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ON EULER'S QUARTIC SURFACE

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1.

We are concerned in this note with the quartic surface given by the equation

$$x^4+y^4+z^4 = t^4. \tag{1}$$

This has recently been shown by Noam D. Elkies (communication to Richard K. Guy dated August 19, 1987) to have the integer solution

$$2682440^4+15365639^4+187960^4 = 20615673^4.$$

Previously, the best result appears to be that of Lander, Parkin, Selfridge [7] who by direct computation showed that there is no non-trivial solution in integers of the Diophantine equation (1), for which $t < 220.000$.

Euler is apparently the first person, at least in print, to consider the question of whether a fourth power could be expressed as the sum of three smaller fourth powers. His remarks [5] are of interest:

> Quum demonstratum sit neque summam neque differentiam duorum biquadratorum quadratum esse posse, multo minus biquadratum esse poterit; haud minori autem fiducia negari solet summam trium adeo biquadratorum umquam biquadratum esse posse, etiamsi hoc nusquam demonstratum reperiatur. Utrum autem quatuor biquadrata reperire liceat, quorum summa sit biquadratum, merito dubitamus, quum a nemine adhuc talia biquadrata sint exhibita.

In later writings, his remarks [6] are stonger:

> Pluribus autem insignibus Geometris visum est haec theoremata latius extendi posse. Quemadmodum enim duo cubi exhiberi nequeunt, quorum summa vel differentia sit

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cubus, ita etiam certum est nequidem exhiberi posse tria biquadrata, quorum summa sit pariter biquadratum, sed ad minimum quatuor biquadrata requiri, ut eorum summa prodire queat biquadratum, quamquam nemo adhuc talia quatuor biquadrata assignare potuerit. Eodem modo etiam affirmari posse videtur non exhiberi posse quatuor potestates quintas, quarum summa etiam esset potestas quinta; similique modo res se habebit in altioribus potestatibus; unde sequentes quoque positiones omnes pro impossibilibus erunt habendae:

$$\text{I.}\quad a^3+b^3=c^3,$$

$$\text{II.}\quad a^4+b^4+c^4=d^4,$$

$$\text{III.}\quad a^5+b^5+c^5+d^5=e^5,$$

$$\text{IV.}\quad a^6+b^6+c^6+d^6+e^6=f^6$$

etc.

Plurimum igitur scientia numerorum promoveri esset censenda, si demonstrationem desideratam etiam ad has formulas extendere liceret.

Despite the assuredness, a proof is lacking. In 1911, Norrie gave a numerical example to show that a fourth power could indeed be a sum of four fourth powers

$$30^4+120^4+272^4+315^4=353^4,$$

as predicted by Euler. However, in 1966, Lander and Parkin found by computer search the identity

$$27^5+84^5+110^5+113^5=144^5,$$

which is a counterexample to the Euler conjecture on fifth powers. The fourth power problem was thus rendered all the more tantalizing; but the example of Elkies shows also that Euler was mistaken.

In the current note, we make no further remarks on this aspect of equation (1). Instead, we concentrate upon the existence of parametrizations of (1) corresponding to curves of geometric genus zero lying on the surface. Segre [9] in his extensive survey of the surface

$$x_1^4+x_2^4+x_3^4+x_4^4=0$$

considers parametrizable curves of small degree, which consequently correspond to such curves on (1) with field of definition $C(\lambda)$. Attention here is restricted to real parametrizations (which in fact turns out to be equivalent to parametrizations defined over $Q(\sqrt{2})$).

There are infinitely many such real parametrizations, yet by a simple application of Galois Theory, no such parametrizations defined over Q. The methods of Swinnerton-Dyer [12] as exemplified in Bremner [1], [2], [3], coupled with the finiteness results of Sterk [11] provide a finite algorithm for generating all such real parametrizations; but the calculations necessary to furnish the relevant finite sets are not a priori effective. See the worked examples of Swinnerton-Dyer [12] and Bremner [1], [2]. We have not undertaken these computations for the surface (1).

As a corollary to knowing all the real parametrizable curves on (1) (at least, of small degree), we exhibit those diagonal quartic surfaces defined over Q which admit of rational parametrizations of degree 2, and of degree 4.

2.

Let V denote the surface (1). Then V is a non-singular quartic surface, and hence K3. So algebraic equivalence of divisors on V is the same as linear equivalence of divisors. By abuse of notation we shall use the same symbol to denote a divisor on V, its linear equivalence class, and the corresponding curve on V.

There are precisely 48 straight lines on V, given in Table 1. The field of definition of these lines is $Q(i, \sqrt{2})$; the action of the Galois group is such that $\sqrt{2} \to -\sqrt{2}$ interchanges Γ_j with Γ_j'', Γ_j' with Γ_j''', and $i \to -i$ interchanges Γ_j with Γ_j''', Γ_j' with Γ_j''. The matrix of their intersection numbers is shown in Matrix 1, and has rank 20. It is (reasonably) well-known that the Néron-Severi group $NS(V, C)$ of V over C is generated as Z-module by the classes of these straight lines (see for example Shioda [10, p. 181]). Accordingly, it follows that the field of definition of $NS(V, C)$ is the field $K = Q(i, \sqrt{2})$.

