrow +\infty$ ,
- where  $\text{Lip}(\psi, \alpha, 2)$  is the  $\psi$ -Bessel Lipschitz class.

The main aim of this paper is to establish a generalization of Theorem 1.5 in the Bessel transform. For this purpose, we use the Bessel translation operator.

## 2. Main Results

In this section we give the main result of this paper. We need first to define  $(\psi, \delta, \beta)$ -Bessel Lipschitz class.

DEFINITION 2.1. *A function  $f \in L_{2,\alpha}(\mathbb{R}^+)$  is said to be in the  $(\psi, \delta, \beta)$ -Bessel Lipschitz class, denote by  $\text{Lip}^2(\psi, \delta, \beta)$ , if*

$$\|\Delta_h^m f(t)\| = O(h^\delta \psi(h^\beta)) \quad \text{as } h \rightarrow 0,$$

where

- (1)  $\delta > m$ ,  $\beta > 0$  and  $m \in \{1, 2, \dots\}$ ,
- (2)  $\psi$  is a continuous increasing function on  $[0, \infty)$ ,
- (3)  $\psi(0) = 0$  and  $\psi(ts) = \psi(t)\psi(s)$  for all  $t, s \in [0, \infty)$ ,
- (4) and

$$\int_0^{1/h} s^{2m-2\delta-1} \psi(s^{-2\beta}) ds = O(h^{2\delta-2m} \psi(h^{2\beta})) \quad \text{as } h \rightarrow 0.$$

**THEOREM 2.2.** *Let  $f \in L_{2,\alpha}(\mathbb{R}^+)$ . Then the following are equivalent*

- (1)  $f \in \text{Lip}^2(\psi, \delta, \beta)$ ,
- (2)  $\int_r^\infty |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda = O(r^{-2\delta} \psi(r^{-2\beta}))$  as  $r \rightarrow +\infty$ .

**PROOF.** (1)  $\implies$  (2): Assume that  $f \in \text{Lip}^2(\psi, \delta, \beta)$ . Then

$$\|\Delta_h^m f(t)\| = O(h^\delta \psi(h^\beta)) \quad \text{as } h \rightarrow 0.$$

Lemma 1.4 gives

$$\|\Delta_h^m f(x)\|^2 = \frac{1}{(2^\alpha \Gamma(\alpha + 1))^2} \int_0^\infty |1 - j_\alpha(\lambda h)|^{2m} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda.$$

If  $\lambda \in [\frac{1}{h}, \frac{2}{h}]$  then  $\lambda h \geq 1$  and Lemma 1.2 implies that

$$1 \leq \frac{1}{c^{2m}} |1 - j_\alpha(\lambda h)|^{2m}.$$

Then

$$\begin{aligned} \int_{1/h}^{2/h} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda &\leq \frac{1}{c^{2m}} \int_{1/h}^{2/h} |1 - j_\alpha(\lambda h)|^{2m} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda \\ &\leq \frac{1}{c^{2m}} \int_0^\infty |1 - j_\alpha(\lambda h)|^{2m} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda \end{aligned}$$

and there exists a positive constant  $C$  such that

$$\int_{1/h}^{2/h} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda \leq C h^{2\delta} \psi(h^{2\beta}).$$

We obtain

$$\int_r^{2r} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda \leq C r^{-2\delta} \psi(r^{-2\beta}).$$

So that

$$\begin{aligned}
 \int_r^\infty |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda &= \left[ \int_r^{2r} + \int_{2r}^{4r} + \int_{4r}^{8r} + \dots \right] |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda \\
 &\leq C(r^{-2\delta} \psi(r^{-2\beta}) + (2r)^{-2\delta} \psi((2r)^{-2\beta}) + \dots) \\
 &\leq Cr^{-2\delta} \psi(r^{-2\beta}) (1 + 2^{-2\delta} \psi(2^{-2\beta}) \\
 &\quad + (2^{-2\delta} \psi(2^{-2\beta}))^2 + (2^{-2\delta} \psi(2^{-2\beta}))^3 + \dots) \\
 &\leq CK_{\delta,\beta} r^{-2\delta} \psi(r^{-2\beta}),
 \end{aligned}$$

where  $K_{\delta,\beta} = (1 - 2^{-2\delta} \psi(2^{-2\beta}))^{-1}$  since  $2^{-2\delta} \psi(2^{-2\beta}) < 1$ . This proves that

$$\int_r^\infty |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda = O(r^{-2\delta} \psi(r^{-2\beta})) \quad \text{as } r \rightarrow +\infty.$$

(2)  $\implies$  (1): Suppose now that

$$\int_r^\infty |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda = O(r^{-2\delta} \psi(r^{-2\beta})) \quad \text{as } r \rightarrow +\infty.$$

We have to show that

$$\int_0^\infty |1 - j_\alpha(\lambda h)|^{2m} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda = O(h^{2\delta} \psi(h^{2\beta})) \quad \text{as } h \rightarrow 0.$$

We write

$$\int_0^\infty |1 - j_\alpha(\lambda h)|^{2m} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda = I_1 + I_2,$$

where

$$I_1 = \int_0^{1/h} |1 - j_\alpha(\lambda h)|^{2m} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda$$

and

$$I_2 = \int_{1/h}^\infty |1 - j_\alpha(\lambda h)|^{2m} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda.$$

Firstly, we have from (1) in Lemma 1.1

$$I_2 \leq 4^m \int_{1/h}^\infty |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda = O(h^{2\delta} \psi(h^{2\beta})).$$

Set

$$g(x) = \int_x^\infty |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda.$$

From (1) and (2) of Lemma 1.1 and integration by parts, we obtain

$$\begin{aligned} I_1 &= \int_0^{1/h} |1 - j_\alpha(\lambda h)|^{2m} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda \\ &\leq 2^m \int_0^{1/h} |1 - j_\alpha(\lambda h)|^m |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda \\ &\leq -C_1 h^{2m} \int_0^{1/h} x^{2m} g'(x) dx \\ &\leq -C_1 g(1/h) + 2m C_1 h^{2m} \int_0^{1/h} x^{2m-1} g(x) dx \\ &\leq C_2 h^{2m} \int_0^{1/h} x^{2m-1} x^{-2\delta} \psi(x^{-2\beta}) dx \\ &\leq C_2 h^{2m} \int_0^{1/h} x^{2m-1-2\delta} \psi(x^{-2\beta}) dx \\ &\leq C_3 h^{2m} h^{2\delta-2m} \psi(h^{2\beta}) \\ &\leq C_3 h^{2\delta} \psi(h^{2\beta}), \end{aligned}$$

where  $C_1$ ,  $C_2$  and  $C_3$  are a positive constants and this ends the proof.  $\square$

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