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ON ESTIMATES OF POISSON KERNELS FOR SYMMETRIC LÉVY PROCESSES

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Abstract. In this paper, using elementary calculus only, we give a simple proof that Green function estimates imply the sharp two-sided pointwise estimates for Poisson kernels for subordinate Brownian motions. In particular, by combining the recent result of Kim and Mimica [5], our result provides the sharp two-sided estimates for Poisson kernels for a large class of subordinate Brownian motions including geometric stable processes.

1. Introduction and main result

The purpose of this paper is to serve as a reference to the sharp two-sided pointwise estimates for Poisson kernel for a large class of symmetric Lévy processes.

Typically, the infinitesimal generators of general Lévy processes in $\mathbb{R}^d$ are not differential operators but non-local (or integro-differential) operators. Although integro-differential operators are also very important in the theory of partial differential equations, general Lévy processes and corresponding integro-differential operators are not easy to deal with. The investigation on fine potential-theoretic properties of Lévy processes corresponding to integro-differential operators in the Euclidean space began in the late 1990's with the study of symmetric stable processes (equivalently, fractional Laplacian). One of the first results obtained in this area was the sharp Green function and Poisson kernel estimates of symmetric α-stable processes in bounded $C^{1,1}$ domains in $\mathbb{R}^d$, $0 < \alpha < 2$, $d \geq 2$ (see [2, 10]). Very recently in [5, 6], Green function estimates are established for a large class of subordinate Brownian motions in bounded $C^{1,1}$ open sets. The goal of this paper is to obtain Poisson kernel estimates for subordinate Brownian motions in bounded $C^{1,1}$ open sets.

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A subordinate Brownian motion in $\mathbb{R}^d$ is a Lévy process which can be obtained by replacing the time of Brownian motion in $\mathbb{R}^d$ by an independent subordinator. More precisely, let $B = (B_t : t \geq 0)$ be a Brownian motion in $\mathbb{R}^d$ (our Brownian motion B runs at twice the usual speed) and $S = (S_t : t \geq 0)$ be a subordinator (i.e., an increasing Lévy process in $\mathbb{R}^d$) independent of B whose Laplace exponent is ϕ, that is, $\mathbb{E}[\exp\{-\lambda S_t\}] = \exp\{-t\phi(\lambda)\}$, $\lambda > 0$. The process $X = (X_t : t \geq 0)$ defined by $X_t = B_{S_t}$ is a rotationally invariant Lévy process in $\mathbb{R}^d$ and is called a subordinate Brownian motion. The characteristic exponent Φ of the subordinate Brownian motion X is $\Phi(x) = \phi(|x|^2)$. Subordinate Brownian motions form a very large class of Lévy processes. Nonetheless, compared with general Lévy processes, subordinate Brownian motions are much more tractable. If we take the Brownian motion B as given, then X is completely determined by the subordinator S. For a summary of some of these recent results on subordinate Brownian motion, see [1, 7] and the references therein.

Before stating the recent results in [4, 5, 8] and the main theorem of this paper, we introduce some notations. We use ":=" to denote a definition, which is read as "is defined to be". We denote $a \wedge b := \min\{a, b\}$, $a \vee b := \max\{a, b\}$. $\delta_D(x)$ is the distance between the point x and the boundary of D. We say that $f : \mathbb{R} \to \mathbb{R}$ is increasing if $s \leq t$ implies $f(s) \leq f(t)$ and analogously for a decreasing function. We use notation $f(t) \asymp g(t)$ as $t \to \infty$ (resp., $t \to 0+$) if the quotient $f(t)/g(t)$ stays bounded between two positive constants as $t \to \infty$ (resp., $t \to 0+$).

Recently, in [8], implicitly it is conjectured that for a large class of transient subordinate Brownian motions, Green function $G_D(x, y)$ in D enjoys the following two-sided estimates in terms of ϕ and Green function $G(x, y)$ in $\mathbb{R}^d$:

$$\begin{aligned}(1.1)\qquad & c^{-1}\left(1 \wedge \frac{\phi(|x-y|^{-2})}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(y)^{-2})}}\right) G(x,y) \\ & \leq G_D(x,y) \leq c\left(1 \wedge \frac{\phi(|x-y|^{-2})}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(y)^{-2})}}\right) G(x,y).\end{aligned}$$

This conjecture has been proved in [8] for the case when ϕ varies regularly with index $\alpha \in (0, 2)$ and D in bounded $C^{1,1}$ open sets. Very recently in [5], jointly with Ante Mimica, the second named author proved this conjecture for the case when ϕ is a complete Bernstein function satisfying some scaling assumptions (see **(A1)**–**(A5)** below), which is milder than the ones in [8]; the Green function $G_D(x, y)$ of X in D satisfies the following estimates:

$$\begin{aligned}(1.2)\qquad & C_0^{*\,-1}\left(1 \wedge \frac{\phi(|x-y|^{-2})}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(y)^{-2})}}\right) \frac{\phi'(|x-y|^{-2})}{|x-y|^{d+2}\phi(|x-y|^{-2})^2} \\ & \leq G_D(x,y)\end{aligned}$$

$$\leq C_0^* \left(1 \wedge \frac{\phi(|x-y|^{-2})}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(y)^{-2})}}\right) \frac{\phi'(|x-y|^{-2})}{|x-y|^{d+2}\phi(|x-y|^{-2})^2}.$$

Note, under even milder assumptions, it is shown in [4] that

$$G(x,y) = g(|x-y|) \asymp \frac{\phi'(|x-y|^{-2})}{|x-y|^{d+2}\phi(|x-y|^{-2})^2} \quad \text{as } |x-y| \to 0.$$

Thus (1.1) holds.

The Laplace exponent $\phi : (0,\infty) \to (0,\infty)$ of a subordinator S is a Bernstein function with $\phi(0+) = 0$. Thus it is of the form

$$\phi(\lambda) = b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda t})\, \mu(dt)\,, \quad \lambda > 0\,, \tag{1.3}$$

where $b \geq 0$ and μ is a measure on $(0,\infty)$ satisfying $\int_{(0,\infty)} (1 \wedge t)\, \mu(dt) < \infty$, called the Lévy measure.

The infinitesimal generator of the subordinate Brownian motion X is $\phi(\Delta) := -\phi(-\Delta)$, which on $C_b^2(\mathbb{R}^d)$, the collection of bounded C^2 functions in $\mathbb{R}^d$ with bounded derivatives, turns out to be an integro-differential operator of the type

$$b\Delta f(x) + \int_{\mathbb{R}^d} \left(f(x+y) - f(x) - \nabla f(x) \cdot y \mathbf{1}_{\{|y|\leq 1\}}\right) J(y)\, dy\,,$$

where $J(x) = j(|x|)$ with $j : (0,\infty) \to (0,\infty)$ given by

$$j(r) = \int_0^\infty (4\pi t)^{-d/2} e^{-r^2/(4t)}\, \mu(dt)\,.$$

Note that the function $r \mapsto j(r)$ is strictly positive, continuous and decreasing on $(0,\infty)$. We will assume that $b = 0$ so that our subordinate Brownian motion is a pure jump process.

We will consider the following properties of j, which hold under the assumptions **(A1)**–**(A4)** (see [4]).

(1) There exists $C_1^* > 0$ such that

$$j(r) \leq C_1^* j(r+1), \quad r > 1. \tag{1.4}$$

(2) For every $M > 0$, there exists $C_2^* = C_2^*(M) > 1$ such that

$$(C_2^*)^{-1} \frac{\phi'(r^{-2})}{r^{d+2}} \leq j(r) \leq C_2^* \frac{\phi'(r^{-2})}{r^{d+2}}, \quad r \leq 3M. \tag{1.5}$$

Note that (1.5) implies that for any $T > 0$, there exists $c > 0$ such that

$$j(r) \leq c j(2r), \quad r \in (0,T). \tag{1.6}$$

By the result of Ikeda and Watanabe (see [3, Theorem 1]), we know that for every bounded open subset D and every $f \geq 0$ and $x \in D$,

$$\mathbb{E}_x \left[f(X_{\tau_D});\, X_{\tau_D -} \neq X_{\tau_D}\right] = \int_{\overline{D}^c} \int_D G_D(x,y) J(y-z) dy f(z) dz. \tag{1.7}$$

Now, we define the Poisson kernel by

$$K_D(x,z) := \int_D G_D(x,y)J(y-z)dy, \qquad (x,z) \in D \times \overline{D}^c. \tag{1.8}$$

Then (1.7) can be written as

$$\mathbb{E}_x\left[f(X_{\tau_D});\, X_{\tau_D-} \neq X_{\tau_D}\right] = \int_{\overline{D}^c} K_D(x,z)f(z)dz.$$

In this paper we use CS_z to denote an orthonormal coordinate system CS_z: $y = (y_1, \dots, y_{d-1}, y_d) := (\tilde{y},\, y_d)$ with origin at $z \in \mathbb{R}^d$. We say $\mathcal{C}(x,r,\eta)$ is a cone with vertex $x \in \mathbb{R}^d$, angle $\eta > 0$ and radius $r > 0$ when $\mathcal{C}(x,r,\eta) = \{y = (\tilde{y}, y_d) \in B(0,r) \text{ in } CS_x : y_d > 0, |\tilde{y}| < \eta y_d\}$.

Definition 1.1. An open set $D \subset \mathbb{R}^d$ is said to satisfy the cone condition if there exist constants $R > 0$ and $\eta \in (0,2]$ such that the following holds:

(1) For any $x \in \overline{D}$, $\overline{\mathcal{C}}(x,R,\eta) \setminus \{x\} \subset D$ for some orthonormal coordinate system CS_x, where $\overline{\mathcal{C}}(x,R,\eta)$ is a closure of $\mathcal{C}(x,R,\eta)$.
(2) For any $z \in \overline{D}^c$ with $\delta_D(z) < R/4$, there exist $z_0 \in \partial D$ such that $\delta_D(z) \le |z - z_0| \le 2\delta_D(z)$ and a corresponding cone $\mathcal{C}(z_0,R,\eta)$, which is contained in D for some coordinate system CS_{z_0}. In particular, $\tilde{z} = \tilde{0}$ in CS_{z_0}.

The pair (R,η) is called the cone characteristic constant of the open set D.

Note that Lipschitz open set satisfies the above cone condition. For an open set D, we denote $d_D := \mathrm{diam}(D) := \sup\{|x-y| : x,y \in D\}$.

We are now in a position to state the main result of this paper.