It is readily checked that the 20 divisors

(2) $\Gamma_1, \Gamma_1', \Gamma_1'', \Gamma_1''', \Gamma_2, \Gamma_2', \Gamma_2'', \Gamma_2'''; \Gamma_3, \Gamma_3''', \Gamma_5, \Gamma_5''', \Gamma_6, \Gamma_6''', \Gamma_7, \Gamma_7''', \Gamma_9, \Gamma_9''', \Gamma_{10}, \Gamma_{10}'''$

form a basis for $NS(V, C)$ over Z. The respective combinations of the 48 lines on V in terms of this basis are given in Table 2.

Put

(3) $$L_i = \Gamma_i + \Gamma_i''', \quad i = 1, \ldots, 12$$

TABLE 1

	Γ_j	Γ'_j	Γ''_j	Γ'''_j
Γ_1	$t = x$ $y = \varepsilon z$	$t = x$ $y = \varepsilon^3 z$	$t = x$ $y = -\varepsilon z$	$t = x$ $y = -\varepsilon^3 z$
Γ_2	$t = \varepsilon^2 x$ $y = \varepsilon z$	$t = -\varepsilon^2 x$ $y = \varepsilon^3 z$	$t = \varepsilon^2 x$ $y = -\varepsilon z$	$t = -\varepsilon^2 x$ $y = -\varepsilon^3 z$
Γ_3	$t = \varepsilon^2 x$ $y = \varepsilon^3 z$	$t = -\varepsilon^2 x$ $y = \varepsilon z$	$t = \varepsilon^2 x$ $y = -\varepsilon^3 z$	$t = -\varepsilon^2 x$ $y = -\varepsilon z$
Γ_4	$t = -x$ $y = \varepsilon z$	$t = -x$ $y = \varepsilon^3 z$	$t = -x$ $y = -\varepsilon z$	$t = -x$ $y = -\varepsilon^3 z$
Γ_5	$t = y$ $x = \varepsilon z$	$t = y$ $x = \varepsilon^3 z$	$t = y$ $x = -\varepsilon z$	$t = y$ $x = -\varepsilon^3 z$
Γ_6	$t = \varepsilon^2 y$ $x = \varepsilon z$	$t = -\varepsilon^2 y$ $x = \varepsilon^3 z$	$t = \varepsilon^2 y$ $x = -\varepsilon z$	$t = -\varepsilon^2 y$ $x = -\varepsilon^3 z$
Γ_7	$t = \varepsilon^2 y$ $x = \varepsilon^3 z$	$t = -\varepsilon^2 y$ $x = \varepsilon z$	$t = \varepsilon^2 y$ $x = -\varepsilon^3 z$	$t = -\varepsilon^2 y$ $x = -\varepsilon z$
Γ_8	$t = -y$ $x = \varepsilon z$	$t = -y$ $x = \varepsilon^3 z$	$t = -y$ $x = -\varepsilon z$	$t = -y$ $x = -\varepsilon^3 z$
Γ_9	$t = z$ $x = \varepsilon y$	$t = z$ $x = \varepsilon^3 y$	$t = z$ $x = -\varepsilon y$	$t = z$ $x = -\varepsilon^3 y$
Γ_{10}	$t = \varepsilon^2 z$ $x = \varepsilon y$	$t = -\varepsilon^2 z$ $x = \varepsilon^3 y$	$t = \varepsilon^2 z$ $x = -\varepsilon y$	$t = -\varepsilon^2 z$ $x = -\varepsilon^3 y$
Γ_{11}	$t = \varepsilon^2 z$ $x = \varepsilon^3 y$	$t = -\varepsilon^2 z$ $x = \varepsilon y$	$t = \varepsilon^2 z$ $x = -\varepsilon^3 y$	$t = -\varepsilon^2 z$ $x = -\varepsilon y$
Γ_{12}	$t = -z$ $x = \varepsilon y$	$t = -z$ $x = \varepsilon^3 y$	$t = -z$ $x = -\varepsilon y$	$t = -z$ $x = -\varepsilon^3 y$

N.B. $\varepsilon = \dfrac{1+i}{\sqrt{2}}$ is a fourth root of -1.

and let a bar denote conjugacy under the K-automorphism induced by $\sqrt{2} \to -\sqrt{2}$, so that

$$\bar{L}_i = \Gamma'_i + \Gamma''_i, \quad i = 1, \ldots, 12. \tag{4}$$

Then L_i, $\bar{L}_i \in \mathrm{NS}(V, \mathsf{R})$, where the field of definition of $\mathrm{NS}(V, \mathsf{R})$ is $\mathsf{Q}(\sqrt{2})$. The matrix of intersection of the L_i and $\bar{L}_i$ is shown in Matrix 2, and is of rank 10.

