On Exact Kalman Filtering of Polynomial Systems

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Abstract

A closed-form state estimator for some polynomial nonlinear systems is derived in this paper. Exploiting full Taylor series expansion we first give exact matrix expressions to compute mean and covariance of any random variable distribution that has been transformed through a polynomial function. An original discrete-time Kalman filtering implementation relying on this exact polynomial transformation is proposed. The important problem of chaotic synchronization of Chebyshev maps is then considered to illustrate the significance of these results. Mean Square Error (MSE) between synchronized signals and consistency criteria are chosen as performance measures under various signal-to-noise ratios (SNR). Comparisons to the popular Extended Kalman Filter (EKF) and to the recent Unscented Kalman Filter (UKF) are also conducted to show the pertinence of our filtering formulation.

Index Terms - Nonlinear Statistical Transformation, Nonlinear Kalman Filter, Closed-Form Computations, Polynomial Models, Chaos Synchronization, Chebyshev Maps

I. Introduction

The Kalman Filter (KF) is considered as a reference technique for linear estimation, yielding the Minimum Mean Squared Error (MMSE) solution in a recursive manner thanks to state-space modeling of the underlying dynamical system [1][2]. In case of nonlinear dynamics, it is well-known that the problem of state estimation becomes difficult. In this context no filtering scheme clearly outperforms all other strategies. As the full probability density function is required at each recursion step, the optimal Bayesian estimator is usually not used in practice. Only a small class of nonlinear systems can be estimated in this way [3][4]. Among the wide variety of approximate nonlinear filtering, the EKF remains a popular solution as it avoids the large computational cost usually required by nearly optimal approaches such as particle filters [5]. The EKF relies on a propagation of the state distribution, considered as a Gaussian random variable, through a first order linearization of the nonlinear system, hence keeping the structure of the KF. Although the EKF can cope with weakly nonlinear systems, large errors in the true posterior mean and covariance can occur in presence of severe nonlinearities as a consequence of successive linearizations. This problem can be mitigated in many ways while keeping the linearization approach of the EKF, among others via Iterated EKF or higher order EKF, but one limitation for practical purposes is the increased computational complexity. Another disadvantage of EKF-like filters is their difficulty of implementation due to analytical derivative computations.

Some attractive nonlinear Kalman filtering methods have recently been proposed to avoid previous limitations of the EKF without any significant additional computation cost. The UKF, first introduced by Julier *et al.* [6] in the context of nonlinear control and further extended by Wan and van der Merwe [7], addresses the approximation issues of the EKF. The state distribution is again represented by a Gaussian random variable, but is now specified using a minimal set of carefully chosen sample points (the *sigma points*). At each step of the recursion, these sample points are propagated through the true nonlinear functions of the model (*Unscented* Transformation), hence avoiding Jacobians computation. Following this approach, posterior mean and covariance are captured up to the third or second order terms of the Taylor series expansion, whatever the nonlinearity is. The approximation of higher order moments is enabled by changing the *sigma points* weights. The UKF is a good choice for real-time applications, as it gives a trade-off between accurate state estimation (always better consistency than the EKF), limited computational cost and ease of implementation. Norgaard *et al*. [8] have proposed another derivative-free nonlinear

Kalman filter. The authors make use of a polynomial interpolation of the system functions using Stirling's formula in order to compute analytically mean and covariance of the state distribution. No Taylor series truncation is then operated. This original nonlinear transformation is then exploited in a recursive manner owing to a Kalman filter structure. Two variants are suggested: the first one (DD1), being a generalization of the Central Difference Kalman Filter (CDKF) of Schei [10], relies on a first order polynomial approximation of the system nonlinearities; the second one (DD2) gives better results due to a second order approximation. This DD2 filter seems to perform very well as its *a priori* state estimation matches that derived through *Unscented* transformation and its covariance estimate is slightly more accurate. Lefebvre *et al*. have proposed in [9] a unified interpretation of some of these recent nonlinear Kalman filters (UKF, CDKF, DD1) using the concept of linear statistical regression and a comparison of performances is presented. A separate analysis of the process update and measurement update steps is also provided to show that mixing filtering techniques can increase the overall filter efficiency.

One motivation for the present paper was to extend the idea of polynomial-based transform suggested in [8] to some systems that intrinsically present polynomial functions. Whereas the DD2 filter of Norgaard *et al.* can be suitable for a wide variety of engineering applications, for specific cases where the underlying process and/or measurement model is of polynomial form, the overall estimation performance can significantly be increased. To the knowledge of the authors, this question has just been investigated for some continuous-time systems in a recent paper of Basin [11], under the assumption of a linear observation model. So, the present paper aims to address the discrete-time case where both process and observation models have a polynomial form; Another contribution is the derivation of a set of matrix formulas giving the exact mean and covariance of any stochastic variable which has undergone a polynomial transformation, considering full Taylor series expansion. Thanks to this closed-form computations, the implementation of the proposed polynomial Kalman filter is made easier and with no approximation. To show the significance of this filtering scheme we consider a direct application to synchronization of chaotic signals [12] generated through Chebyshev maps [31]. There has been significant interest in recent years in exploiting chaos in communication systems [14]-[16]. Due to its random-like behavior and its wideband characteristics, a chaotic dynamical system can be very helpful to secure or encrypt a transmission. Another motivation in using chaos in this context is the potential decrease of the receiver design complexity thanks to chaos synchronization property. Synchronization of chaos, introduced both by Pecora and Caroll in [12] and Fujisaka and Yamada in [13], is the ability of retrieving any original chaotic signal from noisy measurements using the knowledge of the chaotic dynamics only. Many chaos synchronization approaches have been developed over the past. Many papers demonstrate the pertinence of various Kalman filtering schemes to accomplish this task [17]-[29]. In particular, Leung and Zhu have recently derived important results about chaos synchronization through Extended Kalman Filtering (EKF); the authors showed that the EKF-based technique is a generalization of two conventional schemes (unidirectionally coupled and drive-response methods). It is also shown that the EKF-based synchronization approaches the averaged Cramer-Rao Lower Bound at high Signal-to-Noise-Ratios (SNR).

The paper is organized as follows. In section II, we derive the general matrix formulas giving the second-order statistics of any random variable which has been transformed through a polynomial function. Then, in section III, an Exact Kalman Filter relying on these analytical results is formulated. An application to synchronizing chaotic sequences constructed via Chebyshev maps is considered in section IV. Finally, before the conclusions, some remarks about stability are considered for the particular case of second order Chebyshev polynomial and the performances of the proposed Kalman filter is examined and compared to that of the EKF and UKF through Monte-Carlo simulations.

II. Exact Polynomial Transformation

The nonlinear transformation of an *a priori* distribution is the heart of all methods of nonlinear Kalman filtering, and as a consequence there is a growing interest in giving the transformation that offers the best approximation to the moments.

In this paper we will consider the case of a family of nonlinear functions, which are characterized by a single dimensional polynomial form:

$$f(x)=\sum_{n=0}^{N}a_n x^n \tag{1}$$

The purpose of the transformation is to find the mean and variance of a random variable y resulting from the propagation of the random variable x through the nonlinear polynomial function mentioned above:

$$y=f(x) \tag{2}$$

At the moment, no restrictions will be supposed about the probability density functions. The Gaussian assumption will be considered later for the Kalman filtering model (section III).

In general, we can write the first two moments of the transformed distribution of the random variable y using the Taylor series expansion. So we consider the initial distribution written in the following form:

$$x=\bar{x}+\Delta x \tag{3}$$

where Δx is a random variable having a zero mean distribution. Now we can write the Taylor expansion for y as

$$y=f\left(\bar{x}\right)+\sum_{n=1}^{N}\frac{\left(\Delta x\right)^n}{n!}\left.\frac{d^n f}{dx^n}\right|_{x=\bar{x}} \tag{4}$$

and we can determine the first moment:

$$\begin{aligned}\bar{y} &= E\left[y\right]\\ &= f\left(\bar{x}\right)+\sum_{n=2}^{N}\frac{m_n}{n!}\left.\frac{d^n f}{dx^n}\right|_{x=\bar{x}}\end{aligned} \tag{5}$$

where m_n denotes the nth - order moment of the random variable Δx.