Theorem 1.2. *Suppose $M > 0$ and that $X = (X_t : t \ge 0)$ is a Lévy process whose characteristic exponent is given by $\Phi(\theta) = \phi(|\theta|^2)$, $\theta \in \mathbb{R}^d$, where $\phi : (0,\infty) \to (0,\infty)$ is a Bernstein function with $\phi(0+) = 0$ and $\lim_{t\to\infty} \phi(t) = \infty$. We assume that there exists an increasing function $\psi : ((5M)^{-2},\infty) \to (0,\infty)$ and a constant $c_1 \ge 1$ such that*

$$c_1^{-1}\psi(\lambda) \le \lambda^{1+d/2}\phi'(\lambda)/\phi(\lambda) \le c_1\psi(\lambda), \quad \lambda \in ((5M)^{-2},\infty). \tag{1.9}$$

Then (1.2), (1.4) and (1.5) imply that if a bounded open set D satisfies the cone condition with cone characteristic constant (R,η) and $d_D < M$, then there exists $c = c(c_1, C_0^*, C_1^*, C_2^*, R/d_D, \eta, M, d) > 1$ such that

(1.10)

$$\begin{aligned} & c^{-1}\frac{\phi(\delta_D(z)^{-2})^{1/2}}{\phi(\delta_D(x)^{-2})^{1/2}\phi(|x-z|^{-2})(1+\phi(d_D^{-2})^{1/2}\phi(\delta_D(z)^{-2})^{-1/2})}j(|x-z|) \\ & \le K_D(x,z) \\ & \le c\,\frac{\phi(\delta_D(z)^{-2})^{1/2}}{\phi(\delta_D(x)^{-2})^{1/2}\phi(|x-z|^{-2})(1+\phi(d_D^{-2})^{1/2}\phi(\delta_D(z)^{-2})^{-1/2})}j(|x-z|), \end{aligned}$$

where C_0^*, C_1^* and C_2^* are constants satisfying (1.2), (1.4) and (1.5).

The assumption (1.9) is very mild. For example, if ϕ is a special Bernstein function ($\lambda \to \lambda/\phi(\lambda)$ is a Bernstein function), then $\lambda \to \lambda^2\phi'(\lambda)/\phi(\lambda)^2$ is increasing for all $\lambda > 0$ (see [4, Lemma 3.1]). Moreover, if $G(x, y) = g(|x-y|) \asymp \frac{\phi'(|x-y|^{-2})}{|x-y|^{d+2}\phi(|x-y|^{-2})^2}$ as $|x-y| \to 0$, then (1.9) is always true because $g(\lambda)$ is decreasing. Note that the term $1 + \phi(d_D^{-2})^{1/2}\phi(\delta_D(z)^{-2})^{-1/2}$ appears in (1.10) since the constant c in Theorem 1.2 depends on R/d_D, but neither on R nor d_D.

Although (1.10) follows from direct integration and estimation, due to our general formulation, it is not straightforward. Nevertheless, assumptions on the set D are mild; it may be just a bounded Lipschitz or $C^{1,\beta}$ open set for some $\beta \in (0, 1)$. It is worth mentioning that the constant c in Theorem 1.2 depends on R/d_D, thereby allowing uniform estimates of Poisson kernels of balls with constant not depending on the radii of balls (cf. Corollary 2.7).

Recall that an open set D in $\mathbb{R}^d$ (when $d \geq 2$) is said to be $C^{1,1}$ if there exist a localization radius $R > 0$ and a constant $\Lambda > 0$ such that for every $z \in \partial D$, there exist a $C^{1,1}$-function $\phi = \phi_z : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\phi(0) = 0$, $\nabla\phi(0) = (0, \dots, 0)$, $\|\nabla\phi\|_\infty \leq \Lambda$, $|\nabla\phi(x) - \nabla\phi(z)| \leq \Lambda|x - z|$, and an orthonormal coordinate system CS_z: $y = (y_1, \dots, y_{d-1}, y_d) := (\tilde{y}, y_d)$ with origin at z such that $B(z, R) \cap D = \{y = (\tilde{y}, y_d) \in B(0, R) \text{ in } CS_z : y_d > \phi(\tilde{y})\}$. We call the pair (R, Λ) the $C^{1,1}$ characteristic of the open set D. By a $C^{1,1}$ open set in $\mathbb{R}$ we mean an open set which can be written as the union of disjoint intervals so that the minimum of the lengths of all these intervals is positive and the minimum of the distances between these intervals is also positive.

In [5], the following conditions on the Laplace exponent ϕ of the subordinator S are considered:

(A-1) ϕ is a complete Bernstein function, i.e., the Lévy density μ of ϕ has a completely monotone density;

(A-2) the Lévy density μ of ϕ is infinite, i.e., $\mu(0, \infty) = \infty$;

(A-3) there exist constants $\sigma > 0$, $\lambda_0 > 0$ and $\delta \in (0, 1]$ such that

$$\frac{\phi'(\lambda x)}{\phi'(\lambda)} \leq \sigma\, x^{-\delta} \quad \text{for all} \quad x \geq 1 \quad \text{and} \quad \lambda \geq \lambda_0.$$

(A-4) If $d \leq 2$, then we assume that the constant δ in **(A-3)** satisfies $d + 2\delta - 2 > 0$ and that there are $\sigma_0 > 0$ and

$$\delta_0 \in \left(1 - \tfrac{d}{2}, (1 + \tfrac{d}{2}) \wedge (2\delta + \tfrac{d-2}{2})\right)$$

such that

$$\frac{\phi'(\lambda x)}{\phi'(\lambda)} \geq \sigma_0\, x^{-\delta_0} \quad \text{for all} \quad x \geq 1 \quad \text{and} \quad \lambda \geq \lambda_0.$$

(A-5) If $d \geq 2$ and the constant δ in **(A-3)** satisfies $0 < \delta \leq \frac{1}{2}$, then we assume that there exist constants $\sigma_1 > 0$ and $\delta_1 \in [\delta, 1)$ such that

$$\frac{\phi(\lambda x)}{\phi(\lambda)} \geq \sigma_1\, x^{1-\delta_1} \quad \text{for all} \quad x \geq 1 \quad \text{and} \quad \lambda \geq \lambda_0\,.$$

Due to [4, 5], under these assumptions, (1.2)–(1.5) hold and $G(x,y) = g(|x-y|) \asymp \frac{\phi'(|x-y|^{-2})}{|x-y|^{d+2}\phi(|x-y|^{-2})^2}$ as $|x-y| \to 0$ so that (1.9) also holds. Therefore, applying Theorem 1.2, we have the sharp two-sided estimates for Poisson kernel for a large class of subordinate Brownian motions including geometric stable process.

Theorem 1.3. *Suppose that $X = (X_t, \mathbb{P}_x : t \geq 0, x \in \mathbb{R}^d)$ is a transient subordinate Brownian motion whose characteristic exponent is given by $\Phi(\theta) = \phi(|\theta|^2)$, $\theta \in \mathbb{R}^d$, satisfying* **(A-1)**–**(A-5)**. *Then for every bounded $C^{1,1}$ open set D in $\mathbb{R}^d$ with characteristics (R, Λ), there exists $c = c(d_D, R, \Lambda, \phi, d) > 1$ such that*

$$\begin{aligned} &c^{-1}\frac{\phi(\delta_D(z)^{-2})^{1/2}}{\phi(\delta_D(x)^{-2})^{1/2}\phi(|x-z|^{-2})(1+\phi(\delta_D(z)^{-2})^{-1/2})}j(|x-z|) \\ &\leq K_D(x,z) \leq c\,\frac{\phi(\delta_D(z)^{-2})^{1/2}}{\phi(\delta_D(x)^{-2})^{1/2}\phi(|x-z|^{-2})(1+\phi(\delta_D(z)^{-2})^{-1/2})}j(|x-z|). \end{aligned}$$

Example 1.4. When the subordinator has the Laplace exponent

$$\phi(\lambda) = \log(1+\lambda^{\alpha/2}) \qquad (0 < \alpha \leq 2, d > \alpha),$$

by [9, Lemma 3.3] and our Theorem 1.3, we have

$$K_D(x,z) \asymp \begin{cases} \frac{(\log(1+\delta_D(z)^{-\alpha}))^{1/2}}{(\log(1+\delta_D(x)^{-\alpha}))^{1/2}(1+(\log(1+\delta_D(z)^{-\alpha}))^{-1/2})}\frac{1}{(\log(1+|x-z|^{-\alpha}))^{1/2}|x-z|^d} \\ \text{when } \delta_D(z) \leq 2d_D \\ \frac{\delta_D(x)^{\alpha/2}}{\delta_D(z)^{\alpha/2}(1+\delta_D(z)^{\alpha/2})}|x-z|^{-d} \quad \text{when } \delta_D(z) > 2d_D. \end{cases}$$

Note that when $\phi(\lambda) = \lambda^{\alpha/2}$, it is known that

$$K_D(x,z) \asymp \frac{\delta_D(x)^{\alpha/2}}{\delta_D(z)^{\alpha/2}(1+\delta_D(z)^{\alpha/2})}|x-z|^{-d}$$

(see [2, 10]).

In this paper, we will use the following conventions. The values of the constants $\gamma_1, \gamma_2, C_0^*, C_1^*, C_2^*, C_3^*, C_4^*, C_0, C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8$ will remain the same throughout this paper, while $c, c_1, c_2, c_3, \dots$ stand for constants whose values are unimportant and which may change from one appearance to another. All constants are positive finite numbers. The labeling of the constants $c_1, c_2, \dots$ starts anew in the proof of each result. We denote by ω_d the surface area of the unit sphere $\partial B(0,1)$ in $\mathbb{R}^d$.

2. Proof

In order to cover more general Lévy processes, we give the proof under slightly weaker assumptions. From now on, D is a bounded open set with $d_D < M$ for some $M \geq 1$.

We assume the function $\Phi : [0,\infty) \to [0,\infty)$ satisfies the following properties:

(P1) Φ is an increasing C^1-function with $\Phi(0) = 0$ and $\lim_{t\to\infty}\Phi(t) = \infty$.

(P2) There exists a constant $C_0 \geq 1$ such that

$$\Phi(t\lambda) \leq C_0 \lambda^2 \Phi(t) \qquad \text{for all } \lambda \geq 1, t > 0. \tag{2.1}$$

(P3) There exists a constant $C_1 > 0$ such that

$$\Phi'(t\lambda) \leq C_1 \lambda \Phi'(t) \qquad \text{for all } \lambda \geq 1, t > 0. \tag{2.2}$$

(P4) There exist an increasing function $\Psi : ((5M)^{-1}, \infty) \to (0, \infty)$ and a constant $C_2 \geq 1$ such that

$$C_2^{-1}\Psi(\lambda) \leq \lambda^{1+d}\frac{\Phi'(\lambda)}{\Phi(\lambda)} \leq C_2\Psi(\lambda), \quad \lambda \in ((5M)^{-1}, \infty).$$

We assume $X := (X_t, \mathbb{P}_x : t \geq 0, x \in \mathbb{R}^d)$ is a purely discontinuous symmetric Lévy process such that the characteristic exponent of X is $\Phi_X(\xi)$ and the Lévy measure of X has a density $J(x)$ and $\mathbb{P}_x(X_0 = x) = 1$. Then

$$\mathbb{E}_x\left[e^{i\xi\cdot(X_t - X_0)}\right] = e^{-t\Phi_X(\xi)}, \quad x \text{ and } \xi \in \mathbb{R}^d,$$

with

$$\Phi_X(\xi) = \int_{\mathbb{R}^d}(1 - \cos(\xi \cdot y))J(y)dy.$$

We further assume that

(J1) There exist a decreasing function $j : (0, \infty) \to (0, \infty)$ and constants $\gamma_1, \gamma_2 > 0$ such that

$$\gamma_1 j(|x|) \leq J(x) \leq \gamma_2 j(|x|). \tag{2.3}$$

Let τ_D be the first exit time of D, i.e., $\tau_D = \inf\{t > 0 : X_t \notin D\}$. We assume that the mean occupation time of X before exiting D

$$U \mapsto \mathbb{E}_x \int_0^{\tau_D} \mathbf{1}_U(X_t)dt, \quad U \subset D$$

has a density, which we denote by $G_D(x, y)$, and will be called the Green function of D (with respect to X).