MATRIX 1

	Γ_1				Γ_2				Γ_3				Γ_4				Γ_5				Γ_6				Γ_7				Γ_8				Γ_9				Γ_{10}				Γ_{11}				Γ_{12}			
Γ_1	-2	1	1	1	1					1			1				1							1	1						1					1	1							1		1		
	1	-2	1	1		1			1					1				1					1			1						1			1			1					1		1			
	1	1	-2	1			1					1			1				1			1					1		1					1					1			1						1
	1	1	1	-2				1			1					1				1	1							1		1			1							1	1						1	
Γ_2	1				-2		1		1	1	1		1							1	1							1		1					1			1					1		1			
		1				-2		1	1	1		1		1					1			1					1		1							1	1							1		1		
			1		1		-2		1		1	1			1			1					1			1						1	1							1	1						1	
				1		1		-2		1	1	1				1	1							1	1						1			1					1			1					1
Γ_3		1			1	1	1		-2		1			1			1							1	1						1			1					1			1					1
	1				1	1		1		-2		1	1					1					1			1						1	1							1	1						1	
				1	1		1	1	1		-2					1			1			1					1		1							1	1							1		1		
			1			1	1	1		1		-2			1					1	1							1		1					1			1					1		1			
Γ_4	1				1					1			-2	1	1	1			1			1					1		1					1					1			1						1
		1				1			1				1	-2	1	1				1	1							1		1			1							1	1						1	
			1				1					1	1	1	-2	1	1							1	1						1					1	1							1		1		
				1				1			1		1	1	1	-2		1					1			1						1			1			1					1		1			
Γ_5	1							1	1						1		-2	1	1	1	1					1			1				1					1					1				1	
		1					1			1						1	1	-2	1	1		1			1					1				1			1							1				1
			1			1					1		1				1	1	-2	1			1					1			1				1					1	1				1			
				1	1							1		1			1	1	1	-2				1			1					1				1			1			1				1		
Γ_6				1	1							1		1			1				-2		1		1	1	1		1					1			1							1				1
			1			1					1		1					1				-2		1	1	1		1		1			1					1					1				1	
		1					1			1						1			1		1		-2		1		1	1			1					1			1			1				1		
	1							1	1						1					1		1		-2		1	1	1				1			1					1	1				1			
Γ_7	1							1	1						1			1			1	1	1		-2		1			1					1					1	1				1			
		1					1			1						1	1				1	1		1		-2		1	1							1			1			1				1		
			1			1					1		1							1	1		1	1	1		-2					1	1					1					1				1	
				1	1							1		1					1			1	1	1		1		-2			1			1			1							1				1
Γ_8			1			1					1		1				1				1					1			-2	1	1	1			1					1	1				1			
				1	1							1		1				1				1			1				1	-2	1	1				1			1			1				1		
	1							1	1						1				1				1					1	1	1	-2	1	1					1					1				1	
		1					1			1						1				1				1			1		1	1	1	-2		1			1							1				1
Γ_9				1			1			1				1			1					1					1				1		-2	1	1	1	1					1			1			
			1					1	1				1					1			1							1				1	1	-2	1	1		1			1					1		
		1			1							1				1			1					1	1				1				1	1	-2	1			1					1			1	
	1					1					1				1					1			1			1				1			1	1	1	-2				1			1					1
Γ_{10}	1					1					1				1			1			1							1				1	1				-2		1		1	1	1		1			
		1			1							1				1	1					1					1				1			1				-2		1	1	1		1		1		
			1					1	1				1							1			1			1				1					1		1		-2		1		1	1			1	
				1			1			1				1					1					1	1				1							1		1		-2		1	1	1				1
Γ_{11}				1			1			1				1					1					1	1				1					1			1	1	1		-2		1			1		
			1					1	1				1							1			1			1				1			1				1	1		1		-2		1	1			
		1			1							1				1	1					1					1				1					1	1		1	1	1		-2					1
	1					1					1				1			1			1							1				1			1			1	1	1		1		-2			1	
Γ_{12}		1			1							1				1			1					1	1				1				1				1					1			-2	1	1	1
	1					1					1				1					1			1			1				1				1				1			1				1	-2	1	1
				1			1			1				1			1					1					1				1				1				1					1	1	1	-2	1
			1					1	1				1					1			1							1				1				1				1			1		1	1	1	-2