Due to the polynomial form (1) for the function $f\left(\cdot\right)$ the derivatives can be expressed as

$$\frac{d^n f}{dx^n} = \sum_{i=n}^{N}a_i\frac{i!}{(i-n)!}x^{i-n} \tag{6}$$

Then, since $m_0=1$, $m_1=0$ and using the binomial coefficients $C_i^n=\frac{i!}{n!\cdot(i-n)!}$, the first order moment becomes:

$$\bar{y} \;=\; \sum_{n=0}^{N} m_n \sum_{i=n}^{N} a_i C_i^n \bar{x}^{i-n} \tag{7}$$

This expression can finally be written in a compact matrix form in order to facilitate the computation in practice:

$$\bar{y} = \mathbf{a}_{0:N}^T \mathbf{C}^{\bar{x}} \mathbf{m}_{0:N}^x \tag{8}$$

where $\mathbf{a}_{i:j}$ stands for $[a_i, a_{i+1}, ..., a_j]^T$, $\mathbf{m}_{i:j}^x = [m_i, m_{i+1}, ..., m_j]^T$ and $\mathbf{C}^{\bar{x}}$ denoting a lower triangular matrix where entries are powers of $\bar{x}$:

$$\mathbf{C}^{\bar{x}} = \begin{bmatrix} C_0^0 \bar{x}^0 & 0 & 0 & ... & 0 \\ C_1^0 \bar{x}^1 & C_1^1 \bar{x}^0 & 0 & ... & 0 \\ C_2^0 \bar{x}^2 & C_2^1 \bar{x}^1 & C_2^2 \bar{x}^0 & ... & 0 \\ ... & ... & ... & & ... \\ C_N^0 \bar{x}^N & C_N^1 \bar{x}^{N-1} & C_N^2 \bar{x}^{N-2} & ... & C_N^N \bar{x}^0 \end{bmatrix} \tag{9}$$

Considering again the Taylor series expansion, the second order centered moment σ_y^2 can be computed following the same developments:

$$\begin{aligned} \sigma_y^2 &= E\left[(y-\bar{y})^2\right] \\ &= E\left[\left(f(\bar{x}) + \sum_{n=1}^{N} \frac{(\Delta x)^n}{n!} \frac{d^n f}{dx^n}\bigg|_{x=\bar{x}} - f(\bar{x}) - \sum_{n=2}^{N} \frac{m_n}{n!} \frac{d^n f}{dx^n}\bigg|_{x=\bar{x}}\right)^2\right] \end{aligned} \tag{10}$$

After some algebra manipulations, the second order moment can be shown to be:

$$\sigma_y^2 = \sum_{n=1}^{N} \sum_{k=1}^{N} \frac{m_{n+k} - m_n m_k}{n! \cdot k!} \; \frac{d^n f}{dx^n} \frac{d^k f}{dx^k}\bigg|_{x=\bar{x}} \tag{11}$$

This time the product of derivatives can be expressed as:

$$\begin{aligned} \frac{1}{n! \cdot k!} \; \frac{d^n f}{dx^n} \frac{d^k f}{dx^k}\bigg|_{x=\bar{x}} &= \sum_{i=n}^{N} \sum_{j=k}^{N} a_i a_j C_i^n C_j^k \bar{x}^{i+j-n-k} \\ &= \mathbf{a}_{1:N}^T \mathbf{C}_{n,k}^{\bar{x}} \mathbf{a}_{1:N} \end{aligned} \tag{12}$$

where matrix $\mathbf{C}_{n,k}^{\bar{x}}$ takes the following form:

$$\mathbf{C}_{n,k}^{\bar{x}} = \begin{bmatrix} \mathbf{0}_{(n-1)\times(k-1)} & \mathbf{0}_{(n-1)\times(N-k+1)} \\ \mathbf{0}_{(N-n+1)\times(k-1)} & \begin{matrix} C_n^n C_k^k \bar{x}^0 & C_n^n C_{k+1}^k \bar{x}^1 & ... & C_n^n C_N^k \bar{x}^{N-k} \\ C_{n+1}^n C_k^k \bar{x}^1 & C_{n+1}^n C_{k+1}^k \bar{x}^2 & ... & C_{n+1}^n C_N^k \bar{x}^{N-k+1} \\ ... & ... & & ... \\ C_N^n C_k^k \bar{x}^{N-n} & C_N^n C_{k+1}^k \bar{x}^{N-n+1} & ... & C_N^n C_N^k \bar{x}^{2N-n-k} \end{matrix} \end{bmatrix}$$

$\mathbf{0}_{i\times j}$ standing for a matrix of dimension $i \times j$ whose entries are all zero.

From equations (11) and (12) we finally get the second order moment of the random variable y in the matrix form:

$$\sigma_y^2 = \mathbf{1}_N^T \left(\mathcal{M}^x \boxdot \mathcal{C}^{\bar{x}}\right) \mathbf{1}_N - \left(\mathbf{m}_{1:N}^x\right)^T \mathcal{C}^{\bar{x}} \mathbf{m}_{1:N}^x \tag{13}$$

where matrices $\mathcal{C}^{\bar{x}}$ and $\mathcal{M}^x$ are computed as:

$$\mathcal{C}^{\bar{x}} = \begin{bmatrix} \mathbf{a}_{1:N}^T \mathbf{C}_{1,1}^{\bar{x}} \mathbf{a}_{1:N} & \mathbf{a}_{1:N}^T \mathbf{C}_{1,2}^{\bar{x}} \mathbf{a}_{1:N} & \dots & \mathbf{a}_{1:N}^T \mathbf{C}_{1,N}^{\bar{x}} \mathbf{a}_{1:N} \\ \mathbf{a}_{1:N}^T \mathbf{C}_{2,1}^{\bar{x}} \mathbf{a}_{1:N} & \mathbf{a}_{1:N}^T \mathbf{C}_{2,2}^{\bar{x}} \mathbf{a}_{1:N} & \dots & \mathbf{a}_{1:N}^T \mathbf{C}_{2,N}^{\bar{x}} \mathbf{a}_{1:N} \\ \dots & \dots & & \dots \\ \mathbf{a}_{1:N}^T \mathbf{C}_{N,1}^{\bar{x}} \mathbf{a}_{1:N} & \mathbf{a}_{1:N}^T \mathbf{C}_{N,2}^{\bar{x}} \mathbf{a}_{1:N} & \dots & \mathbf{a}_{1:N}^T \mathbf{C}_{N,N}^{\bar{x}} \mathbf{a}_{1:N} \end{bmatrix} \tag{14}$$

$$\mathcal{M}^x = \begin{bmatrix} m_2 & m_3 & \dots & m_{N+1} \\ m_3 & m_4 & \dots & m_{N+2} \\ \dots & \dots & & \dots \\ m_{N+1} & m_{N+2} & \dots & m_{2N} \end{bmatrix} \tag{15}$$

$\boxdot$ denoting the Hadamard product and $\mathbf{1}_N$ standing for a column vector of size N whose entries are all one.

Our objective being to derive a Kalman filter relying on the previous relations, it remains to express the transition covariance P_{xy} between the variables x and y:

$$\begin{aligned} P_{xy} &= E\left[(x-\bar{x})(y-\bar{y})\right] \\ &= E\left[\Delta x \cdot y\right] - E\left[\Delta x \cdot \bar{y}\right] \\ &= \sum_{n=1}^{N} m_{n+1} \sum_{i=n}^{N} a_i C_i^n \bar{x}^{i-n} \end{aligned} \tag{16}$$

Once again we can put the relation above in matrix form:

$$P_{xy} = \mathbf{a}_{0:N}^T \mathbf{C}^{\bar{x}} \mathbf{m}_{1:N+1}^x \tag{17}$$

We would like to point out that the relations (8), (13) and (17) are given for any initial distribution x for which we know the centered moments, and for any polynomial form of the function $f(\cdot)$. This doesn't mean that these relations minimize the computational cost but in exchange offer a general formula to calculate exactly the first two moments $\left(\bar{y}, \sigma_y^2\right)$. We will present in section IV a more particular approach, in case of nonlinear functions represented by Chebyshev's polynomials. A novel Kalman filter implementation is proposed in the next section thanks to the analytical expressions of the transformed random variable first two moments.