We assume that the Green function $G_D(x, y)$ and the function j in **(J1)** satisfies the following estimates:

(G) There exist positive constants C_3 and C_4 such that

$$\begin{aligned}
&C_3\left(1 \wedge \frac{\Phi(|x-y|^{-1})}{\Phi(\delta_D(x)^{-1})}\right)^{1/2}\left(1 \wedge \frac{\Phi(|x-y|^{-1})}{\Phi(\delta_D(y)^{-1})}\right)^{1/2}\frac{\Phi'(|x-y|^{-1})}{|x-y|^{d+1}\Phi(|x-y|^{-1})^2} \\
&\leq G_D(x, y) \\
&\leq C_4\left(1 \wedge \frac{\Phi(|x-y|^{-1})}{\Phi(\delta_D(x)^{-1})}\right)^{1/2}\left(1 \wedge \frac{\Phi(|x-y|^{-1})}{\Phi(\delta_D(y)^{-1})}\right)^{1/2}\frac{\Phi'(|x-y|^{-1})}{|x-y|^{d+1}\Phi(|x-y|^{-1})^2}.
\end{aligned} \tag{2.4}$$

(J2) There exist positive constants $C_5 = C_5(M)$ and $C_6 = C_6(M)$ such that

$$C_5\frac{\Phi'(r^{-1})}{r^{d+1}} \leq j(r) \leq C_6\frac{\Phi'(r^{-1})}{r^{d+1}}, \quad r \in (0, 10M). \tag{2.5}$$

(J3) There exists $C_7 > 0$ such that

$$j(r) \le C_7 j(r+1), \qquad r > 1. \tag{2.6}$$

Note that **(P3)** and **(J2)** imply that there exists $C_8 > 0$ such that

$$j(r) \le C_8 j(2r), \quad r \in (0, 5M). \tag{2.7}$$

In fact,

$$j(r) \le C_6 \frac{\Phi'(r^{-1})}{r^{d+1}} \le 2C_1C_6 \frac{\Phi'(2^{-1}r^{-1})}{r^{d+1}} \le C_1C_5^{-1}C_6 2^{d+2} j(2r), \quad r \in (0, 5M).$$

Also, by using the assumption that Φ is increasing and (2.1), it follows that (2.4) is equivalent to

$$\begin{aligned} & C_3^* \left(1 \wedge \frac{\Phi(|x-y|^{-1})}{\Phi(\delta_D(x)^{-1})^{1/2}\Phi(\delta_D(y)^{-1})^{1/2}}\right) \frac{\Phi'(|x-y|^{-1})}{|x-y|^{d+1}\Phi(|x-y|^{-1})^2} \\ & \le G_D(x,y) \\ & \le C_4^* \left(1 \wedge \frac{\Phi(|x-y|^{-1})}{\Phi(\delta_D(x)^{-1})^{1/2}\Phi(\delta_D(y)^{-1})^{1/2}}\right) \frac{\Phi'(|x-y|^{-1})}{|x-y|^{d+1}\Phi(|x-y|^{-1})^2} \end{aligned} \tag{2.8}$$

for some positive constant C_3^*, C_4^*. Indeed,

$$\left(1 \wedge \frac{\Phi(|x-y|^{-1})}{\Phi(\delta_D(x)^{-1})}\right)\left(1 \wedge \frac{\Phi(|x-y|^{-1})}{\Phi(\delta_D(y)^{-1})}\right) \le \left(1 \wedge \frac{\Phi(|x-y|^{-1})^2}{\Phi(\delta_D(x)^{-1})\Phi(\delta_D(y)^{-1})}\right).$$

Since other cases are similar or easy to check, we will show that

$$\left(1 \wedge \frac{\Phi(|x-y|^{-1})^2}{\Phi(\delta_D(x)^{-1})\Phi(\delta_D(y)^{-1})}\right) \le 4C_0 \left(1 \wedge \frac{\Phi(|x-y|^{-1})}{\Phi(\delta_D(x)^{-1})}\right)\left(1 \wedge \frac{\Phi(|x-y|^{-1})}{\Phi(\delta_D(y)^{-1})}\right) \tag{2.9}$$

when $\delta_D(y) \le |x-y| \le \delta_D(x)$. In this case, $\delta_D(x) \le \delta_D(y) + |x-y| \le 2|x-y|$. So

$$\begin{aligned} 1 \wedge \frac{\Phi(|x-y|^{-1})^2}{\Phi(\delta_D(x)^{-1})\Phi(\delta_D(y)^{-1})} & \le 1 \wedge \frac{\Phi(|x-y|^{-1})^2}{\Phi((2|x-y|)^{-1})\Phi(\delta_D(y)^{-1})} \\ & \le 1 \wedge \frac{4C_0\Phi(|x-y|^{-1})}{\Phi(\delta_D(y)^{-1})} \\ & \le 4C_0 \left(1 \wedge \frac{\Phi(|x-y|^{-1})}{\Phi(\delta_D(y)^{-1})}\right), \end{aligned}$$

which implies (2.9). This shows that (2.8) is equivalent to (2.4).

As in (1.8), we denote the Poisson kernel of X in $D \times \overline{D}^c$ by $K_D(x,z)$.

Remark 2.1. When Φ is of the form $\Phi(\lambda) = \phi(\lambda^2)$, we can check **(P1)–(P4)** for some particular cases of ϕ:

(1) ϕ is a Bernstein function with $\phi(0+) = 0$:

In this case, Φ is an increasing C^∞-function and $\Phi'(\lambda) = 2\lambda\phi'(\lambda^2)$. By concavity, every Bernstein function ϕ satisfies $\phi(t\lambda) \le \lambda\phi(t)$ for all $\lambda \ge 1, t > 0$. So we have **(P2)** with $C_0 = 1$. Since ϕ' is decreasing, we have **(P3)** with $C_1 = 1/2$. So, for a Bernstein function ϕ, **(P2)** and **(P3)** hold. If ϕ further has the property that $\lim_{t\to\infty} \phi(t) = \infty$, then $\lim_{t\to\infty} \Phi(t) = \infty$, which implies **(P1)**. In fact, $\lim_{t\to\infty} \phi(t) = \infty$ holds when Lévy measure of X is infinite.

(2) ϕ is a special Bernstein function, i.e., $\lambda \mapsto \frac{\lambda}{\phi(\lambda)}$ is also a Bernstein function:

By [4, Lemma 3.1], $\lambda \to \lambda^2\phi'(\lambda)/\phi(\lambda)^2$ is increasing for all $\lambda > 0$. Since $\lambda^{1+d}\Phi'(\lambda)/\Phi(\lambda) = 2(\lambda^2)^{1+d/2}\phi'(\lambda^2)/\phi(\lambda^2)$ and ϕ is increasing, (**P4**) holds if $d \geq 2$. Thus for a special Bernstein function, **(P4)** holds for $d \geq 2$. Note that **(P2)** and **(P3)** also hold by (1).

(3) ϕ is a Laplace exponent of subordinator which satisfies the assumptions **(A-1)**–**(A-3)** and **(B)** in [4]:

In this case, Lévy process X is a subordinate Brownian motion with Lévy exponent Φ and ϕ is of the form (1.3) with $\phi(0) = 0$ $(b = 0)$ and $\lim_{t\to\infty} \phi(t) = \infty$. Hence **(P1)**, **(P2)** and **(P3)** hold. By [4, Proposition 4.2], we get **(J2)** and if X is transient, then by [4, Proposition 4.5], $g(r) \asymp r^{-2-d}\phi'(r^{-2})/\phi(r^{-2})^2$ as $r \to 0+$, which implies **(P4)** holds. In fact, [4, Remark 3.1(i)] says ϕ is a special Bernstein function. So we have **(P4)** for $d \geq 2$ without **(B)** and transience of X.

(4) ϕ is a Laplace exponent of subordinator which satisfies assumptions **(A-1)**–**(A-5)**:

(J1), **(J2)** and **(J3)** hold by [5, Proposition 2.6] and the statements that follow. Since ϕ is a Bernstein function of the form (1.3) satisfying **(A-2)**, it can be seen as in (3) that **(P1)**, **(P2)**, **(P3)** hold. When X is transient, we have **(G)** by [5, Theorem 1.2] and $g(\lambda^{-1}) \asymp \lambda^{2+d}\phi'(\lambda^2)/\phi(\lambda^2)^2$, which implies **(P4)** since $g(r)$ is decreasing.

It follows from Remark 2.1 that if ϕ satisfies the assumptions in Theorem 1.2, then $\Phi(\lambda) = \Phi_X(\lambda) = \phi(\lambda^2)$ satisfies **(P1)**–**(P4)**, and (1.2), (1.4), (1.5) imply **(G)**, **(J1)**, **(J2)** and **(J3)**. For the remainder of this section, we assume that Φ satisfies **(P1)**–**(P4)**. We want to estimate $K_D(x, z)$ in terms of Φ when **(G)**, **(J1)**, **(J2)** and **(J3)** hold.

We first consider the case when $\delta_D(z) > 2d_D$.

Proposition 2.2. *If* (2.3), (2.6) *and* (2.7) *hold, then there exist* $c_1 = c_1(\gamma_1, C_7, C_8, M) > 0$ *and* $c_2 = c_2(\gamma_2, C_7, C_8, M) > 0$ *such that for* $z \in \overline{D}^c$ *with* $\delta_D(z) > 2d_D$,

$$c_1 \int_D G_D(x, y)dy\, j(|x - z|) \leq K_D(x, z) \leq c_2 \int_D G_D(x, y)dy\, j(|x - z|). \tag{2.10}$$

In addition, if the upper bound of $G_D(x, y)$ *in* (2.4) *holds, then there exists* $c_3 = c_3(\gamma_2, C_4, C_7, C_8, d, M) > 0$ *such that for* $z \in \overline{D}^c$ *with* $\delta_D(z) > 2d_D$,

$$K_D(x, z) \leq c_3 \frac{j(|x - z|)}{\Phi(d_D^{-1})^{1/2}\Phi(\delta_D(x)^{-1})^{1/2}}. \tag{2.11}$$

Proof. We note that

(2.12)
$$|y-z|-d_D \le |y-z|-|x-y| \le |x-z| \le |y-z|+|x-y| \le |y-z|+d_D.$$

We consider two cases, $2d_D < \delta_D(z) \le 2M$ and $\delta_D(z) > 2M$, separately to prove (2.10). First, consider the case when $2d_D < \delta_D(z) \le 2M$. Since $|y-z| > 2d_D$, by (2.12) we have

$$\frac{1}{2}|y-z| < |x-z| < \frac{3}{2}|y-z|.$$

Since $|x-z|, |y-z| \le 2M+d_D < 3M$, (2.10) follows from (2.3) and (2.7) in this case. If $\delta_D(z) > 2M$, then $2M < |y-z|$. Since $|y-z|-d_D < |x-z| < |y-z|+d_D$ and $d_D < M$, we have

$$|y-z|-M < |x-z| < |y-z|+M.$$

This, with (2.3) and (2.6), proves (2.10) since $|y-z|-M > M \ge 1$. Hence for $\delta_D(z) > 2d_D$, (2.10) holds.