TABLE 2

	Γ_1	Γ_1'	Γ_1''	Γ_1'''	Γ_2	Γ_2'	Γ_2''	Γ_2'''	Γ_3	Γ_3'''	Γ_5	Γ_5'''	Γ_6	Γ_6'''	Γ_7	Γ_7'''	Γ_9	Γ_9'''	Γ_{10}	Γ_{10}'''
Γ_1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Γ_2	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
Γ_3	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
	1	1	1	1	0	−1	0	−1	0	−1	0	0	0	0	0	0	0	0	0	0
	1	1	1	1	−1	0	−1	0	−1	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
Γ_4	1	0	0	0	−1	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0
	1	0	1	1	0	−1	0	0	−1	0	0	0	0	0	0	0	0	0	0	0
	1	1	0	1	0	0	−1	0	0	−1	0	0	0	0	0	0	0	0	0	0
	0	0	0	−1	1	0	1	−1	1	0	0	0	0	0	0	0	0	0	0	0
Γ_5	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
	0	0	1	1	0	0	0	0	−1	1	−1	0	0	0	−1	1	0	0	0	0
	1	1	0	0	0	0	0	0	1	−1	0	−1	0	0	1	−1	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
Γ_6	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
	0	−1	−1	0	1	−1	0	0	0	0	1	1	1	0	0	0	0	0	0	0
	0	−1	−1	0	0	0	−1	1	0	0	1	1	0	1	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
Γ_7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
	1	2	2	1	−1	1	0	0	0	0	−1	−1	−1	−1	0	−1	0	0	0	0
	1	2	2	1	0	0	1	−1	0	0	−1	−1	−1	−1	−1	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
Γ_8	0	−1	−1	0	1	−1	0	0	0	0	0	1	0	1	0	1	0	0	0	0
	1	2	1	0	−1	1	0	0	1	−1	0	−1	−1	0	0	−1	0	0	0	0
	0	1	2	1	0	0	1	−1	−1	1	−1	0	0	−1	−1	0	0	0	0	0
	0	−1	−1	0	0	0	−1	1	0	0	1	0	1	0	1	0	0	0	0	0
Γ_9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
	1	1	0	0	0	0	0	0	1	−1	0	0	−1	1	1	−1	−1	0	−1	1
	0	0	1	1	0	0	0	0	−1	1	0	0	1	−1	−1	1	0	−1	1	−1
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
Γ_{10}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
	1	0	0	1	−1	0	−1	0	−1	−1	0	0	0	0	0	0	1	1	1	0
	1	0	0	1	0	−1	0	−1	−1	−1	0	0	0	0	0	0	1	1	0	1
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
Γ_{11}	0	1	1	0	0	1	0	0	0	1	0	0	0	−1	−1	0	0	0	0	−1
	0	0	0	0	1	0	0	0	0	1	0	0	1	0	0	1	−1	−1	0	−1
	0	0	0	0	0	0	0	1	1	0	0	0	0	1	1	0	−1	−1	−1	0
	0	1	1	0	0	0	1	0	1	0	0	0	−1	0	0	−1	0	0	−1	0
Γ_{12}	1	1	1	1	−1	0	0	0	0	−1	0	0	−1	0	0	−1	0	1	−1	1
	−1	−1	0	0	1	−1	1	0	0	1	0	0	1	0	0	1	0	−1	0	0
	0	0	−1	−1	0	1	−1	1	1	0	0	0	0	1	1	0	−1	0	0	0
	1	1	1	1	0	0	0	−1	−1	0	0	0	0	−1	−1	0	1	0	1	−1

MATRIX 2

	L_1	$\bar{L}_1$	L_2	$\bar{L}_2$	L_3	$\bar{L}_3$	L_4	$\bar{L}_4$	L_5	$\bar{L}_5$	L_6	$\bar{L}_6$	L_7	$\bar{L}_7$	L_8	$\bar{L}_8$	L_9	$\bar{L}_9$	L_{10}	$\bar{L}_{10}$	L_{11}	$\bar{L}_{11}$	L_{12}	$\bar{L}_{12}$
L_1	−2	4	2			2	2		2		2		2			2	2		2		2			2
	4	−2		2	2			2		2		2		2	2			2		2		2	2	
L_2	2		−4	2	2	4	2		2		2		2			2		2		2		2	2	
		2	2	−4	4	2		2		2		2		2	2		2		2		2			2
L_3		2	2	4	−4	2		2	2		2		2			2		2		2		2	2	
	2		4	2	2	−4	2			2		2		2	2		2		2		2			2
L_4	2		2			2	−2	4		2		2		2	2			2		2		2	2	
		2		2	2		4	−2	2		2		2			2	2		2		2			2
L_5	2		2		2			2	−2	4	2			2	2		2			2		2		2
		2		2		2	2		4	−2		2	2			2		2	2		2		2	
L_6	2		2		2			2	2		−4	2	2	4	2			2	2		2		2	
		2		2		2	2			2	2	−4	4	2		2	2			2		2		2
L_7	2		2		2			2		2	2	4	−4	2		2		2	2		2		2	
		2		2		2	2		2		4	2	2	−4	2		2			2		2		2
L_8		2		2		2	2		2		2			2	−2	4		2	2		2		2	
	2		2		2			2		2		2	2		4	−2	2			2		2		2
L_9	2			2		2		2	2			2		2		2	−2	4	2			2	2	
		2	2		2		2			2	2		2		2		4	−2		2	2			2
L_{10}	2			2		2		2		2	2		2		2		2		−4	2	2	4	2	
		2	2		2		2		2			2		2		2		2	2	−4	4	2		2
L_{11}	2			2		2		2		2	2		2		2			2	2	4	−4	2		2
		2	2		2		2		2			2		2		2	2		4	2	2	−4	2	
L_{12}		2	2		2		2			2	2		2		2		2		2			2	−2	4
	2			2		2		2	2			2		2		2		2		2	2		4	−2