III. Exact Polynomial Kalman Filter (ExPKF)

To develop our polynomial transform based Kalman filter, we consider the following one-dimensional process and observation models:

$$\begin{aligned} x_{k+1} &= f(x_k) + v_k \\ y_k &= h(x_k) + n_k \end{aligned} \tag{18}$$

where $\{f(\cdot), h(\cdot)\}$ denote nonlinear functions and where additive noise terms $\{v_k, n_k\}$ are supposed zero mean, Gaussian and uncorrelated between each other and with the states:

$$\begin{aligned} E[v_k] &= E[n_k] = 0 \\ E[v_k v_j] &= Q\delta_{kj} \\ E[n_k n_j] &= R\delta_{kj} \\ E[v_k n_j] &= 0 \\ E[v_k x_j] &= E[n_k x_j] = 0 \end{aligned} \tag{19}$$

Assuming also the Gaussianity of the prior estimate x_k and of the current observation y_{k+1}, we can express the optimal estimate for x_{k+1} as:

$$\hat{x}_{k+1} = \hat{x}_{k+1|k} + K_{k+1}\left(y_{k+1} - \hat{y}_{k+1|k}\right) \tag{20}$$

where

$$\begin{aligned} \hat{x}_{k+1|k} &= E[f(x_k) + v_k] = E[f(x_k)] \\ \hat{y}_{k+1|k} &= E\left[h\left(x_{k+1|k}\right) + n_{k+1}\right] = E\left[h\left(x_{k+1|k}\right)\right] \\ K_{k+1} &= P_{x_{k+1|k} y_{k+1|k}} P^{-1}_{y_{k+1|k} y_{k+1|k}} \end{aligned} \tag{21}$$

The covariances $P_{x_{k+1|k} y_{k+1|k}}$ and $P_{y_{k+1|k} y_{k+1|k}}$ take the following form considering the model (18):

$$\begin{aligned} P_{x_{k+1|k} y_{k+1|k}} &= E\left[\left(x_{k+1|k} - \hat{x}_{k+1|k}\right)\left(y_{k+1|k} - \hat{y}_{k+1|k}\right)\right] \\ &= E\left[\left(x_{k+1|k} - \hat{x}_{k+1|k}\right)\left(h\left(x_{k+1|k}\right) - E\left[h\left(\hat{x}_{k+1|k}\right)\right]\right)\right] \end{aligned} \tag{22}$$

$$\begin{aligned} P_{y_{k+1|k}y_{k+1|k}} &= E\left[\left(y_{k+1|k}-\hat{y}_{k+1|k}\right)\left(y_{k+1|k}-\hat{y}_{k+1|k}\right)\right] \\ &= E\left[\left(h\left(x_{k+1|k}\right)-\hat{y}_{k+1|k}\right)^2\right]+R \end{aligned} \tag{23}$$

The Gaussian assumption for estimates and observations limits the initial hypothesis of knowing general moments to the first two orders and in this case, the moments vector is given by:

$$\begin{aligned} \mathbf{m}^x_{0:N} &= \left[m_0, m_1, m_2, m_3, m_4, ...\right]^T \\ &= \left[1, 0, \sigma_x^2, 0, 3\sigma_x^4, ...\right]^T \end{aligned} \tag{24}$$

Now thanks to the Exact Polynomial Transform developed in section II, an Exact Kalman Filter algorithm can be stated when both process and observation functions are of polynomial form, i.e. $f(x)=\sum_{n=0}^{N} a_n x^n$, $h(x)=\sum_{k=0}^{K} b_k x^k$.

At the time-update step, the mean and the covariance of the predicted state will be computed as follows, owing to (8) and (13) respectively:

$$\begin{aligned} \hat{x}_{k+1|k} &= E\left[f\left(x_k\right)\right] \\ &= \mathbf{a}^T_{0:N}\mathbf{C}^{\hat{x}_k}\mathbf{m}^{x_k}_{0:N} \end{aligned} \tag{25}$$

$$\begin{aligned} P_{k+1|k} &= E\left[\left(x_{k+1|k}-\hat{x}_{k+1|k}\right)^2\right] \\ &= \mathbf{1}^T_N\left(\mathcal{M}^{x_k}\boxdot\mathcal{C}^{\hat{x}_k}\right)\mathbf{1}_N-\left(\mathbf{m}^{x_k}_{1:N}\right)^T\mathcal{C}^{\hat{x}_k}\mathbf{m}^{x_k}_{1:N}+Q \end{aligned} \tag{26}$$

At the measurement-update step, the predicted observation and the transition/innovation covariances will be computed thanks to (8), (17) and (13):

$$\begin{aligned} \hat{y}_{k+1|k} &= E\left[h\left(x_{k+1|k}\right)\right] \\ &= \mathbf{b}^T_{0:K}\mathbf{C}^{\hat{x}_{k+1|k}}\mathbf{m}^{x_{k+1|k}}_{0:N} \end{aligned} \tag{27}$$

$$\begin{aligned} P_{x_{k+1|k}y_{k+1|k}} &= E\left[\left(x_{k+1|k}-\hat{x}_{k+1|k}\right)\left(h\left(x_{k+1|k}\right)-E\left[h\left(\hat{x}_{k+1|k}\right)\right]\right)\right] \\ &= \mathbf{b}^T_{0:K}\mathbf{C}^{\hat{x}_{k+1|k}}\mathbf{m}^{x_{k+1|k}}_{1:K+1} \end{aligned} \tag{28}$$

$$P_{y_{k+1|k}y_{k+1|k}} = E\left[\left(h\left(x_{k+1|k}\right) - \hat{y}_{k+1|k}\right)^2\right] + R$$
$$= \mathbf{1}_K^T \left(\mathcal{M}^{x_{k+1|k}} \boxdot \mathcal{C}^{\hat{x}_{k+1|k}}\right) \mathbf{1}_K - \left(\mathbf{m}_{1:K}^{x_{k+1|k}}\right)^T \mathcal{C}^{\hat{x}_{k+1|k}} \mathbf{m}_{1:K}^{x_{k+1|k}} + R \quad (29)$$

As the result of the $(k+1)$th recursion step, the mean and covariance of x_{k+1} are computed as:

$$\hat{x}_{k+1} = \hat{x}_{k+1|k} + K_{k+1}\left(y_{k+1} - \hat{y}_{k+1|k}\right) \quad (30)$$

$$P_{k+1} = P_{k+1|k} - K_{k+1}^2 P_{y_{k+1|k}y_{k+1|k}} \quad (31)$$

where the Kalman gain K_{k+1} is expressed as in (21).

Again, we remind that the relations above give only a general formulation and are not optimized to reduce the computation time. For a particular model there can be given a low computational cost algorithm, as will be shown in the next section.

Remark 1: As mentioned in the present paper introduction, the issue of filtering polynomial systems have already been considered in recent works. However, it should be noticed that either the problem formulation or the proposed approach was different. The paper of Norgaard *et al.* [8] investigates the use of a polynomial approximation (limited to the second order) rather than Taylor series expansion to achieve the nonlinear Kalman filtering; In particular, the best choice of the interval length when applying the Stirling's formula is discussed. Our paper does not exactly address the same problem as the nonlinear system is supposed to already have a polynomial structure (the state vector is one-dimensional but the polynomial is of any order). Then, exploiting the full Taylor series expansion, we compute the second-order statistics in closed-form without any approximation. Compact general expressions are derived to facilitate the implementation of polynomial Kalman filters in matrix-oriented programming language such as *Matlab*. Our paper shares the same objective as that of Basin in [11] but in addition to the general matrix expressions described above, our approach avoids the constraint of having a linear observation model. Also we address the filtering problem in discrete-time and not in the continuous-time case.

IV. Chaotic Synchronization using the Exact Polynomial Kalman Filter

The class of Chebyshev maps is proved to be a very particular one, that is characterized by a polynomial form, the Chebyshev polynomials. The properties of these polynomials made them doubtless one of the most used in modelization, filtering and signal processing [31]. In what concerns the chaos, it is known that all Chebyshev maps generate chaotic sequences which have limit densities $\rho(x) = \frac{1}{\pi\sqrt{1-x^2}}$ [32], and are characterized by a Lyapunov exponent $\ln p$ [30][35], where p is the order of the polynomial $T_p(x)$ which generates the chaotic sequence $\{x_k, k = 0, 1, ...\}$:

$$x_{k+1} = f\left(x_k\right) = T_p\left(x_k\right) \quad (32)$$

Due to their favorable correlation properties, Chebyshev polynomials have been applied successfully to Code Division

p	$y = T_p(x)$	$\bar{y} = E[y]$	$\sigma_y^2 = E\left[(y-\bar{y})^2\right]$
2	$2x^2 - 1$	$2\sigma_x^2 + 2\bar{x}^2 - 1$	$8\sigma_x^2\left(\sigma_x^2 + 2\bar{x}^2\right)$
3	$4x^3 - 3x$	$\bar{x}\left(12\sigma_x^2 + 4\bar{x}^2 - 3\right)$	$3\sigma_x^2\left(80\sigma_x^4 + 192\sigma_x^2\bar{x}^2 + ... \right.$ $\left. +48\bar{x}^4 - 24\sigma_x^2 - 24\bar{x}^2 + 3\right)$
4	$8x^4 - 8x^2 + 1$	$24\sigma_x^4 + 48\sigma_x^2\bar{x}^2 + ...$ $+8\bar{x}^4 - 8\sigma_x^2 - 8\bar{x}^2 + 1$	$128\sigma_x^2\left(48\sigma_x^6 + 192\sigma_x^4\bar{x}^2 + ... \right.$ $+84\sigma_x^2\bar{x}^4 + 8\bar{x}^6 - 12\sigma_x^4 - ...$ $\left. -36\sigma_x^2\bar{x}^2 - 8\bar{x}^4 + \sigma_x^2 + 2\bar{x}^2\right)$

TABLE I

FIRST AND SECOND MOMENTS OF THE RANDOM VARIABLE RESULTING FROM A CHEBYSHEV POLYNOMIAL TRANSFORMATION OF A GAUSSIAN DISTRIBUTION (POLYNOMIALS OF DEGREE 2, 3 AND 4)

Multiple Access digital communications [33], [34].