Now we further assume that the upper bound of $G_D(x,y)$ in (2.4) holds. Then

$$\begin{aligned}
&\int_D G_D(x,y)dy \\
&\le C_4 \int_D \left(1 \wedge \frac{\Phi(|x-y|^{-1})}{\Phi(\delta_D(x)^{-1})}\right)^{1/2} \left(1 \wedge \frac{\Phi(|x-y|^{-1})}{\Phi(\delta_D(y)^{-1})}\right)^{1/2} \frac{\Phi'(|x-y|^{-1})}{|x-y|^{d+1}\Phi(|x-y|^{-1})^2} dy \\
&\le C_4 \int_D \frac{\Phi'(|x-y|^{-1})}{\Phi(\delta_D(x)^{-1})^{1/2}|x-y|^{d+1}\Phi(|x-y|^{-1})^{3/2}} dy \\
&\le \frac{C_4\omega_d}{\Phi(\delta_D(x)^{-1})^{1/2}} \int_0^{d_D} 2(\Phi(r^{-1})^{-1/2})' dr = \frac{2C_4\omega_d}{\Phi(\delta_D(x)^{-1})^{1/2}\Phi(d_D^{-1})^{1/2}}.
\end{aligned}$$

The last equality follows from $\lim_{t\to\infty}\Phi(t) = \infty$. □

We now give the upper bound of $K_D(x,z)$ when $\delta_D(z) \le 2d_D$.

Proposition 2.3. *Assume* (2.3) *and suppose that the upper bounds of* $G_D(x,y)$ *and* $j(|x|)$ *are given by* (2.4) *and* (2.5)*, respectively. Then there exists* $c = c(\gamma_2, C_0, C_1, C_2, C_4, C_6, d) > 0$ *such that for every* $x \in D$ *and* $z \in \overline{D}^c$ *with* $\delta_D(z) \le 2d_D$*,*

$$K_D(x,z) \le c\frac{\Phi(\delta_D(z)^{-1})^{1/2}}{\Phi(\delta_D(x)^{-1})^{1/2}} \frac{\Phi'(|x-z|^{-1})}{|x-z|^{d+1}\Phi(|x-z|^{-1})}.$$

Proof. By (1.8), we have

$$\begin{aligned}
K_D(x,z) &= \int_D G_D(x,y)J(y-z)dy \\
&= \int_{\{y\in D: |x-z|<2|x-y|\}} G_D(x,y)J(y-z)dy
\end{aligned}$$

$$+\int_{\{y\in D:|x-z|\geq 2|x-y|\}} G_D(x,y)J(y-z)dy =: I+II.$$

By (2.4), we have the following estimate.

$$G_D(x,y)\leq C_4\frac{\Phi'(|x-y|^{-1})}{\Phi(\delta_D(x)^{-1})^{1/2}\Phi(\delta_D(y)^{-1})^{1/2}|x-y|^{d+1}\Phi(|x-y|^{-1})}, \tag{2.13}$$

$$G_D(x,y)\leq C_4\frac{\Phi'(|x-y|^{-1})}{\Phi(\delta_D(x)^{-1})^{1/2}|x-y|^{d+1}\Phi(|x-y|^{-1})^{3/2}}. \tag{2.14}$$

When $|x-z|<2|x-y|$, by using **(P4)**, (2.2) and the assumption that Φ is increasing,

$$\begin{aligned}\frac{\Phi'(|x-y|^{-1})}{|x-y|^{d+1}\Phi(|x-y|^{-1})}&\leq C_2^2\frac{2^{d+1}\Phi'(2|x-z|^{-1})}{|x-z|^{d+1}\Phi(2|x-z|^{-1})}\\&\leq c_1\frac{\Phi'(|x-z|^{-1})}{|x-z|^{d+1}\Phi(|x-z|^{-1})},\end{aligned} \tag{2.15}$$

where $c_1=C_1C_2^2 2^{d+2}$. Since $|y-z|\leq 3d_D<3M$, by (2.5),

$$j(|y-z|)\leq C_6\Phi'(|y-z|^{-1})/|y-z|^{d+1}$$

holds. Using this, (2.3), (2.13), (2.15) and polar coordinates,

$$\begin{aligned}I&\leq \gamma_2C_4c_1C_6\frac{1}{\Phi(\delta_D(x)^{-1})^{1/2}}\frac{\Phi'(|x-z|^{-1})}{|x-z|^{d+1}\Phi(|x-z|^{-1})}\\&\quad\times\int_{\{y\in D:|x-z|<2|x-y|\}}\frac{1}{\Phi(\delta_D(y)^{-1})^{1/2}}\frac{\Phi'(|y-z|^{-1})}{|y-z|^{d+1}}dy\\&\leq \gamma_2C_4c_1C_6\frac{1}{\Phi(\delta_D(x)^{-1})^{1/2}}\frac{\Phi'(|x-z|^{-1})}{|x-z|^{d+1}\Phi(|x-z|^{-1})}\\&\quad\times\int_D\frac{1}{\Phi(|y-z|^{-1})^{1/2}}\frac{\Phi'(|y-z|^{-1})}{|y-z|^{d+1}}dy\\&\leq \gamma_2C_4c_1C_6\omega_d\frac{1}{\Phi(\delta_D(x)^{-1})^{1/2}}\frac{\Phi'(|x-z|^{-1})}{|x-z|^{d+1}\Phi(|x-z|^{-1})}\\&\quad\times\int_{\delta_D(z)}^{\delta_D(z)+d_D}\frac{1}{\Phi(r^{-1})^{1/2}}\frac{\Phi'(r^{-1})}{r^{d+1}}r^{d-1}dr\\&\leq 2\gamma_2C_4c_1C_6\omega_d\frac{1}{\Phi(\delta_D(x)^{-1})^{1/2}}\frac{\Phi'(|x-z|^{-1})}{|x-z|^{d+1}\Phi(|x-z|^{-1})}\\&\quad\times\int_{\delta_D(z)}^{\infty}-(\Phi(r^{-1})^{1/2})'dr\\&\leq 2\gamma_2C_4c_1C_6\omega_d\frac{\Phi(\delta_D(z)^{-1})^{1/2}}{\Phi(\delta_D(x)^{-1})^{1/2}}\frac{\Phi'(|x-z|^{-1})}{|x-z|^{d+1}\Phi(|x-z|^{-1})}.\end{aligned}$$

The second inequality follows from the fact that $\delta_D(y)\leq|y-z|$ and the last inequality follows from $\Phi(0)=0$.

On the other hand, when $|x-z| \geq 2|x-y|$, we have

$$|y-z| \geq |x-z| - |x-y| \geq \frac{1}{2}|x-z| \geq |x-y|. \tag{2.16}$$

Thus by using **(P4)**, (2.2) and the assumption that Φ is increasing,

$$\frac{\Phi'(|y-z|^{-1})}{|y-z|^{d+1}} \leq c_1 \Phi(|y-z|^{-1}) \frac{\Phi'(|x-z|^{-1})}{|x-z|^{d+1}\Phi(|x-z|^{-1})} \tag{2.17}$$

as in (2.15). From (2.3), (2.5), (2.14) and (2.17), we get

$$\begin{aligned} II &\leq \gamma_2 C_4 c_1 C_6 \frac{1}{\Phi(\delta_D(x)^{-1})^{1/2}} \frac{\Phi'(|x-z|^{-1})}{|x-z|^{d+1}\Phi(|x-z|^{-1})} \\ &\quad \times \int_{\{y \in D : |x-z| \geq 2|x-y|\}} \frac{\Phi'(|x-y|^{-1})}{|x-y|^{d+1}\Phi(|x-y|^{-1})^{3/2}} \Phi(|y-z|^{-1}) dy. \end{aligned} \tag{2.18}$$

Let $a := |x-z|$. By the triangle inequality and (2.16),

$$\begin{aligned} &\int_{\{y \in D : |x-z| \geq 2|x-y|\}} \frac{\Phi'(|x-y|^{-1})}{|x-y|^{d+1}\Phi(|x-y|^{-1})^{3/2}} \Phi(|y-z|^{-1}) dy \\ \leq &\int_{\{y \in D : |x-z| \geq 2|x-y|\}} \frac{\Phi'(|x-y|^{-1})}{|x-y|^{d+1}\Phi(|x-y|^{-1})^{3/2}} \\ &\quad \times \Phi((||x-z|-|x-y||)^{-1} \wedge |x-y|^{-1}) dy \\ \leq &\, \omega_d \int_0^{d_D} \frac{\Phi'(r^{-1})}{r^{d+1}\Phi(r^{-1})^{3/2}} \Phi(|a-r|^{-1} \wedge r^{-1}) r^{d-1} dr \\ = &\, \omega_d \int_0^{d_D} \frac{\Phi'(r^{-1})}{r^{2}\Phi(r^{-1})^{3/2}} \Phi(|a-r|^{-1} \wedge r^{-1}) dr. \end{aligned}$$

We split the above integral as

$$\begin{aligned} &\int_0^{d_D} \frac{\Phi'(r^{-1})}{r^{2}\Phi(r^{-1})^{3/2}} \Phi(|a-r|^{-1} \wedge r^{-1}) dr \\ \leq &\int_0^{\frac{a}{2}} \frac{\Phi'(r^{-1})}{r^{2}\Phi(r^{-1})^{3/2}} \Phi(|a-r|^{-1}) dr + \int_{\frac{a}{2}}^{\infty} \frac{\Phi'(r^{-1})}{r^{2}\Phi(r^{-1})^{3/2}} \Phi(r^{-1}) dr \\ \leq &\, \Phi(2a^{-1}) \int_0^{\frac{a}{2}} \frac{\Phi'(r^{-1})}{r^{2}\Phi(r^{-1})^{3/2}} dr + \int_{\frac{a}{2}}^{\infty} \frac{\Phi'(r^{-1})}{r^{2}\Phi(r^{-1})^{1/2}} dr. \end{aligned}$$