Take now $\Gamma \in \mathrm{NS}(V, \mathsf{R})$, so that $\Gamma \sim \tilde{\Gamma}$, where $\sim$ denotes conjugacy by $i \to -i$. From (2), there exists integers $a_1, a'_1, \ldots, a_{10}, a'''_{10}$ such that

$$\Gamma \sim a_1\Gamma_1 + a'_1\Gamma'_1 + \ldots + a_{10}\Gamma_{10} + a'''_{10}\Gamma'''_{10}$$

and it follows that

$$a_1\Gamma_1 + a'_1\Gamma'_1 + \ldots + a_{10}\Gamma_{10} + a'''_{10}\Gamma'''_{10} \sim a_1\Gamma'''_1 + a'_1\Gamma''_1 + \ldots + a_{10}\Gamma'''_{10} + a'''_{10}\Gamma_{10}$$

whence

$$(a_1 - a'''_1)(\Gamma_1 - \Gamma'''_1) + (a'_1 - a''_1)(\Gamma'_1 - \Gamma''_1) + \ldots + (a_{10} - a'''_{10})(\Gamma_{10} - \Gamma'''_{10}) \sim 0.$$

Thus from (2), $a_1 = a'''_1, a'_1 = a''_1, \ldots, a_{10} = a'''_{10}$, and

$$\Gamma \sim a_1L_1 + a'_1\bar{L}_1 + \ldots + a_{10}L_{10}.$$

So the ten divisors

(5) $$L_1, \bar{L}_1, L_2, \bar{L}_2, L_3, L_5, L_6, L_7, L_9, L_{10}$$

form a basis for NS(V, R) as Z-module. The respective combinations of $L_1, \bar{L}_1, \ldots, L_{12}, \bar{L}_{12}$ in terms of this basis, are given in Table 3.

TABLE 3

	L_1	$\bar{L}_1$	L_2	$\bar{L}_2$	L_3	L_5	L_6	L_7	L_9	L_{10}
L_1	1	0	0	0	0	0	0	0	0	0
$\bar{L}_1$	0	1	0	0	0	0	0	0	0	0
L_2	0	0	1	0	0	0	0	0	0	0
$\bar{L}_2$	0	0	0	1	0	0	0	0	0	0
L_3	0	0	0	0	1	0	0	0	0	0
$\bar{L}_3$	2	2	−1	−1	−1	0	0	0	0	0
L_4	−1	0	0	1	1	0	0	0	0	0
$\bar{L}_4$	2	1	0	−1	−1	0	0	0	0	0
L_5	0	0	0	0	0	1	0	0	0	0
$\bar{L}_5$	1	1	0	0	0	−1	0	0	0	0
L_6	0	0	0	0	0	0	1	0	0	0
$\bar{L}_6$	0	−2	1	−1	0	2	1	0	0	0
L_7	0	0	0	0	0	0	0	1	0	0
$\bar{L}_7$	2	4	−1	1	0	−2	−2	1	0	0
L_8	0	−2	1	−1	0	1	1	1	0	0
$\bar{L}_8$	1	3	−1	1	0	−1	−1	−1	0	0
L_9	0	0	0	0	0	0	0	0	1	0
$\bar{L}_9$	1	1	0	0	0	0	0	0	−1	0
L_{10}	0	0	0	0	0	0	0	0	0	1
$\bar{L}_{10}$	2	0	−1	−1	−2	0	0	0	2	1
L_{11}	0	2	0	1	1	0	−1	−1	0	−1
$\bar{L}_{11}$	0	0	1	0	1	0	1	1	−2	−1
L_{12}	2	2	−1	0	−1	0	−1	−1	1	0
$\bar{L}_{12}$	−1	−1	1	0	1	0	1	1	−1	0

Finally, put

(6) $$\Lambda_i = L_i + \bar{L}_i, \quad i = 1, \ldots, 12$$

so that $\Lambda_i \in$ NS(V, Q). The matrix of intersection of the Λ_i is shown in Matrix 3, and is of rank 4.

Just as above, by equating a Z-linear combination of $L_1, \bar{L}_1, \ldots, L_9, L_{10}$ to its conjugate under $\sqrt{2} \to -\sqrt{2}$, one discovers that the four divisors

$$\Lambda_1, \Lambda_2, \tfrac{1}{2}(\Lambda_2 + \Lambda_6), \tfrac{1}{2}(\Lambda_6 + \Lambda_{10})$$

(where $\frac{1}{2}(\Lambda_2 + \Lambda_6) \sim -\bar{L}_1 + L_2 + L_5 + L_6$; $\frac{1}{2}(\Lambda_6 + \Lambda_{10}) \sim L_1 - \bar{L}_1 - \bar{L}_2 - L_3 + L_5 + L_6 + L_9 + L_{10}$) form a basis for NS($V$, Q) over Z. The latter two