It is well known that the polynomials follow a recursive expression which permits a very simple implementation to determine the coefficients, for any order:

$$T_{p+1}(x) = 2xT_p(x) - T_{p-1}(x) \tag{33}$$

Thanks to this general polynomial form we can proceed to a direct implementation of the proposed Exact Polynomial Kalman Filter (ExPKF) to achieve chaos synchronization. We will make the computation of the moments for some low-order Chebyshev polynomials, to clearly exemplify that a very low-cost Kalman filter algorithm can be obtained. It should be emphasized that chaos synchronization is just considered here to illustrate the pertinence of our ExPKF (which is the main contribution of the paper) with respect to other popular approaches like EKF or UKF. As pointed out in the beginning of the paper, numerous previous works have already consider the problem of synchronizing chaotic time-series thanks to nonlinear Kalman filters.

In case of the synchronization of a mono-dimensional chaotic polynomial map, the nonlinear model simplifies due to the linearity of the function $h(\cdot)$:

$$\begin{aligned} x_{k+1} &= f(x_k) + v_k \\ y_k &= x_k + n_k \end{aligned} \tag{34}$$

and we can write the equations of the proposed ExPKF algorithm using the analytical formulas of $\left\{\bar{y}, \sigma_y^2\right\}$. As an example, for a second order Chebyshev sequence synchronization, the filter is implemented as follows, once the second order statistics have been computed analytically (see Table I).

The time-update general equations (25) and (26) give:

$$\hat{x}_{k+1|k} = E[f(x_k)] = 2P_k + 2\hat{x}_k^2 - 1 \tag{35}$$

$$P_{k+1|k} = E\left[\left(x_{k+1|k} - \hat{x}_{k+1|k}\right)^2\right] = 8P_k^2 + 16P_k\hat{x}_k^2 + Q \tag{36}$$

Also, considering the linearity of the function $h(\cdot)$ and the independence of the model and observation noises between them and with the states, the measurement-update equations become:

$$\hat{y}_{k+1|k} = E\left[h\left(x_{k+1|k}\right)\right] = \hat{x}_{k+1|k} \tag{37}$$

$$\begin{aligned} P_{x_{k+1|k}y_{k+1|k}} &= E\left[\left(x_{k+1|k} - \hat{x}_{k+1|k}\right)\left(y_{k+1|k} - \hat{y}_{k+1|k}\right)\right] \\ &= P_{k+1|k} \end{aligned} \tag{38}$$

$$\begin{aligned} P_{y_{k+1|k}y_{k+1|k}} &= E\left[\left(y_{k+1|k} - \hat{y}_{k+1|k}\right)\left(y_{k+1|k} - \hat{y}_{k+1|k}\right)\right] \\ &= P_{k+1|k} + R \end{aligned} \tag{39}$$

$$K_{k+1} = \frac{P_{k+1|k}}{P_{k+1|k} + R} \tag{40}$$

$$\hat{x}_{k+1} = \hat{x}_{k+1|k} + K_{k+1}\left(y_{k+1} - \hat{x}_{k+1|k}\right) \tag{41}$$

$$P_{k+1} = P_{k+1|k} - K_{k+1}^2 P_{y_{k+1|k}y_{k+1|k}} = \frac{P_{k+1|k}R}{P_{k+1|k} + R} = K_{k+1}R \tag{42}$$

As can be seen by the relations (37) - (42), the Exact Kalman Filtering algorithm applied to the chaos synchronization of a second order Chebyshev polynomial model, offers if not the best, one of the most cost-effective solutions. So complementary with the general matrix form, for particular system models, some low computational cost implementations with very good performances can be expressed, as it will be confirmed by the numerical results presented in the next section.

V. Performance Evaluation of the Proposed Filter

As performance analysis we have considered four directions. The first one is the study of nonlinear transformation of a Gaussian distribution using different orders of Chebyshev polynomials as nonlinear function. The second one makes a comparative study of the various nonlinear Kalman filters from a stability point of view. The third one concentrates on the performances in chaos synchronization of those filters implementations by numerical simulations, and the last part deals with the consistency for different observation noise levels.

A. Study of nonlinear transformations

To make the comparison through graphical representation easier in presence of various polynomial orders, we will consider an initial bi-dimensional random Gaussian vector $\mathbf{x} \sim N\left(\bar{\mathbf{x}}, \mathbf{P}_{xx}\right)$ with mutually independent components. This random vector will then be transformed through the two dimensional function $\mathbf{f}$ as:

$$\begin{aligned} \mathbf{y} &= \mathbf{f}(\mathbf{x}) \\ &= \begin{bmatrix} f_1(x_1) \\ f_2(x_2) \end{bmatrix} \\ &= \begin{bmatrix} T_p(x_1) \\ T_q(x_2) \end{bmatrix} \end{aligned} \tag{43}$$

where $\{p, q\}$ represent the orders of the Chebyshev polynomials chosen as nonlinearities for the X-axis and Y-axis respectively.

After making the point on the specific hypothesis for the vector distribution and the nonlinear transformation model we consider a comparative study with two popular transformations, namely the first order linearization through Taylor series truncation and the *Unscented* Transform, in order to show the effectiveness of our Exact Polynomial Transform.

The linearization through Taylor series truncation, which is used by the EKF, yields the following moments:

$$\begin{aligned} \bar{y} &= \mathbf{f}(\bar{\mathbf{x}}) \\ \mathbf{P}_{yy} &= \nabla\mathbf{f} \cdot \mathbf{P}_{xx} \cdot \nabla\mathbf{f} \end{aligned} \tag{44}$$

The *Unscented* Transformation [6], in opposition with the linearization method, uses a sum of *sigma points* $\{\chi_i\}$ with weights $\{W_i\}$ to calculate the moments:

$$\begin{array}{lll} \chi_0 = \bar{\mathbf{x}}, & W_0 = \kappa/(n_x+\kappa) & \\ \chi_i = \bar{\mathbf{x}} + \left(\sqrt{(n_x+\kappa)\,\mathbf{P}_{xx}}\right)_i, & W_i = 1/\{2(n_x+\kappa)\}, & i = 1, ..., n_x \\ \chi_i = \bar{\mathbf{x}} + \left(\sqrt{(n_x+\kappa)\,\mathbf{P}_{xx}}\right)_i, & W_i = 1/\{2(n_x+\kappa)\}, & i = n_x+1, ..., 2n_x \end{array} \tag{45}$$

where n_x is the dimension of the random variable x, and κ is a scaling parameter.

Then each sample is propagated through the nonlinear function:

$$\mathbf{Y}_i = f(\chi_i) \quad i = 0, ..., 2n_x \tag{46}$$

and the resulting statistics are derived as:

$$\bar{\mathbf{y}} = \sum_{i=0}^{2n_x} W_i \mathbf{Y}_i \tag{47}$$

$$\mathbf{P}_{yy} = \sum_{i=0}^{2n_x} W_i (\mathbf{Y}_i - \bar{\mathbf{y}})(\mathbf{Y}_i - \bar{\mathbf{y}})^T \tag{48}$$

To illustrate the performances of these three methods we will consider the following bi-dimensional random Gaussian vector $\mathbf{x}$, being transformed by the two-dimensional function $\mathbf{f}(\cdot)$ with $p = 2$ and $q = 5$ (Eqn. 43):

$$\begin{aligned} \bar{\mathbf{x}} &= \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} \\ \mathbf{P}_{xx} &= \begin{bmatrix} 0.5 & 0 \\ 0 & 0.1 \end{bmatrix} \end{aligned} \tag{49}$$

Thanks to the considered bi-dimensional transformation we can examine in a graphical way how the drawbacks of each of the three approaches (Exact Polynomial, First order linearization and Unscented Transform) evolve with respect to the polynomial order. To compute the true distribution, resulting from the application of the nonlinear function (43) to the stochastic variable (49), a total number of 10 millions samples have been propagated through the nonlinear model. To guarantee no cross-correlation between the components of the bi-dimensional vector, each one has been initialized with a different seed.