By using $\lim_{t\to\infty} \Phi(t) = \infty$ and $\Phi(0) = 0$ respectively, we have

$$\int_0^{\frac{a}{2}} \frac{\Phi'(r^{-1})}{r^{2}\Phi(r^{-1})^{3/2}} dr = 2\int_0^{\frac{a}{2}} (\Phi(r^{-1})^{-1/2})' dr = 2\Phi(2a^{-1})^{-1/2}$$

and

$$\int_{\frac{a}{2}}^{\infty} \frac{\Phi'(r^{-1})}{r^{2}\Phi(r^{-1})^{1/2}} dr = 2\int_{\frac{a}{2}}^{\infty} -(\Phi(r^{-1})^{1/2})' dr = 2\Phi(2a^{-1})^{1/2}.$$

So by using **(P2)**,

$$\int_{\{y\in D:|x-z|\geq 2|x-y|\}} \frac{\Phi'(|x-y|^{-1})}{|x-y|^{d+1}\Phi(|x-y|^{-1})^{3/2}}\Phi(|y-z|^{-1})dy$$
$$\leq 4\omega_d\Phi(2|x-z|^{-1})^{1/2} \leq 8\omega_d C_0^{1/2}\Phi(|x-z|^{-1})^{1/2} \leq 8\omega_d C_0^{1/2}\Phi(\delta_D(z)^{-1})^{1/2}.$$

Combining this with (2.18), we have

$$II \leq 8c_1C_0^{1/2}\gamma_2C_4C_6\omega_d\frac{\Phi(\delta_D(z)^{-1})^{1/2}}{\Phi(\delta_D(x)^{-1})^{1/2}}\frac{\Phi'(|x-z|^{-1})}{|x-z|^{d+1}\Phi(|x-z|^{-1})}.$$

Thus

$$K_D(x,z) = I + II \leq c\frac{\Phi(\delta_D(z)^{-1})^{1/2}}{\Phi(\delta_D(x)^{-1})^{1/2}}\frac{\Phi'(|x-z|^{-1})}{|x-z|^{d+1}\Phi(|x-z|^{-1})}$$

for some $c = c(\gamma_2, C_0, C_1, C_2, C_4, C_6, d) > 0$. This finishes the proof. □

Note that in Proposition 2.3, we do not need the cone condition of D. In the remainder of this paper, we assume further that the bounded open set D satisfies the cone condition with cone characteristic constant (R,η) (cf. Definition 1.1).

Proposition 2.4. *Suppose that* (2.3), (2.6) *and* (2.7) *hold and that the lower bound of $G_D(x,y)$ in* (2.4) *holds. Then there exists $c = c(\gamma_1, C_0, C_3, C_7, C_8, R/d_D, \eta, M, d) > 0$ such that for $z \in \overline{D}^c$ with $\delta_D(z) > 2d_D$,*

$$K_D(x,z) \geq c\frac{j(|x-z|)}{\Phi(\delta_D(x)^{-1})^{1/2}\Phi(d_D^{-1})^{1/2}}.$$

Proof. By (2.10), we only need to show that

(2.19)
$$h(x) := \int_D \left(1\wedge\frac{\Phi(|x-y|^{-1})}{\Phi(\delta_D(x)^{-1})}\right)^{1/2}\left(1\wedge\frac{\Phi(|x-y|^{-1})}{\Phi(\delta_D(y)^{-1})}\right)^{1/2}\frac{\Phi'(|x-y|^{-1})}{|x-y|^{d+1}\Phi(|x-y|^{-1})^2}dy$$
$$\geq \frac{c}{\Phi(\delta_D(x)^{-1})^{1/2}\Phi(d_D^{-1})^{1/2}}.$$

Since D satisfies the cone condition and $x \in D$, there exists a cone $\mathcal{C}(x,R,\eta) \subset D$ for some coordinate system CS_x. So $E_x := \mathcal{C}(x,R,\eta/2)$ is also in D in the same coordinate system CS_x. Then there exists a constant $c_1 = c_1(\eta) \in (0,1]$ such that $c_1|x-y| \leq \delta_D(y)$ for $y \in E_x$. This and (2.1) imply that $\Phi(\delta_D(y)^{-1})^{1/2} \leq C_0^{1/2}c_1^{-1}\Phi(|x-y|^{-1})^{1/2}$ for $y \in E_x$. Let $c_2 = C_0^{1/2}c_1^{-1} \geq 1$. Since $\delta_D(x) < d_D$ and $|x-y| \leq d_D$ for all $y \in D$, on E_x we have

$$\left(1\wedge\frac{\Phi(|x-y|^{-1})}{\Phi(\delta_D(x)^{-1})}\right)^{1/2}\left(1\wedge\frac{\Phi(|x-y|^{-1})}{\Phi(\delta_D(y)^{-1})}\right)^{1/2}$$
$$= \frac{1}{\Phi(\delta_D(x)^{-1})}(\Phi(\delta_D(x)^{-1})\wedge\Phi(|x-y|^{-1}))^{1/2}$$

$$\times \left(\Phi(\delta_D(x)^{-1}) \wedge \frac{\Phi(\delta_D(x)^{-1})\Phi(|x-y|^{-1})}{\Phi(\delta_D(y)^{-1})}\right)^{1/2}$$
$$\geq \frac{1}{\Phi(\delta_D(x)^{-1})}\Phi(d_D^{-1})^{1/2}(\Phi(d_D^{-1})/c_2)^{1/2}.$$

Thus using (2.1) with $c_3 = c_2^{1/2}$, we get

$$\begin{aligned} h(x) &\geq \frac{\Phi(d_D^{-1})}{c_3\Phi(\delta_D(x)^{-1})}\int_{E_x}\frac{\Phi'(|x-y|^{-1})}{|x-y|^{d+1}\Phi(|x-y|^{-1})^2}dy \\ &\geq \frac{c_4\omega_d\Phi(d_D^{-1})}{c_3\Phi(\delta_D(x)^{-1})}\int_0^R \frac{\Phi'(r^{-1})}{r^2\Phi(r^{-1})^2}dr = \frac{c_4\omega_d\Phi(d_D^{-1})}{c_3\Phi(\delta_D(x)^{-1})}\int_0^R (1/\Phi(r^{-1}))'dr \\ &= \frac{c_4\omega_d\Phi(d_D^{-1})}{c_3\Phi(\delta_D(x)^{-1})\Phi(R^{-1})} \geq \frac{c_4\omega_d(R/d_D)^2}{c_3C_0\Phi(\delta_D(x)^{-1})} \end{aligned} \tag{2.20}$$

for some $c_4 = c_4(\eta) > 0$.

Take $c_5 = R/(4d_D)$ and define $V_x := \{y \in \mathcal{C}(x,R,\eta/2) : c_5\delta_D(x) < |x-y|\}$. Note that $2c_5\delta_D(x) < R$ since $\delta_D(x) < d_D$. So for $y \in V_x$, $C_0^{1/2}c_5^{-1}\Phi(\delta_D(x)^{-1})^{1/2} \geq \Phi(|x-y|^{-1})^{1/2}$. Since $V_x \subset E_x$, $\Phi(\delta_D(y)^{-1})^{1/2} \leq C_0^{1/2}c_1^{-1}\Phi(|x-y|^{-1})^{1/2}$ for $y \in V_x$. From these facts, for some $c_6 = c_6(\eta) > 0$, we have

$$\begin{aligned} h(x) &\geq \frac{c_1c_5}{C_0}\int_{V_x}\frac{1}{\Phi(\delta_D(x)^{-1})^{1/2}}\frac{\Phi'(|x-y|^{-1})}{|x-y|^{d+1}\Phi(|x-y|^{-1})^{3/2}}dy \\ &\geq \frac{c_1c_5c_6\omega_d}{C_0\Phi(\delta_D(x)^{-1})^{1/2}}\int_{c_5\delta_D(x)}^R \frac{\Phi'(r^{-1})}{r^2\Phi(r^{-1})^{3/2}}dr \\ &= \frac{2c_1c_5c_6\omega_d}{C_0\Phi(\delta_D(x)^{-1})^{1/2}}\int_{c_5\delta_D(x)}^R (\Phi(r^{-1})^{-1/2})'dr \\ &= \frac{2c_1c_5c_6\omega_d}{C_0\Phi(\delta_D(x)^{-1})^{1/2}}\left(\frac{1}{\Phi(R^{-1})^{1/2}} - \frac{1}{\Phi(c_5^{-1}\delta_D(x)^{-1})^{1/2}}\right). \end{aligned} \tag{2.21}$$

Let $c_7 := c_4\omega_d 2^{-1}c_3^{-1}C_0^{-1}(R/d_D)^2$ and choose $c_8 := c_1c_5c_6\omega_d C_0{}^{-1} \wedge c_7$. Then by (2.20) and (2.21),

$$\begin{aligned} h(x) &= \frac{1}{2}h(x) + \frac{1}{2}h(x) \\ &\geq \frac{c_7}{\Phi(\delta_D(x)^{-1})} + \frac{c_8}{\Phi(\delta_D(x)^{-1})^{1/2}}\left(\frac{1}{\Phi(R^{-1})^{1/2}} - \frac{1}{\Phi(c_5^{-1}\delta_D(x)^{-1})^{1/2}}\right) \\ &= \frac{c_8}{\Phi(\delta_D(x)^{-1})^{1/2}\Phi(R^{-1})^{1/2}} \\ &\quad + \frac{1}{\Phi(\delta_D(x)^{-1})^{1/2}}\left(\frac{c_7}{\Phi(\delta_D(x)^{-1})^{1/2}} - \frac{c_8}{\Phi(c_5^{-1}\delta_D(x)^{-1})^{1/2}}\right) \\ &\geq \frac{c_8}{\Phi(\delta_D(x)^{-1})^{1/2}\Phi(R^{-1})^{1/2}} \geq \frac{c_8R}{C_0^{1/2}d_D\Phi(\delta_D(x)^{-1})^{1/2}\Phi(d_D^{-1})^{1/2}}. \end{aligned}$$

The penultimate inequality follows from the facts that $c_5 < 1$ and Φ is increasing. The claim (2.19) is proved. $\square$

Proposition 2.5. *Assume* (2.3) *and suppose that the lower bounds of* $G_D(x,y)$ *and* $j(|x|)$ *are given by* (2.4) *and* (2.5), *respectively. Then there exists* $c = c(\gamma_1, C_0, C_1, C_2, C_3, C_5, \eta, R/d_D, d) > 0$ *such that for every* $x \in D$ *and* $z \in \overline{D}^c$ *with* $\delta_D(z) \le 2d_D$,

$$K_D(x,z) \ge c \frac{1}{\Phi(\delta_D(x)^{-1})^{1/2}} \frac{\Phi'(|x-z|^{-1})}{|x-z|^{d+1}\Phi(|x-z|^{-1})^{1/2}}.$$

Proof. Since $|x-z| \ge \delta_D(x)$ and Φ is increasing, we have

$$\begin{aligned}\left(1 \wedge \frac{\Phi(|x-y|^{-1})}{\Phi(\delta_D(x)^{-1})}\right) &= \frac{\Phi(|x-z|^{-1})}{\Phi(\delta_D(x)^{-1})}\left(\frac{\Phi(\delta_D(x)^{-1})}{\Phi(|x-z|^{-1})} \wedge \frac{\Phi(|x-y|^{-1})}{\Phi(|x-z|^{-1})}\right) \\ &\ge \frac{\Phi(|x-z|^{-1})}{\Phi(\delta_D(x)^{-1})}\left(1 \wedge \frac{\Phi(|x-y|^{-1})}{\Phi(|x-z|^{-1})}\right).\end{aligned}$$