MATRIX 3

	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5	Λ_6	Λ_7	Λ_8	Λ_9	Λ_{10}	Λ_{11}	Λ_{12}
Λ_1	4	4	4	4	4	4	4	4	4	4	4	4
Λ_2	4	−4	12	4	4	4	4	4	4	4	4	4
Λ_3	4	12	−4	4	4	4	4	4	4	4	4	4
Λ_4	4	4	4	4	4	4	4	4	4	4	4	4
Λ_5	4	4	4	4	4	4	4	4	4	4	4	4
Λ_6	4	4	4	4	4	−4	12	4	4	4	4	4
Λ_7	4	4	4	4	4	12	−4	4	4	4	4	4
Λ_8	4	4	4	4	4	4	4	4	4	4	4	4
Λ_9	4	4	4	4	4	4	4	4	4	4	4	4
Λ_{10}	4	4	4	4	4	4	4	4	4	−4	12	4
Λ_{11}	4	4	4	4	4	4	4	4	4	12	−4	4
Λ_{12}	4	4	4	4	4	4	4	4	4	4	4	4

divisors are indeed effective, and are realisable as curves of genus 1 on V.

Put

(7) $$(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (\Lambda_1, \Lambda_2, \tfrac{1}{2}(\Lambda_2+\Lambda_6), \tfrac{1}{2}(\Lambda_6+\Lambda_{10})).$$

Then the matrix of intersection of the γ_i is given in Matrix 4.

MATRIX 4

	γ_1	γ_2	γ_3	γ_4
γ_1	4	4	4	4
γ_2	4	−4	0	4
γ_3	4	0	0	2
γ_4	4	4	2	0

Consider first a curve Γ on V defined over Q. Then there exist integers $n_1, \ldots, n_4$ such that

$$\Gamma \sim n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3 + n_4\gamma_4.$$

The formula for the arithmetic genus of Γ is given by (see, for example, Safarevic [8, p. 5])

$$p_a(\Gamma) = \tfrac{1}{2}(\Gamma\cdot\Gamma)+1$$

so that

(8) $$p_a(\Gamma) = 2n_1^2-2n_2^2+4n_1n_2+4n_1n_3+4n_1n_4+4n_2n_4+2n_3n_4+1.$$

Further, the degree of Γ is given by

(9) $$\begin{aligned}\deg(\Gamma) &= \deg(n_1\gamma_1)+\ldots+\deg(n_4\gamma_4)\\ &= 4(n_1+n_2+n_3+n_4).\end{aligned}$$

The formulae at (8) and (9) immediately imply the following theorem.

THEOREM.

(i) *$p_a(\Gamma)$ is odd.*
(ii) $\deg(\Gamma)$ *is divisible by* 4.

COROLLARY. *There are no curves on V, defined over* Q, *of arithmetic genus* 0.

REMARK. The corollary does not imply the total non-existence of rational parametric solutions to the equation (1). Such a parametrization corresponds on V to a curve of geometric genus 0, and it may be the case that such a curve actually has arithmetic genus greater than 0. See Swinnerton-Dyer [12] for further discussion.

3.

Consider now a curve Γ on V, defined over R. From (5), there exist integers m_i, $i = 1, \ldots, 10$, with

(10) $$\Gamma \sim m_1L_1+m_2\bar{L}_1+m_3L_2+m_4\bar{L}_2+m_5L_3+m_6L_5+ +m_7L_6+m_8L_7+m_9L_9+m_{10}L_{10}.$$

We know all parametrizations over C of degree 1, since these correspond to straight lines on V; and there are no such parametrizations over R.

For parametrizations of degree 2, we can avail ourselves of the enumeration by Segre [9] of quadrics defined over C on the surface. It turns out that there are precisely eight conics on (1) which are defined over R, and precisely two such conics up to symmetry, namely

$$(11)\qquad \begin{aligned} z &= x+y \\ t^2 &= \sqrt{2}(z^2-xy) \end{aligned}$$

and

$$(11\text{bis})\qquad \begin{aligned} z &= x+y \\ t^2 &= -\sqrt{2}(z^2-xy). \end{aligned}$$

Since (11bis) possesses no real points, there is essentially just the one quadratic (11) affording a real quadratic parametrization, viz.

$$(12)\qquad x:y:z:t = -3\lambda^2+2\lambda+1:3\lambda^2+2\lambda-1:4\lambda:\sqrt[4]{2}(3\lambda^2+1).$$

From (10), the degree of Γ is even, and so we can restrict attention to curves Γ with $\deg(\Gamma) = 2d$, $d \in \mathrm{Z}$. From (10),

$$(13)\qquad \sum_{i=1}^{10} m_i = d.$$

The Hodge-Index Theorem tells us the signature of the quadratic form of self-intersection $(\Gamma \cdot \Gamma)$, at least over C. Over R, then using (13) and a simple process of completing the square, we have the following equation, in which $\delta = \frac{1}{2}d$.