Fig. 1 gives a comparative illustration of the second-order statistics obtained through each of the three methods. We can observe that even for a weak nonlinearity (second order polynomial), the linearization method gives a bad estimate for the mean as for the variance. As the polynomial order increases (fourth order in this case), the performances become very poor. The Unscented approach clearly improves the performances, succeeding for the second order polynomial to obtain the true mean and variance, but making a bad estimate for higher polynomial orders. In contrast with these results, the proposed Exact Polynomial Transformation, by using exact analytical formulas for the resulting mean and variance, permits to obtain the best possible estimate of the transformed distribution statistics.

B. Some stability results

Regarding the results of the previous subsection one could demand if this accuracy of the Exact and *Unscented* transformations actually will solve the well known stability issues presented by the EKF which relies on a first order linearization. We will show in the following proposition that, considering equal estimated states for the EKF, UKF and ExPKF, an ordered relation between the predicted covariances exists in the particular case of synchronizing second order Chebyshev sequences.

Proposition 1. In case of second order Chebyshev dynamics, for any state $\hat{x}_k^{EKF} = \hat{x}_k^{UKF} = \hat{x}_k^{ExPKF}$ and error covariance $P_k^{EKF} = P_k^{UKF} = P_k^{ExPKF}$ the synchronization model proposed in (34), yields the following inequality for the predicted covariances:

$$P_{k+1|k}^{EKF} \leq P_{k+1|k}^{UKF} \leq P_{k+1|k}^{ExPKF} \tag{50}$$

Proof: As previously shown, the predicted covariance can be written in the following forms:

$$
\begin{aligned}
P_{k+1|k} &= \sum_{n=1}^{N}\sum_{k=1}^{N}\frac{m_{n+k}-m_n m_k}{n!\cdot k!}\sum_{i=n}^{N}\sum_{j=k}^{N} a_i a_j C_i^n C_j^k \bar{x}^{i+j-n-k}+Q \\
&= \mathbf{1}_N^T\left(\mathcal{M}^x \boxdot \mathcal{C}^{\bar{x}}\right)\mathbf{1}_N-\left(\mathbf{m}_{1:N}^x\right)^T\mathcal{C}^{\bar{x}}\mathbf{m}_{1:N}^x+Q
\end{aligned} \tag{51}
$$

where $\bar{x}$ stands for $\hat{x}_k$ and Q denotes the model covariance noise.

We can express the matrices $\mathcal{M}^x$ and the vectors $\mathbf{m}_{0:N}^x$ in the particular cases of 1-st order linearization and Unscented transforms as:

- first order linearization

$$\mathbf{m}_{0:N}^x \triangleq [1,0,0,0,...]^T$$

$$\mathcal{M}^x \triangleq \begin{bmatrix} m_2 & \mathbf{0}_{1\times(N-1)} \\ \mathbf{0}_{(N-1)\times 1} & \mathbf{0}_{(N-1)\times(N-1)} \end{bmatrix}$$

We give the general moments vector used in the transformation (starting with m_0), as the definitions above are used also for the first order propagation.

- Unscented transform

$$\mathbf{m}_{0:N}^x \triangleq [1,0,m_2,0,...]^T$$

$$\mathcal{M}^x \triangleq \begin{bmatrix} \begin{matrix} m_2 & 0 \\ 0 & m_4 \end{matrix} & \mathbf{0}_{2\times(N-2)} \\ \mathbf{0}_{(N-2)\times 2} & \mathbf{0}_{(N-2)\times(N-2)} \end{bmatrix}$$

The definitions above are a direct consequence on how the original transformation is defined: the *sigma points* are chosen to express the posterior statistics up to the second or third order terms in Taylor expansion. In case of the Exact transform, all the moments are taken into account for the calculus of the first and second order moments.

It is easy to prove that with the considered moments vector above and limiting to the case of Chebyshev polynomials (or to the case of any other polynomial with alternating zero and non-zero coefficients), only the multiple of two powers of $\bar{x}$ will be retained in (51) and so the covariance $P_{k+1|k}$ will be independent of the sign of $\bar{x}$ with the same initial variance σ_x^2. Another important property associated with the moments for the Gaussian distribution is that $m_{n+k}-m_n m_k \geq 0$ for any $n,k=1..N$.

For the case where the polynomial coefficients a_i are all positive for any $i \geq 1$, and considering the notations $P_{k+1|k}^{EKF}$, $P_{k+1|k}^{UKF}$and $P_{k+1|k}^{ExPKF}$ for the predicted covariance at time step $k+1$ and with the supposition of equality between the estimated states at time step k, we finally get:

$$P_{k+1|k}^{EKF} \leq P_{k+1|k}^{UKF} \leq P_{k+1|k}^{ExPKF} \tag{52}$$

□

Although no general proof (i.e. valid for any nonlinear dynamics) of this result has been derived, the numerous simulations that were conducted confirmed that the relation (52) is generally observed.

An immediate consequence of proposition one is that EKF will always yield a more optimistic estimation (its self assessment error will in average be smaller than the actual error) than that obtained through any other Kalman implementation considered in the present paper.

As shown by the next proposition this behavior can result in some local divergence during the filtering process.

Proposition 2. When synchronizing a second order Chebyshev sequence through EKF method, there exists $\hat{x}_k$ such as the filter becomes inconsistent, and gives local divergence.

Proof: The proposition will be demonstrated as a direct consequence of the predicted variance behavior and this local divergence is connected to the choice of the ratio between model and observation noise variances Q/R.

Considering again the mono-dimensional synchronization model (34) the predicted variance using the EKF can be expressed as:

$$P_{k+1|k} = \left.\frac{df}{dx}\right|_{x=\hat{x}_k}^{2} P_k + Q$$

We consider the recursive equation that can define the Kalman gain for the particular case of the 2nd order Chebyshev polynomial:

- EKF

$$\begin{aligned} K_{k+1} &= \frac{P_{k+1|k}}{P_{k+1|k}+R} \\ &= \frac{\left.\frac{df}{dx}\right|_{x=\hat{x}_k}^{2} P_k + Q}{\left.\frac{df}{dx}\right|_{x=\hat{x}_k}^{2} P_k + Q + R} \\ &= \frac{\left.\frac{df}{dx}\right|_{x=\hat{x}_k}^{2} K_k + \frac{Q}{R}}{\left.\frac{df}{dx}\right|_{x=\hat{x}_k}^{2} K_k + \frac{Q}{R} + 1} \\ &= \frac{16\hat{x}_k^2 K_k + \frac{Q}{R}}{16\hat{x}_k^2 K_k + \frac{Q}{R} + 1} \end{aligned} \quad (53)$$

- ExPKF and UKF

$$\begin{aligned} K_{k+1} &= \frac{P_{k+1|k}}{P_{k+1|k}+R} \\ &= \frac{8P_k\left(P_k + 2\hat{x}_k^2\right) + Q}{8P_k\left(P_k + 2\hat{x}_k^2\right) + Q + R} \\ &= \frac{8K_k\left(RK_k + 2\hat{x}_k^2\right) + \frac{Q}{R}}{8K_k\left(RK_k + 2\hat{x}_k^2\right) + \frac{Q}{R} + 1} \end{aligned} \quad (54)$$

If we study the behavior of the Kalman gains, we can easily observe that the minimal value is obtained in both cases for $\hat{x}_k = 0$:

- EKF

$$K_{k+1} = \frac{\frac{Q}{R}}{\frac{Q}{R}+1}$$

- ExPKF and UKF

$$K_{k+1} = \frac{8RK_k^2 + \frac{Q}{R}}{8RK_k^2 + \frac{Q}{R} + 1}$$

To offer a more complete view of the Kalman gain behavior on different situations we have represented in Figures 2, 3 and 4, this recursive forms for different values of the ratio Q/R and previous estimated states. For the ExPKF and UKF we have considered also a variation on the observed noise variance.