Thus by (2.3), (2.4) and (2.5), there exists a constant $c_1 = c_1(\gamma_1, C_3, C_5)$ such that

(2.22)

$$\begin{aligned}&K_D(x,z) \\ &\ge c_1 \int_D \left(1 \wedge \frac{\Phi(|x-y|^{-1})}{\Phi(\delta_D(x)^{-1})}\right)^{1/2}\left(1 \wedge \frac{\Phi(|x-y|^{-1})}{\Phi(\delta_D(y)^{-1})}\right)^{1/2} \frac{\Phi'(|x-y|^{-1})}{|x-y|^{d+1}\Phi(|x-y|^{-1})^2}\frac{\Phi'(|y-z|^{-1})}{|y-z|^{d+1}}dy \\ &\ge c_1 \frac{\Phi(|x-z|^{-1})^{1/2}}{\Phi(\delta_D(x)^{-1})^{1/2}|x-z|^d} A(x,z),\end{aligned}$$

where

$$\begin{aligned}&A(x,z) \\ &:= \int_D \left(1 \wedge \frac{\Phi(|x-y|^{-1})}{\Phi(|x-z|^{-1})}\right)^{1/2}\left(1 \wedge \frac{\Phi(|x-y|^{-1})}{\Phi(\delta_D(y)^{-1})}\right)^{1/2} \frac{\Phi'(|x-y|^{-1})\Phi'(|y-z|^{-1})|x-z|^d}{|x-y|^{d+1}\Phi(|x-y|^{-1})^2|y-z|^{d+1}}dy.\end{aligned}$$

Let $a = |x-z|$ and $D_a := a^{-1}(D-x)$. Note that $0 \in D_a$ and $(3d_D)^{-1} < a^{-1} < \infty$. By change of variable $y - x = |x-z|\hat{y}$ and using the triangle inequality $|y-z| \le |x-z| + |y-x| = (1+|\hat{y}|)|x-z| < 4M$, we have $|y-x|^{-1} = a^{-1}|\hat{y}|^{-1}$ and $|y-z|^{-1} \ge a^{-1}(1+|\hat{y}|)^{-1} > (4M)^{-1}$. Also, $\delta_D(y) = a\delta_{D_a}(\hat{y})$, where $\delta_{D_a}(\hat{y}) = \mathrm{dist}(\hat{y}, \partial D_a)$. Then

$$\begin{aligned}C_2^2 \frac{\Phi'(|y-z|^{-1})}{|y-z|^{d+1}} &\ge \Phi(|y-z|^{-1}) \frac{\Phi'(a^{-1}(1+|\hat{y}|)^{-1})}{a^{d+1}(1+|\hat{y}|)^{d+1}\Phi(a^{-1}(1+|\hat{y}|)^{-1})} \\ &\ge \frac{\Phi'(a^{-1}(1+|\hat{y}|)^{-1})}{a^{d+1}(1+|\hat{y}|)^{d+1}},\end{aligned}$$

where the first inequality follows from **(P4)** and the second inequality holds since Φ is increasing. This implies that

$$A(x,z) \ge a^{-2}C_2^{-2}\int_{D_a}\frac{\Phi'(a^{-1}(1+|\hat y|)^{-1})}{(1+|\hat y|)^{d+1}}\frac{\Phi'(a^{-1}|\hat y|^{-1})}{\Phi(a^{-1}|\hat y|^{-1})^2|\hat y|^{d+1}}$$

$$\times\Big(1\wedge\frac{\Phi(a^{-1}|\hat y|^{-1})}{\Phi(a^{-1})}\Big)^{1/2}\Big(1\wedge\frac{\Phi(a^{-1}|\hat y|^{-1})}{\Phi(a^{-1}\delta_{D_a}(\hat y)^{-1})}\Big)^{1/2}d\hat y. \tag{2.23}$$

Since D satisfies the cone condition with cone characteristics (R,η), there is a cone $\mathcal{C}(w,R,\eta)\subset D$ for all $w\in D$. So $\hat{\mathcal{C}}(0,R/a,\eta)=a^{-1}(\mathcal{C}(x,R,\eta)-x)\subset D_a$. Since $a\le 3d_D$, we have $\hat{\mathcal{C}}(0,R/3d_D,\eta)\subset D_a$. By taking $r_1=R/3d_D\le 1/3$, we have $P:=\hat{\mathcal{C}}(0,r_1,\eta/2)\subset D_a$ in some coordinate system CS_0. Then there exists $c_2=c_2(\eta)\in(0,1]$ such that $c_2|\hat y|\le\delta_{D_a}(\hat y)$ and $|\hat y|\le r_1$ for $\hat y\in P$. Hence by (2.1) and the assumption that Φ is increasing,

$$\begin{aligned}\Phi(a^{-1}\delta_{D_a}(\hat y)^{-1})=\Phi(c_2^{-1}c_2a^{-1}\delta_{D_a}(\hat y)^{-1})&\le C_0(c_2^{-1})^2\Phi(a^{-1}c_2\delta_{D_a}(\hat y)^{-1})\\&\le C_0(c_2^{-1})^2\Phi(a^{-1}|\hat y|^{-1}).\end{aligned}$$

Thus for $\hat y\in P$,

$$\begin{aligned}&\Big(1\wedge\frac{\Phi(a^{-1}|\hat y|^{-1})}{\Phi(a^{-1})}\Big)^{1/2}\Big(1\wedge\frac{\Phi(a^{-1}|\hat y|^{-1})}{\Phi(a^{-1}\delta_{D_a}(\hat y)^{-1})}\Big)^{1/2}\\&\ge\Big(1\wedge\frac{\Phi(a^{-1}(r_1)^{-1})}{\Phi(a^{-1})}\Big)^{1/2}\Big(1\wedge c_2^2/C_0\Big)^{1/2}=c_3,\end{aligned}$$

where $c_3=c_2/C_0^{1/2}$. By (2.2),

$$\Phi'(a^{-1})=\Phi'((1+|\hat y|)(1+|\hat y|)^{-1}a^{-1})\le C_1(1+|\hat y|)\Phi'(a^{-1}(1+|\hat y|)^{-1}),$$

which implies

$$\frac{\Phi'(a^{-1}(1+|\hat y|)^{-1})}{(1+|\hat y|)^{d+1}}\ge C_1^{-1}\frac{\Phi'(a^{-1})}{(1+|\hat y|)^{d+2}}\ge C_1^{-1}\frac{\Phi'(a^{-1})}{(1+r_1)^{d+2}}.$$

Let $c_4=C_1^{-1}/(1+r_1)^{d+2}$. Then for some $c_5=c_5(C_0,C_1,\eta,R/d_D,d)>0$,

$$\begin{aligned}A(x,z)&\ge c_3c_4a^{-2}\Phi'(a^{-1})\int_P\frac{\Phi'(a^{-1}|\hat y|^{-1})}{\Phi(a^{-1}|\hat y|^{-1})^2|\hat y|^{d+1}}d\hat y\\&\ge c_5\omega_d a^{-2}\Phi'(a^{-1})\int_0^{r_1}\frac{\Phi'(a^{-1}r^{-1})}{\Phi(a^{-1}r^{-1})^2r^2}dr\\&=c_5\omega_d a^{-1}\Phi'(a^{-1})\int_0^{r_1}\frac{\partial}{\partial r}\left(\frac{1}{\Phi(a^{-1}r^{-1})}\right)dr\\&=c_5\omega_d\frac{\Phi'(a^{-1})}{a\Phi(a^{-1}r_1^{-1})}\ge c_5\omega_d r_1^2\frac{\Phi'(a^{-1})}{a\Phi(a^{-1})},\end{aligned}\tag{2.24}$$

where the last inequality follows from (2.1) and $r_1<1$.

Therefore, from (2.22)–(2.24), we conclude that

$$K_D(x,z)\ge c_6\frac{\Phi(|x-z|^{-1})^{1/2}}{\Phi(\delta_D(x)^{-1})^{1/2}}\frac{\Phi'(|x-z|^{-1})}{|x-z|^{d+1}\Phi(|x-z|^{-1})}$$

for $c_6 = c_6(\gamma_1, C_0, C_1, C_2, C_3, C_5, \eta, R/d_D, d) > 0$. □

We now restate and prove the main result.

Theorem 2.6. *Let D be a bounded open set which satisfies the cone condition with cone characteristic constant (R, η) and $d_D < M$ for some $M \geq 1$. Furthermore, assume that there exist a function Φ satisfying* **(P1)**-**(P4)** *and a decreasing function j such that* **(G)**, **(J1)**, **(J2)**, **(J3)** *hold. Then there exists $c = c(\gamma_1, \gamma_2, C_0, C_1, C_2, C_3, C_4, C_5, C_6, C_7, R/d_D, \eta, M, d) > 1$ such that for every $x \in D$ and $z \in \overline{D}^c$,*

(2.25)
$$\begin{aligned}
&c^{-1}\frac{\Phi(\delta_D(z)^{-1})^{1/2}}{\Phi(\delta_D(x)^{-1})^{1/2}\Phi(|x-z|^{-1})(1+\Phi(d_D^{-1})^{1/2}\Phi(\delta_D(z)^{-1})^{-1/2})}j(|x-z|)\\
&\leq K_D(x,z)\\
&\leq c\frac{\Phi(\delta_D(z)^{-1})^{1/2}}{\Phi(\delta_D(x)^{-1})^{1/2}\Phi(|x-z|^{-1})(1+\Phi(d_D^{-1})^{1/2}\Phi(\delta_D(z)^{-1})^{-1/2})}j(|x-z|).
\end{aligned}$$

Proof. When $z \in \overline{D}^c$ with $\delta_D(z) \leq 2d_D$, by **(J2)**, (2.25) is equivalent to

$$\begin{aligned}
&c^{-1}\frac{\Phi(\delta_D(z)^{-1})^{1/2}\Phi'(|x-z|^{-1})}{|x-z|^{d+1}\Phi(\delta_D(x)^{-1})^{1/2}\Phi(|x-z|^{-1})}\\
&\leq K_D(x,z) \leq c\frac{\Phi(\delta_D(z)^{-1})^{1/2}\Phi'(|x-z|^{-1})}{|x-z|^{d+1}\Phi(\delta_D(x)^{-1})^{1/2}\Phi(|x-z|^{-1})}.
\end{aligned} \tag{2.26}$$

Indeed, when $\delta_D(z) \leq 2d_D$,

$$1 \leq 1 + \left(\frac{\Phi(d_D^{-1})}{\Phi(\delta_D(z)^{-1})}\right)^{1/2} \leq 1 + \left(\frac{\Phi(d_D^{-1})}{\Phi((2d_D)^{-1})}\right)^{1/2} \leq 1 + 2C_0^{1/2}.$$

From this and **(J2)**, we have

$$\begin{aligned}
&\frac{C_6^{-1}\Phi(\delta_D(z)^{-1})^{1/2}}{\Phi(\delta_D(x)^{-1})^{1/2}\Phi(|x-z|^{-1})(1+\Phi(d_D^{-1})^{1/2}\Phi(\delta_D(z)^{-1})^{-1/2})}j(|x-z|)\\
&\leq \frac{\Phi(\delta_D(z)^{-1})^{1/2}\Phi'(|x-z|^{-1})}{|x-z|^{d+1}\Phi(\delta_D(x)^{-1})^{1/2}\Phi(|x-z|^{-1})}\\
&\leq C_5^{-1}C_6^{-1}(1+2C_0^{1/2})\\
&\quad\times\frac{\Phi(\delta_D(z)^{-1})^{1/2}}{\Phi(\delta_D(x)^{-1})^{1/2}\Phi(|x-z|^{-1})(1+\Phi(R^{-1})^{1/2}\Phi(\delta_D(z)^{-1})^{-1/2})}j(|x-z|),
\end{aligned}$$

which implies the equivalence between (2.25) and (2.26) for $z \in \overline{D}^c$ with $\delta_D(z) \leq 2d_D$.