$$(14)\qquad \begin{aligned} &(m_1+m_4+m_6+m_8-\delta)^2+(m_1+m_4+m_5+m_6+m_{10}-\delta)^2+ \\ &+(m_1+m_4+m_5+m_7+m_8+m_9-\delta)^2+(m_1+m_3+m_5+m_9-\delta)^2+ \\ &+(m_1+m_3+m_5+m_{10}-\delta)^2+(m_1+m_4+m_7-\delta)^2+ \\ &+(m_3+m_4-m_5)^2+(m_7-m_8)^2+m_{10}^2 = 2\delta^2-\tfrac{1}{2}(\Gamma\cdot\Gamma). \end{aligned}$$

For curves Γ of genus 0 then $(\Gamma \cdot \Gamma) = -2$; and (14) for a prescribed value of δ gives an effective means of computing all divisors of the chosen degree and of genus 0.

For curves of degree 4, where $\delta = 1$, the constant on the right hand side of (14) is equal to 3, and one can (even by hand) tabulate the solutions of this quadratic form equation. It is appropriate to remark that the symmetries of V reduce the actual amount of computation, as follows. The group of rational symmetries is of order 48, generated by changing the signs of x, y, z, and permuting x, y, z. It is straightforward to verify that the class of divisors equivalent under symmetry to Γ at (10) always contains a divisor for which the coefficients m_i satisfy

$$(15) \qquad 0 \leqq m_{10} \leqq -m_7+m_8 \leqq m_3+m_4-m_5.$$

Hence it suffices to impose these extra conditions (15) on solutions of (14).

The divisors corresponding to these solutions are guaranteed to be effective, but in practice most turn out to be reducible: in this case, representing a conic pair. More generally, if Γ is of the form $A+B$ where A and B are effective divisors, then certainly Γ is reducible. Of course, only irreducible curves are of interest to us, and it turns out here that up to symmetry there is only one such curve which is not actually the sum of two conics, given by the divisor $\Gamma \sim -\bar{L}_1+L_2+L_6+L_7$.

The parametrization of the corresponding curve is given by

$$(16) \qquad x:y:z:t = \lambda^4-2:2\lambda^3:2\sqrt{2}\lambda:\lambda^4+2$$

and this curve together with the curve obtained by $\lambda \to -\lambda$ are cut out on V by the quadratic

$$(17) \qquad x^2-t^2+\sqrt{2}yz = 0.$$

Similarly, for curves of degree 6 with $\delta = 3/2$, it turns out that there are precisely three divisors up to symmetry with self-intersection -2. They are

$$-\bar{L}_1+L_2+\bar{L}_2+L_3+L_6;$$

$$-L_1+L_2+\bar{L}_2+L_3+L_5+L_6-L_9;$$

$$\bar{L}_1+\bar{L}_2+L_5-L_7+L_9.$$

Of these divisors, only the latter two are irreducible, the first representing a twisted cubic over C and its conjugate (easily identified from a listing the author made of all twisted cubics over C!).

THEOREM. *Up to symmetry, there is of* (1) *just one real parametrization (corresponding to a curve on V of arithmetic genus* 0) *of degree* 4. *There are just two such parametrizations of degree* 6.

One may continue the search to curves of higher degree, but now the problem of determining whether a given divisor is irreducible or not takes considerably more effort. One approach is to compile a list of all the C-divisors corresponding to curves of small degree on V defined over C. The techniques are exactly as above, applied to the quadratic form of intersection derived from the basis (2). This listing will then enable one to identify which divisors over R are reducible. But this method is tedious in the extreme, and hardly seems feasible for curves even of moderate degree. Alternatively, and more constructively, one can follow the ideas of Swinnerton-Dyer, and using fibrations of the surface into elliptic pencils, produce a set of automorphisms of the surface from which all curves of arithmetic genus 0 may be successively generated from a finite set S of such curves. By Sterk [11] there exists a finite set of automorphisms with this property. But we have not carried out what will inevitably be very extensive computations.

4.

There do indeed exist rational curves on V of genus 1. By (8), such curves correspond to solutions of the equation

$$2n_1^2 - 2n_2^2 + 4n_1n_2 + 4n_1n_3 + 4n_1n_4 + 4n_2n_4 + 2n_3n_4 = 0,$$

i.e., to solutions of

$$(2n_2 + n_3)^2 + (n_3 + n_4)^2 + n_4^2 = 2(n_1 + n_2 + n_3 + n_4)^2. \tag{18}$$

The smallest degree for such curves is 4, when $n_1 + n_2 + n_3 + n_4 = 1$; and then there are precisely twelve solutions to (18). The curves corresponding to these twelve divisors are all equivalent under symmetry, and so there is essentially a unique divisor of degree 4 and of genus 1. It may be realised as a pencil of curves given by the intersection of two quadratics

$$\begin{aligned} t_2 - ix^2 + y^2 + z^2 &= \theta(t - iy)(t - iz) \\ t^2 + ix^2 + y^2 + z^2 &= \bar{\theta}(t + iy)(t + iz), \end{aligned} \tag{19}$$

where $\theta \in \mathbf{C}$, $\theta\bar{\theta} = 2$. See also Dem'yanenko [4].