We remark that recursive Kalman gain characteristic for the EKF does not depend on R and is a function only of Q/R ratio, from the process statistics point of view. Figures 3 and 4 show the characteritics similarity for the ExPKF and UKF with low observation noise values, but when R increases the filter adapts the process to cope with the increased uncertainity.

In the following developements we will prove this stability problem for a particular case. We consider now the function $g\left(K_k, \hat{x}_k\right)$ defined as

$$g\left(K_k, \hat{x}_k\right) = K_{k+1}\left(K_k, \hat{x}_k\right)$$

The derivative of this function with respect to K_k is expressed, in case of EKF, as:

$$\frac{\partial}{\partial K_k} g\left(K_k, \hat{x}_k\right) = \frac{16\hat{x}_k^2}{\left(16\hat{x}_k^2 K_k + \frac{Q}{R} + 1\right)^2}$$

and with the condition of bounded estimated state $\exists a > 0$ such as $\hat{x}_k^2 \leq a$, we can write:

$$K_{k+1} \leq 16aK_k \tag{55}$$

With the relation (53) if $\hat{x}_k \simeq 0$ then $K_{k+1} \simeq \frac{Q}{Q+R}$ and the following inequality holds:

$$\begin{aligned} K_{k+M} &\leq 16^{M-1} a^{M-1} K_{k+1} \\ &\leq 16^{M-1} a^{M-1} \frac{Q}{Q+R} \end{aligned}$$

where M denotes the number of processed time steps.

If the observation noise is relatively small we can consider $a = 1$ and as a consequence $K_{k+M} \leq 16^{M-1} \frac{Q}{Q+R}$. Considering well known chaotic dynamics, the model noise covariance respects the relation $Q \ll R$; With this hypothesis we have $\frac{Q}{Q+R} \simeq 0$ and $M = 1$ is not sufficient for the filter gain to move away from its local minimum. The consequence is that the estimated state is highly weighted by the predicted one and the observation is almost neglected. Actually this will translate in an inconsistency problem, the estimated error covariance defined by equation (42) being totally unadapted to the current error, and the filter behavior can be characterized as too optimistic.□

Figures 5 and 6 give an illustrative example of such comportment for the EKF filter in case of second order Chebyshev polynomial with the parameters considered as $Q/R = 10^{-10}$ and $R = 10^{-2}$.

If we refer now to the ExPKF and UKF a similar relation as (52) can be derived taking in calculus the higher order terms in the Taylor series. This will yield to a more pessimistic estimation than that given by the EKF. For a second order Chebyshev polynomial and considering again equal states at time step k for the different filters, we can write from the proposition 1 an inequality between the Kalman Gains:

$$K_{k+1}^{EKF} \leq K_{k+1}^{UKF} \leq K_{k+1}^{ExPKF} \tag{56}$$

C. *MSE performance for chaos synchronization problem*

We are now going to examine the ability of the proposed ExPKF Filter to synchronize chaotic sequences generated via Chebyshev maps. One of the most popular criteria to measure an estimator performances is the Mean Square Error (MSE), which can be defined as

$$MSE = \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} (x_k - \hat{x}_k)^2 \tag{57}$$

where x_k stands for the true state at time step k and $\hat{x}_k$ being its estimate.

Monte Carlo simulations have been conducted to get the Mean Square Error (MSE) between the original chaotic signal and the synchronized one, owing to the proposed ExPKF, under various noise conditions. For purpose of comparison we also consider the results obtained through EKF and UKF. For all the methods we have considered sequences of 10^5 samples length with a transition period of 10^3 samples. The filters were all initialized with $x_0 = 0.3$ and $P_0 = 0.25$, and are simulated with the same noise sequences. In order to bring out the denoising capability of the various filters, we show in Fig. 7 how the MSE, normalized to the observation noise variance R, evolves. A $4th$ order Chebyshev polynomial is considered for this simulation, because for smaller orders the performances of the filters are close, the influence of higher-order statistical moments being limited in this case. To cope with the stability problem presented by the EKF we have considered a Q/R ratio equal to 10^{-1} for all the Kalman implementations.

Whereas the EKF performs well for high SNR, as demonstrated recently by Leung and Zhu in [20], its performances degrades rapidly in very noisy conditions. It is seen on Fig. 7 that the EKF is a suitable method for synchronizing fourth order Chebyshev sequences only in presence of noise having a variance less than 10^{-3}.

For the UKF we have considered two implementations: the first one respects the definitions given in the beginning of this section, and the second one uses an algorithm (*scaled*) developed by Julier *et al.* [6], [7] to cope with some monotonic spreading of the *sigma points*. As noted on the same figure, the classical UKF achieves good performances for all the considered noise levels, but always remaining under the performances of ExPKF. Regarding the *scaled* implementation of UKF for observation noise variances up to about $3E-2$ it gives a normalized error very close to the one obtained via our Exact Polynomial Filter but for larger noises, the proposed filter offers a significant advantage. In particular, the ExPKF has its performances insensitive to the noise level in the interval $[10^{-2}, 10^{-1}]$ as the curve slope is close to zero.

For a stronger nonlinearity (keeping Chebyshev dynamics) the difference between various methods becomes more pronounced.

D. Consistency evaluation

The consistency (or credibility) of any estimator/filter is an important question for practical issues. This question refers to checking any significant deviation between the actual estimation error (bias and MSE) and the one provided by the estimator itself (computed bias and error covariance). It is well known that any significant distance between these two information sets can lead to poor filtering performances and even to divergence.

To evaluate our polynomial filter consistency, we consider again a total of N independent Monte Carlo simulations, each of these giving an estimate $\hat{x}_k$ of the true state x_k at time step k, together with an estimated error covariance P_k. The Normalized Estimation Error Squared (NEES) is frequently used when dealing with credibility measure of a filter [2]. At time step k, and considering a particular Monte Carlo run (denoted as the ith run), the NEES is defined by

$$\epsilon_k^{(i)} = \left(x_k - \hat{x}_k\right)^T P_k \left(x_k - \hat{x}_k\right) \tag{58}$$

By averaging the NEES over the whole set of Monte Carlo runs we then get the statistical test $\bar{\epsilon}_k$ for examining the filter consistency :

$$\bar{\epsilon}_k = \frac{1}{nN} \sum_{i=1}^{N} \epsilon_k^{(i)} \tag{59}$$

n standing for the state vector dimension; A value $\bar{\epsilon}_k$ close to one indicates a reliable filter, having an actual estimation error in concordance with its self-assessment.

Assuming a Gaussian distribution for the estimation error $\tilde{x}_k$, the averaged NEES is known to be chi-square distributed (with $n = 1$ degree of freedom in our case). Then, to verify the filter consistency, we can define a confidence region which could correspond, for example, to a two-sided 95 % probability value (significance level $\alpha = 0.05$) :

$$\left[\chi^2(0.25), \chi^2(0.975)\right] = [0.74, 1.30] \tag{60}$$

For the Monte Carlo evaluation of the acceptance interval we have considered one hundred runs ($N = 100$), conducted for 4th order Chebyshev chaotic sequences of 300 samples length (after a transition sequence of 1000 samples), in case of a low observation noise ($R = 10^{-4}$) and also for a more noisy experiment ($R = 10^{-1}$). A model uncertainty such as the ratio Q/R equals 10^{-4} was considered each time. The averaged NEES obtained for these two simulations are depicted in Figs. 8 and 9, respectively. For purpose of comparisons, we consider again the results obtained through EKF and UKF. For declaring whether or not these filters are consistent we verify if the values $\bar{\epsilon}_k$ remain in the interval $[0.74, 1.30]$ along the filtering process. In case of a large SNR (Fig. 8), the three filters yield almost the same consistency and the proposed polynomial filter does not offer any clear advantage over the EKF or UKF approaches. On the other hand, for more noisy conditions (Fig. 9), very different behaviors are observed. The EKF becomes inconsistent as the averaged NEES takes almost all its values outside the $[0.74, 1.30]$ interval. This too optimistic characteristic results from large modelization errors when linearizing the process dynamics (which in this case is strongly nonlinear). The UKF still performs very well but a significant difference with respect to averaged NEES obtained through exact polynomial filtering is noted. The proposed filter is the most credible estimator for this second experiment.