When $z \in \overline{D}^c$ with $\delta_D(z) > 2d_D$, we have $\delta_D(z) \leq |x-z| \leq 3\delta_D(z)/2$. So

$$(4/9C_0)\Phi(\delta_D(z)^{-1}) \leq \Phi(|x-z|^{-1}) \leq \Phi(\delta_D(z)^{-1}).$$

Also, we have $0 < \Phi(\delta_D(z)^{-1})^{1/2} < \Phi(d_D^{-1})^{1/2}$ from $\delta_D(z) > 2d_D > d_D$. This implies

$$\begin{aligned}
&\frac{4\Phi(\delta_D(z)^{-1})^{1/2}}{9C_0\Phi(\delta_D(x)^{-1})^{1/2}\Phi(|x-z|^{-1})(1+\Phi(d_D^{-1})^{1/2}\Phi(\delta_D(z)^{-1})^{-1/2})}j(|x-z|)\\
&= \frac{4\Phi(\delta_D(z)^{-1})}{9C_0\Phi(\delta_D(x)^{-1})^{1/2}\Phi(|x-z|^{-1})(\Phi(\delta_D(z)^{-1})^{1/2}+\Phi(d_D^{-1})^{1/2})}j(|x-z|)\\
&\le \frac{1}{\Phi(\delta_D(x)^{-1})^{1/2}\Phi(d_D^{-1})^{1/2}}j(|x-z|)\\
&\le \frac{2\Phi(\delta_D(z)^{-1})}{\Phi(\delta_D(x)^{-1})^{1/2}\Phi(|x-z|^{-1})(\Phi(\delta_D(z)^{-1})^{1/2}+\Phi(d_D^{-1})^{1/2})}j(|x-z|)\\
&= \frac{2\Phi(\delta_D(z)^{-1})^{1/2}}{\Phi(\delta_D(x)^{-1})^{1/2}\Phi(|x-z|^{-1})(1+\Phi(d_D^{-1})^{1/2}\Phi(\delta_D(z)^{-1})^{-1/2})}j(|x-z|).
\end{aligned}$$

Thus (2.25) is equivalent to

$$\begin{aligned}
&c^{-1}\frac{1}{\Phi(\delta_D(x)^{-1})^{1/2}\Phi(d_D^{-1})^{1/2}}j(|x-z|)\\
&\le K_D(x,z) \le c\frac{1}{\Phi(\delta_D(x)^{-1})^{1/2}\Phi(d_D^{-1})^{1/2}}j(|x-z|)
\end{aligned}$$

when $z \in \overline{D}^c$ with $\delta_D(z) > 2d_D$.

Hence by Proposition 2.3, Proposition 2.4 and (2.11), it suffices to show that the lower bound of (2.26) holds for $z \in \overline{D}^c$ with $\delta_D(z) \le 2d_D$. For the remainder of the proof, we assume $z \in \overline{D}^c$ with $\delta_D(z) \le 2d_D$ and consider the following three cases separately.

Case 1. $R/17 \le \delta_D(z) \le 2d_D$:

Since $|x-z| < 3d_D$ and Φ is increasing, Proposition 2.5 implies

$$\begin{aligned}
K_D(x,z) &\ge c_1\frac{\Phi(\delta_D(z)^{-1})^{1/2}}{\Phi((R/17)^{-1})^{1/2}}\frac{\Phi((3d_D)^{-1})^{1/2}}{\Phi(\delta_D(x)^{-1})^{1/2}}\frac{\Phi'(|x-z|^{-1})}{|x-z|^{d+1}\Phi(|x-z|^{-1})}\\
&\ge c_1c_2\frac{\Phi(\delta_D(z)^{-1})^{1/2}}{\Phi(\delta_D(x)^{-1})^{1/2}}\frac{\Phi'(|x-z|^{-1})}{|x-z|^{d+1}\Phi(|x-z|^{-1})},
\end{aligned}\tag{2.27}$$

where $c_2 = R/(C_0^{1/2}51d_D)$. Note that c_2 satisfies the inequality

$$\Phi((R/17)^{-1})^{1/2} = \Phi((R/51d_D)^{-1}(3d_D)^{-1})^{1/2} \le (1/c_2)\Phi((3d_D)^{-1})^{1/2}.$$

Case 2. $|x-z| \le 32\delta_D(z)$ and $\delta_D(z) \le 2d_D$:

In this case, using Proposition 2.5 and (2.1), we have

$$\begin{aligned}
K_D(x,z) &\ge c_1\frac{\Phi((32\delta_D(z))^{-1})^{1/2}}{\Phi(\delta_D(x)^{-1})^{1/2}}\frac{\Phi'(|x-z|^{-1})}{|x-z|^{d+1}\Phi(|x-z|^{-1})}\\
&\ge (c_1/32C_0^{1/2})\frac{\Phi(\delta_D(z)^{-1})^{1/2}}{\Phi(\delta_D(x)^{-1})^{1/2}}\frac{\Phi'(|x-z|^{-1})}{|x-z|^{d+1}\Phi(|x-z|^{-1})}.
\end{aligned}\tag{2.28}$$

Case 3. $32\delta_D(z) < |x-z|$ and $\delta_D(z) < R/17$:

Define $Q := \{y \in D : |y-z| < \frac{1}{2}|x-z|\}$. For $y \in Q$,

$$|x-y| \geq |x-z| - |y-z| > |x-z| - \frac{1}{2}|x-z| > \frac{1}{2}|x-z| > |y-z|.$$

So $|x-y| > \frac{1}{2}(\delta_D(x) \vee \delta_D(y))$. This, with (2.1) and (2.4), implies that for $y \in Q$,

$$G_D(x,y) \geq c_3 \frac{1}{\Phi(\delta_D(x)^{-1})^{1/2}\Phi(\delta_D(y)^{-1})^{1/2}} \frac{\Phi'(|x-y|^{-1})}{|x-y|^{d+1}\Phi(|x-y|^{-1})}$$

for $c_3 = C_3/4C_0$. Thus by (2.3) and (2.5),

(2.29)

$$\begin{aligned}
&K_D(x,z)\\
&= \int_D G_D(x,y)J(y,z)dy\\
&\geq \gamma_1 c_3 C_5 \int_Q \frac{1}{\Phi(\delta_D(x)^{-1})^{1/2}\Phi(\delta_D(y)^{-1})^{1/2}} \frac{\Phi'(|x-y|^{-1})}{|x-y|^{d+1}\Phi(|x-y|^{-1})} \frac{\Phi'(|y-z|^{-1})}{|y-z|^{d+1}}dy\\
&= \gamma_1 c_3 C_5 \frac{\Phi(\delta_D(z)^{-1})^{1/2}}{\Phi(\delta_D(x)^{-1})^{1/2}|x-z|^d}\\
&\quad \times \int_Q \frac{|x-z|^d}{\Phi(\delta_D(z)^{-1})^{1/2}\Phi(\delta_D(y)^{-1})^{1/2}} \frac{\Phi'(|x-y|^{-1})}{|x-y|^{d+1}\Phi(|x-y|^{-1})} \frac{\Phi'(|y-z|^{-1})}{|y-z|^{d+1}}dy\\
&=: \gamma_1 c_3 C_5 \frac{\Phi(\delta_D(z)^{-1})^{1/2}}{\Phi(\delta_D(x)^{-1})^{1/2}|x-z|^d} B(x,z).
\end{aligned}$$

For $y \in Q$, $|x-y| \leq |x-z| + |y-z| \leq \frac{3}{2}|x-z|$. This and **(P4)** imply that

$$B(x,z) \geq (2/3)^{d+1} \frac{1}{\Phi(\delta_D(z)^{-1})^{1/2}|x-z|} \frac{\Phi'((3|x-z|/2)^{-1})}{\Phi((3|x-z|/2)^{-1})} \bar{B}(x,z), \tag{2.30}$$

where

$$\bar{B}(x,z) := \int_Q \frac{1}{|y-z|^{d+1}} \frac{\Phi'(|y-z|^{-1})}{\Phi(\delta_D(y)^{-1})^{1/2}}dy.$$

Since Φ is increasing, by (2.2) and (2.30), we have

$$B(x,z) \geq C_1^{-1}(2/3)^{d+2} \frac{1}{\Phi(\delta_D(z)^{-1})^{1/2}|x-z|} \frac{\Phi'(|x-z|^{-1})}{\Phi(|x-z|^{-1})} \bar{B}(x,z). \tag{2.31}$$

Since D satisfies the cone condition and $\delta_D(z) < R/17 < R/4$, as in (2) in the Definition 1.1, there exist $z_0 \in \partial D$ and a cone $\mathcal{C}(z_0, R, \eta) \subset D$ so that $\tilde{z} = \tilde{0}$ in coordinate system CS_{z_0}. Note that $|z - z_0| \leq 2\delta_D(z)$ and $|z-z_0| = -z_d \geq 0$ in CS_{z_0}. Since $\delta_D(z) < R/17$, we have $|z-z_0| \leq 2\delta_D(z) < 2R/17 < R/8$.