If $\theta \in \mathbf{Q}$, then of course (19) defines a curve Γ of genus 1 over the rationals. If Γ were to contain a rational point, then of course this would furnish a numerical counterexample to the Euler conjecturė. For interest, we obtained the necessary and sufficient conditions on θ for Γ to possess points in every completion of $\mathbf{Q}$. The "smallest" such θ (in the sense of smallest denominator) turn out to be

$$\theta = 1 \pm i$$

$$\theta = \frac{73 \pm 263i}{193}$$

$$\theta = \frac{313 + 103i}{233}$$

In the case of $\theta = 1 \pm i$, then Γ does possess rational points, corresponding to the trivial points on the surface (1); and a straight-forward descent argument shows that Γ has no further rational points.

For the other three values of θ above, a small computer search did not find any rational point on the corresponding Γ.

5.

Suppose we consider diagonal quartic surfaces of type

$$ax^4 + by^4 + cz^4 = dt^4, \quad a,b,c,d \in \mathbb{Q}, a,b,c,d > 0, \tag{20}$$

where a, b, c, d are fourth-power free, and ask for rational parametrizations. From section 3, there is up to symmetry just one parametrization of (20) over $\mathbb{R}$ of degree 2; and taking without loss of generality $a = 1$ in (20), this quadratic parametrization is provided by (11) in the form

$$x + b^{1/4}y - c^{1/4}z = 0$$
$$b^{1/4}xy - c^{1/2}z^2 + (d/2)^{1/2}t^2 = 0.$$

Consequently, for a rational parametrization, it is necessary that the following ratios be rational:

$$1 : b^{1/4} : c^{1/4} \quad \text{and} \quad b^{1/4} : c^{1/2} : (d/2)^{1/2}.$$

This is equivalent to $b,c \in \mathbb{Q}^4$, $d/2 \in \mathbb{Q}^2$, say $d = 2\delta^2$; and the surface (20) becomes equivalent to

$$x^4 + y^4 + z^4 = 2\delta^2 t^4$$

with parametrization from (12) given by

$$x : y : z : t = -3\lambda^2 + 2\lambda + 1 : 3\lambda^2 + 2\lambda - 1 : 4\lambda : (3\lambda^2 + 1)\delta^{-1/2}.$$

Thus, for a rational parametrization, $\delta \in \mathbb{Q}^2$, and we have the following result.

THEOREM. *Let V be a diagonal quartic surface of type* (20). *If V possesses a rational parametrization of degree* 2, *then V is given by the equation*

$$x^4 + y^4 + z^4 = 2t^4,$$

with parametrization

$$x : y : z : t = -3\lambda^2 + 2\lambda + 1 : 3\lambda^2 + 2\lambda - 1 : 4\lambda : 3\lambda^2 + 1.$$

Once can similarly ask about parametrizations of (20) of degree 4. Again, from section 3, there is essentially a unique parametrizable quartic curve of arithmetic genus 0 on V, which is cut out by the quadric

$$a^{1/2}x^2 - d^{1/2}t^2 + (4bc)^{1/4}yz = 0.$$

So the quartic curve can be defined over Q only if

$$a^{1/2} : d^{1/2} : (4bc)^{1/4}$$

are rational ratios.

Taking $a = 1$, then it follows that $4bc \in \mathbf{Q}^4$, $d \in \mathbf{Q}^2$, $d = r^2$ say. Then (20) is equivalent to the surface

$$x^4 + by^4 + \frac{4}{b}z^4 = r^2t^4, \tag{22}$$

with the quartic cut out by the quadric

$$x^2 - rt^2 + 2yz = 0. \tag{23}$$

It follows from (22) and (23) that points on the quartic satisfy

$$(by^2 + 2z^2)^2 = 4bryzt^2 \tag{24}$$

and so for a rational parametrization, $bryz$ is a rational square.

Put $by = \alpha p^2$, $rz = \alpha q^2$, so that (24) gives

$$\frac{2pq}{\alpha}t = \frac{p^4}{b} + \frac{2q^4}{r^2},$$

and, from (23),

$$\frac{4p^2q^2}{\alpha^2}x^2 = r\left[\frac{p^4}{b} - \frac{2q^4}{r^2}\right]^2.$$

The latter implies $r = e^2$, $e \in \mathbf{Q}$, and the following result is immediate.

THEOREM. *Let V be a diagonal quartic surface of type* (20). *If V possesses a rational parametrization of degree* 4, *corresponding to a curve of arithmetic genus* 0 *on V, then V is given by an equation*

$$x^4 + by^4 + \frac{4}{b}z^4 = t^4$$

with parametrization

$$x:y:z:t = \frac{1}{b}\lambda^4 - 2 : \frac{2}{b}\lambda^3 : 2\lambda : \frac{1}{b}\lambda^4 + 2.$$

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