VI. Conclusions

The problem of nonlinear filtering of discrete-time mono-dimensional polynomial signal models have been considered in this paper. Considering full Taylor series expansion we first derived analytical formulas giving exact second-order statistics of any stochastic variable that has undergone a polynomial transformation. An original Kalman filtering scheme is then formulated owing to these results. Thanks to compact matrix expressions of the mean and covariance at each recursion step, a rapid and general implementation of this Exact Polynomial Kalman Filter is enabled. As an application, we considered the important problem of synchronizing chaotic sequences generated via Chebyshev maps. Mean Square Error together with Normalized Estimation Error Squared have been evaluated under various Signal-to-Noise-Ratios to show the effectiveness of the proposed filter. A comparison to the results obtained using the popular Extended Kalman Filter and the promising Unscented Kalman Filter is also conducted to confirm these good performances.

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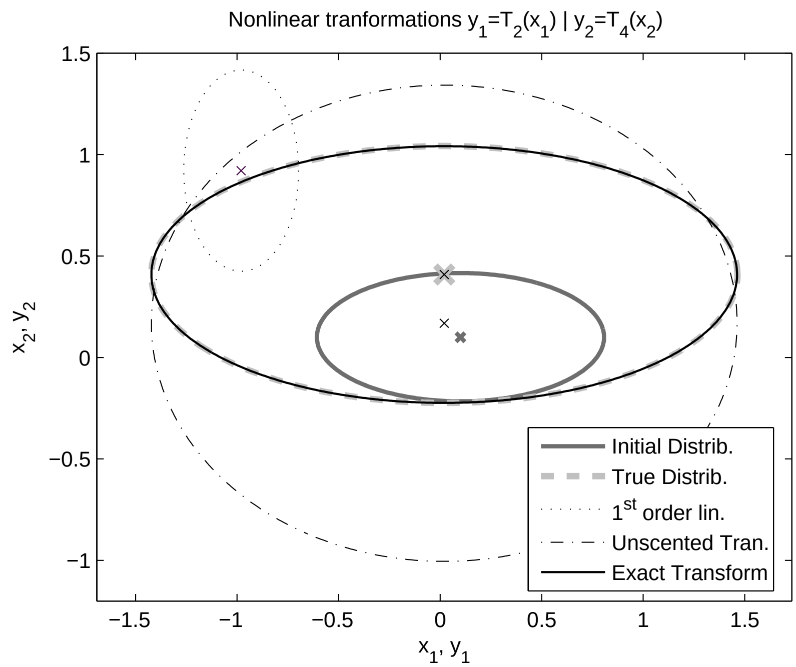

Fig. 1. Nonlinear Transformations using two Chebyshev polynomials, respectively $y_1 = T_2(x_1) \mid y_2 = T_4(x_2)$

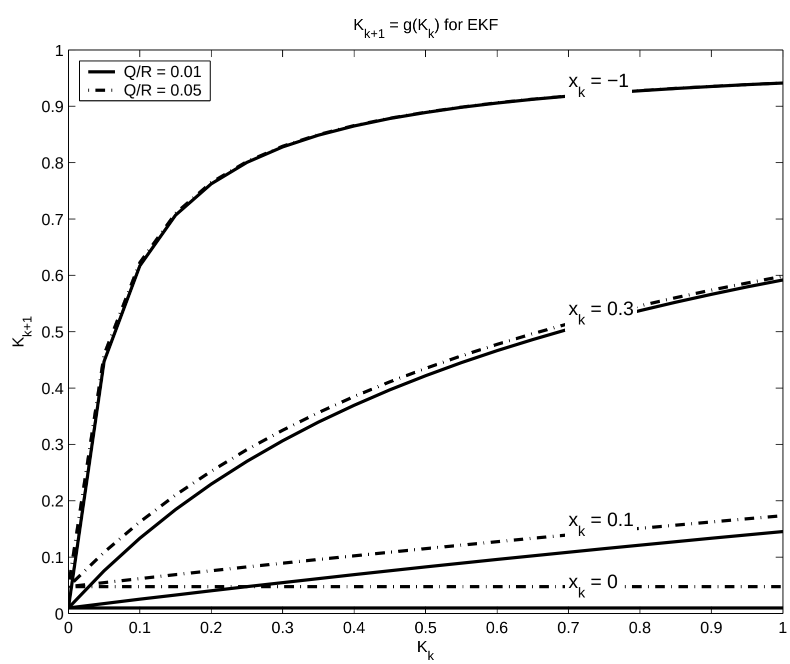

Fig. 2. Recursive evolution of the Kalman gain for the EKF

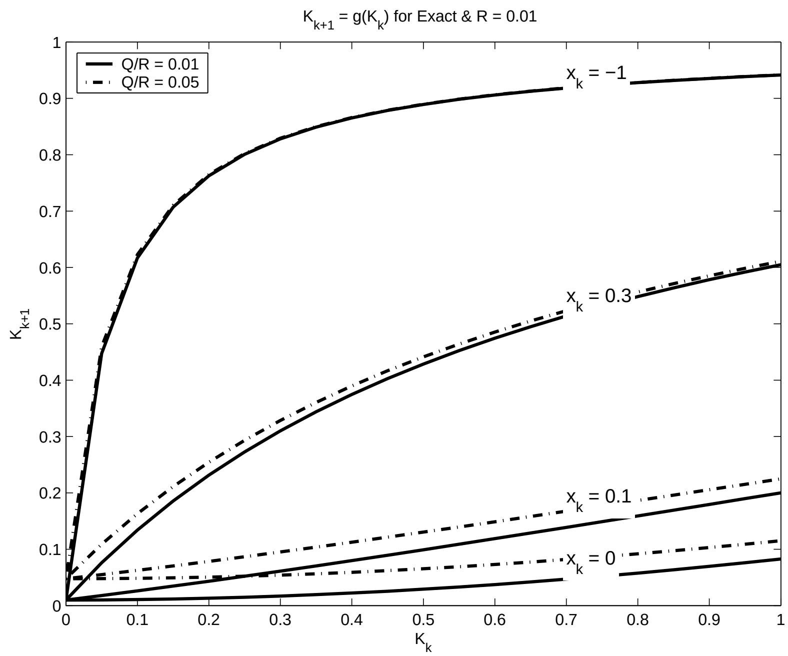

Fig. 3. Recursive evolution of the Kalman gain for the ExPKF and UKF, $R = 10^{-2}$

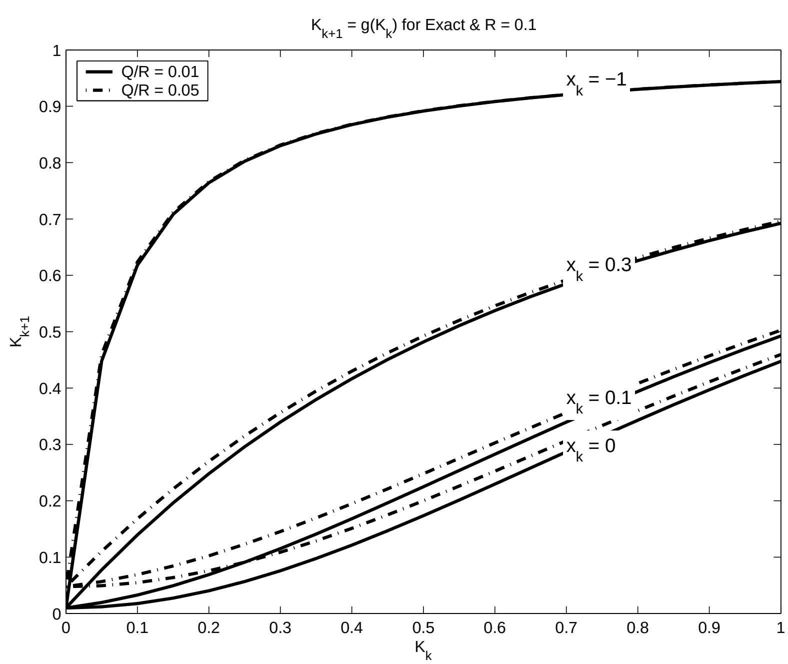

Fig. 4. Recursive evolution of the Kalman gain for the ExPKF and UKF, $R = 10^{-1}$