We will choose $\eta' > 0$ such that

$$\begin{aligned} W &:= \{y \in B(z, (R \wedge |x-z|)/2) \setminus B(z, 2|z-z_0|) : |\tilde{y}| < \eta'(y_d - z_d)\}\\ &\subset \mathcal{C}(z_0, R, \eta/2) \cap Q. \end{aligned} \tag{2.32}$$

Let $\kappa = (\sqrt{3\eta^4 + 16\eta^2} - 2\eta)/(4 + \eta^2)$ so that $4 = (1 + 2\kappa/\eta)^2 + \kappa^2$. Note that κ is a constant such that $\{(\tilde{y}, y_d) \in \partial\mathcal{C}(z_0, R, \eta/2) : |\tilde{y}| = \kappa|z - z_0|\} = \partial\mathcal{C}(z_0, R, \eta/2) \cap \partial B(z, 2|z - z_0|)$. Let

$$1/\eta' := 1/\kappa + 2/\eta = (4 + \eta^2)/(\sqrt{3\eta^4 + 16\eta^2} - 2\eta) + 2/\eta.$$

Suppose $y \in W$. First, we note that, since $|y - z| < (R \wedge |x - z|)/2 < R/2$,

$$|y - z_0| \le |z - z_0| + |y - z| < 2\delta_D(z) + R/2 < R.$$

Now, we will prove $2|\tilde{y}| < \eta y_d$ for $y \in W$. If $|\tilde{y}| \ge \kappa|z - z_0|$, then clearly $2|\tilde{y}|/\eta \le |\tilde{y}|/\eta' + z_d < y_d$. Suppose $|\tilde{y}| < \kappa|z - z_0|$ and $2\kappa|z - z_0|/\eta \ge y_d$. Then using the fact that $2\kappa|z - z_0|/\eta = \kappa|z - z_0|/\eta' + z_d$, we have in CS_{z_0},

$$|y - z| = (|\tilde{y}|^2 + |y_d - z_d|^2)^{1/2} < (\kappa^2|z - z_0|^2 + (2\kappa|z - z_0|/\eta - z_d)^2)^{1/2} = 2|z - z_0|.$$

This is a contradiction to $y \in W$. So for $|\tilde{y}| < \kappa|z - z_0|$, we have $2|\tilde{y}|/\eta < 2\kappa|z - z_0|/\eta < y_d$. Hence $y \in \mathcal{C}(z_0, R, \eta/2)$, which finishes the proof of (2.32).

(2.32) implies that there exists a constant $c_4(\eta) \in (0, 1]$ such that $\delta_D(y) \ge c_4|y - z_0|$ for $y \in W$. Also, by the definition of W, we have $|y - z| > 2|z - z_0|$ for $y \in W$. From these facts, for all $y \in W$, we have

$$\delta_D(y) \ge c_4|y - z_0| \ge c_4(|y - z| - |z - z_0|) \ge c_5|y - z|, \tag{2.33}$$

where $c_5 = c_4/2$. Thus by (2.32) and (2.33),

(2.34)

$$\begin{aligned}
\bar{B}(x, z) &= \int_Q \frac{1}{|y - z|^{d+1}} \frac{\Phi'(|y - z|^{-1})}{\Phi(\delta_D(y)^{-1})^{1/2}} dy \\
&\ge \int_W \frac{1}{|y - z|^{d+1}} \frac{\Phi'(|y - z|^{-1})}{\Phi(\delta_D(y)^{-1})^{1/2}} dy \\
&\ge c_6\omega_d \int_{2|z-z_0|}^{(R\wedge|x-z|)/2} \frac{1}{r^2} \frac{\Phi'(r^{-1})}{\Phi(c_5^{-1}r^{-1})^{1/2}} dr \\
&= c_5c_6\omega_d C_0^{-1/2} \int_{2|z-z_0|}^{(R\wedge|x-z|)/2} -(\Phi(r^{-1})^{1/2})' dr \\
&= c_5c_6\omega_d C_0^{-1/2} \Big(\Phi\big((2|z - z_0|)^{-1}\big)^{1/2} - \Phi\big(2(R \wedge |x - z|)^{-1}\big)^{1/2}\Big)
\end{aligned}$$

for some constant $c_6(\eta) > 0$.

For simplicity, we define

$$F(x, z) := \frac{\Phi(\delta_D(z)^{-1})^{1/2}\Phi'(|x - z|^{-1})}{|x - z|^{d+1}\Phi(\delta_D(x)^{-1})^{1/2}\Phi(|x - z|^{-1})}. \tag{2.35}$$

Combining Proposition 2.5, (2.29), (2.31), (2.34) and (2.35), for $32\delta_D(z) < |x - z|$ and $\delta_D(z) < R/17$,

(2.36)

$K_D(x, z)$

$$
\begin{aligned}
&= \frac{1}{2}K_D(x,z) + \frac{1}{2}K_D(x,z) \\
&\geq c_7 F(x,z)\frac{\Phi(|x-z|^{-1})^{1/2}}{\Phi(\delta_D(z)^{-1})^{1/2}} \\
&\quad + c_8 F(x,z)\left(\frac{1}{\Phi(\delta_D(z)^{-1})^{1/2}}\left(\Phi\big((2|z-z_0|)^{-1}\big)^{1/2} - 2\Phi\big((R\wedge|x-z|)^{-1}\big)^{1/2}\right)\right) \\
&\geq c_7 F(x,z)\frac{\Phi((|x-z|\wedge 3d_D)^{-1})^{1/2}}{\Phi(\delta_D(z)^{-1})^{1/2}} \\
&\quad + c_9 F(x,z)\left(\frac{1}{\Phi(\delta_D(z)^{-1})^{1/2}}\left(\Phi\big((2|z-z_0|)^{-1}\big)^{1/2} - 2\Phi\big((R\wedge|x-z|)^{-1}\big)^{1/2}\right)\right) \\
&\geq c_9\, F(x,z)\frac{\Phi\big((2|z-z_0|)^{-1}\big)^{1/2}}{\Phi(\delta_D(z)^{-1})^{1/2}} \geq c_{10}\, F(x,z).
\end{aligned}
$$

In the second inequality, the constant c_9 is chosen as follows. For this, we use $|x-z| < 3d_D$. For the case when $|x-z| \leq R$, take c_{11} so that $2c_{11} \leq c_7$. For $|x-z| > R$, take c_{12} sufficiently small so that $c_7 > 2c_{12}c_{13}$, where $c_{13} = R/(3d_D C_0^{1/2})$, which satisfies $\Phi((3d_D)^{-1})^{1/2} \geq c_{13}\Phi(R^{-1})^{1/2}$. Define $c_9 = c_8 \wedge c_{11} \wedge c_{12}$. Then the third inequality holds. For the last inequality, we use $\delta_D(z) \leq |z-z_0| \leq 2\delta_D(z)$ and so $c_{10} = c_9/4C_0^{1/2}$. Hence we get (2.36).

Therefore, by (2.27), (2.28) and (2.36), we have for $\delta_D(z) \leq 2d_D$,

$$
K_D(x,z) \geq c_{14}\frac{\Phi(\delta_D(z)^{-1})^{1/2}\Phi'(|x-z|^{-1})}{|x-z|^{d+1}\Phi(\delta_D(x)^{-1})^{1/2}\Phi(|x-z|^{-1})},
$$

where $c_{14} = c_{14}(\gamma_1, C_0, C_1, C_3, C_4, C_5, C_6, C_7, M, R/d_D, \eta, d)$. □

Corollary 2.7. *Suppose that $M \geq 1$ and that D is a ball with radius $r < M/2$. Furthermore, assume that there exist a function Φ satisfying* **(P1)**-**(P4)** *and a decreasing function j such that* **(G)**, **(J1)**, **(J2)**, **(J3)** *hold. Then there exists $c = c(\gamma_1, \gamma_2, C_0, C_1, C_2, C_3, C_4, C_5, C_6, C_7, M, d) > 1$ such that*

(2.37)

$$
\begin{aligned}
&c^{-1}\frac{\Phi(\delta_D(z)^{-1})^{1/2}}{\Phi(\delta_D(x)^{-1})^{1/2}\Phi(|x-z|^{-1})(1+\Phi(d_D^{-1})^{1/2}\Phi(\delta_D(z)^{-1})^{-1/2})}j(|x-z|) \\
&\leq K_D(x,z) \\
&\leq c\frac{\Phi(\delta_D(z)^{-1})^{1/2}}{\Phi(\delta_D(x)^{-1})^{1/2}\Phi(|x-z|^{-1})(1+\Phi(d_D^{-1})^{1/2}\Phi(\delta_D(z)^{-1})^{-1/2})}j(|x-z|)
\end{aligned}
$$

holds for every $x \in D$ and $z \in \overline{D}^c$. In particular, when the constants C_3, C_4 in **(G)** *are independent of $r < M/2$, then* (2.37) *holds for all balls with radius $r < M/2$ with the same constant c.*

Proof. For any $r < M/2$, a ball with radius r satisfies the cone condition with cone characteristic constant $(r, 1)$. So the ratio $R/d_D = 1/2$ and (2.37) holds for some $c = c(\gamma_1, \gamma_2, C_0, C_1, C_2, C_3, C_4, C_5, C_6, C_7, M, d) > 1$. Except for C_3 and C_4, all other constants are independent of r. Thus if C_3, C_4 are independent of r, then the constant c is independent of the radius of the ball. □

3. Remark

We first record a simple fact.

Lemma 3.1 ([5, Lemma 1.3]). *Suppose there exist constants $\sigma_1 > 0$ and $\delta_1 > 0$ such that*

$$\frac{\phi(\lambda x)}{\phi(\lambda)} \geq \sigma_1 x^{\delta_1} \quad \text{for all} \quad x \geq 1 \quad \text{and} \quad \lambda \geq \lambda_0 .$$

Then there exists a constant $c > 0$ such that $\phi(\lambda) \leq c\lambda\phi'(\lambda)$ for all $\lambda \geq \lambda_0$.

Moreover, by concavity, we see that

$$\phi(t\lambda) \leq \lambda\phi(t), \quad \lambda \geq 1, \ t > 0. \tag{3.1}$$

Thus combining Theorem 1.3, Lemma 3.1 and (3.1), we obtain a familiar form of the Poisson kernel estimates.

Corollary 3.2. *Suppose that $X = (X_t : t \geq 0)$ is a transient subordinate Brownian motion whose characteristic exponent is given by $\Phi(\theta) = \phi(|\theta|^2)$, $\theta \in \mathbb{R}^d$, where $\phi : (0, \infty) \to [0, \infty)$ is a complete Bernstein function such that*

$$c_1 x^{\alpha/2} \leq \frac{\phi(\lambda x)}{\phi(\lambda)} \leq c_2 x^{\beta/2} \quad \text{for all} \quad x \geq 1 \quad \text{and} \quad \lambda \geq \lambda_1$$

for some constants $c_1, c_2, \lambda_1 > 0$, $\alpha, \beta \in (0, 2)$ and $\alpha \leq \beta$. We further assume that **(A-4)** *holds with $\delta = 1 - \beta/2$.*

Then for every bounded $C^{1,1}$ open set D in $\mathbb{R}^d$ with characteristics (R, Λ), there exists $c = c(d_D, R, \Lambda, \phi, d) > 1$ such that for $z \in \overline{D}^c$ with $\delta_D(z) \leq 2d_D$,

$$\begin{aligned} &c^{-1} \frac{\phi(\delta_D(z)^{-2})^{1/2}}{\phi(\delta_D(x)^{-2})^{1/2}(1 + \phi(\delta_D(z)^{-2})^{-1/2})} |x - z|^{-d} \\ &\leq K_D(x, z) \leq c \, \frac{\phi(\delta_D(z)^{-2})^{1/2}}{\phi(\delta_D(x)^{-2})^{1/2}(1 + \phi(\delta_D(z)^{-2})^{-1/2})} |x - z|^{-d}. \end{aligned}$$

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