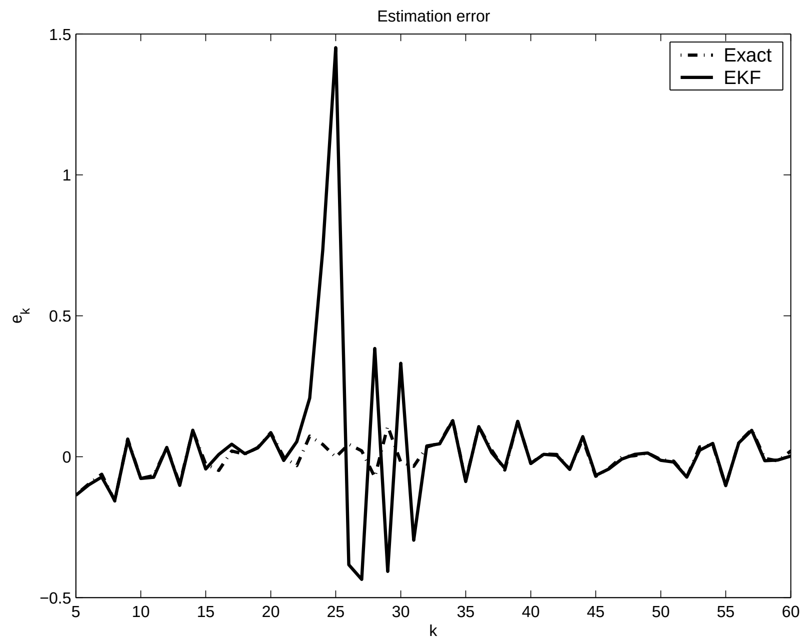

Fig. 5. Local instability of the filter represented by the instantaneous error divergence

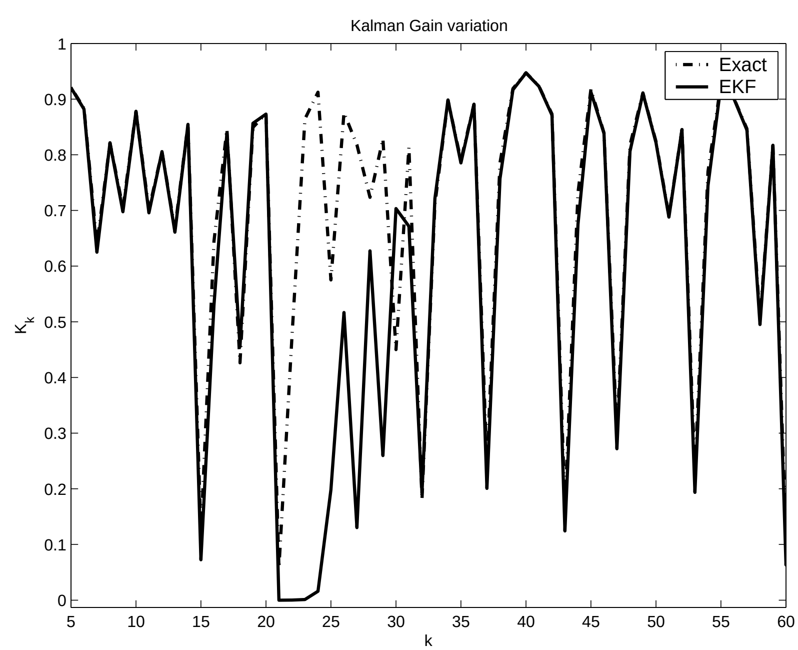

Fig. 6. Kalman Gain variation in the instability region

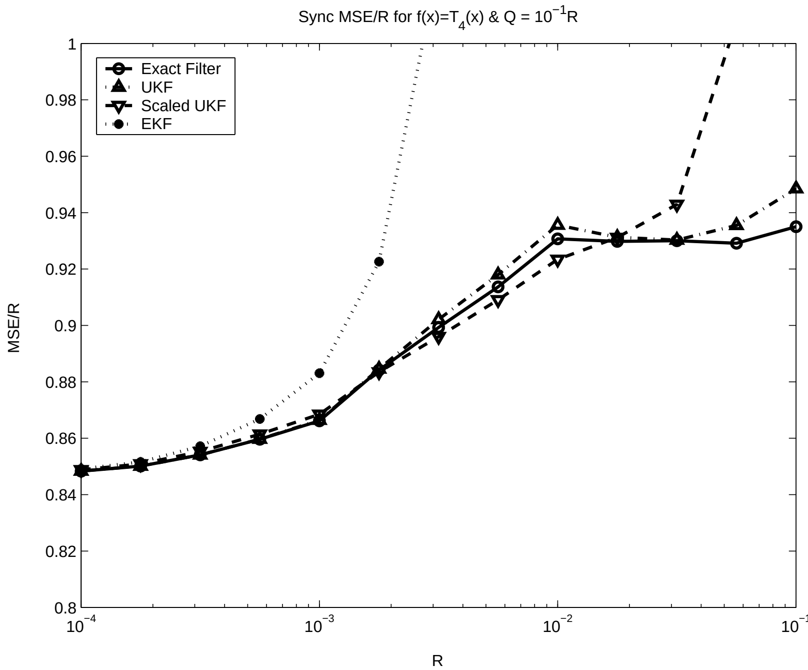

Fig. 7. Synchronization MSE/R for $f(x) = T_4(x)$

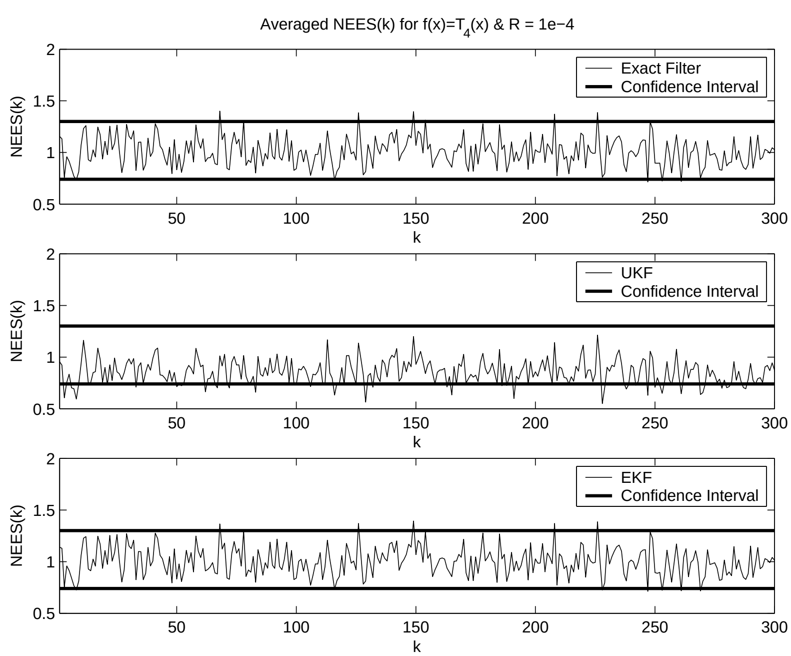

Fig. 8. Averaged NEES for $f(x) = T_4(x)$ and R = 1e-4: a) ExPKF; b) UKF; c) EKF

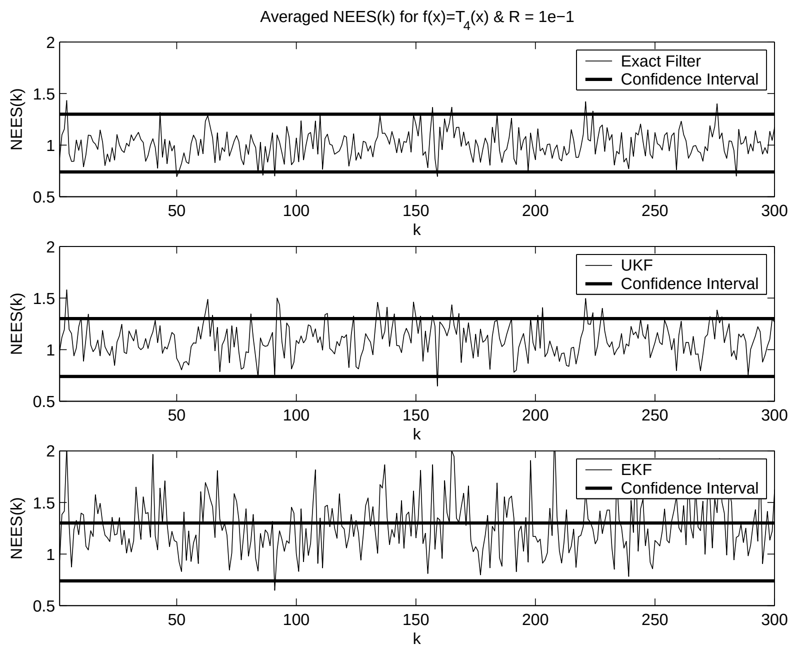

Fig. 9. Averaged NEES for $f(x) = T_4(x)$ and R = 1e-1: a) ExPKF; b) UKF; c) EKF