On exact solutions of some important nonlinear conformable time-fractional differential equations

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Abstract

The nonlinear fractional Boussinesq equations are known as the fractional differential equation class that has an important place in mathematical physics. In this study, a method called $\left(\frac{G'}{G^2}\right)$-extension method which works well and reveals exact solutions is used to examine nonlinear Boussinesq equations with conformable time-fractional derivatives. This method is a very useful approach and extremely useful compared to other analytical methods. With the proposed method, there are three unique types of solutions such as hyperbolic, trigonometric and rational solutions. This approach can similarly be applied to other nonlinear fractional models.

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1. Introduction

Nonlinear partial differential equations (NPDEs) are very useful equations that are important in different fields including engineering sciences, mathematics, fluid dynamics, and physics. Many real world problems have been modeled thanks to NPDEs. Numerous different and robust mathematical methods have been used to obtain exact solutions of NPDEs [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. Fractional calculus is also a new field that has grown in popularity over the past few decades. Various physical phenomena are made easily solvable in fractional differential equations (FDEs), such as viscoelasticity, plasma, solid mechanics, optical fibers, signal processing, electromagnetic waves, fluid dynamics, biomedical sciences and diffusion processes. Researchers have made these equations attractive by using various methods to obtain exact solutions of FDEs. Some of these methods and the important ones in the literature can be listed as $\left(\frac{G'}{G}\right)$-expansion method [12], the improved extended tanh-coth method [13], Lie group analysis method [14], first integral method [15], exp-function method [16], sub-equation method [17], functional variable method [18].

Boussinesq type equations can be considered as the first model for nonlinear, dispersive wave propagation and describe the surface water waves whose horizontal scale is much larger than the depth of the water [19]. They can be noted as a critical class of fractional differential equations in mathematical physics. Recently, different techniques have been used to find analytical and numerical solutions of Boussinesq equations. These can be mentioned as invariant subspace method [20], a newly developed method called the expansion method [21], exp-function method [22].

Conformable time fractional Boussinesq equations, which can be used to describe the propagation of the wave in the magnetic field, describes surface water waves. In addition, it is an important nonlinear model that occurs in physics, hydromechanics and optics. It is also known that it can be used to describe the physical direction of wave propagation in plasma and nonlinear wave [23, 24, 25, 26, 27]. The coupled Boussinesq equations occur in shallow water waves for bilayer fluid flow. This happens when there is an accidental oil spill from a ship, which causes a slick of oil to float above the water slide [28].

The study includes five chapters. In the first chapter, fractional calculations are mentioned along with brief information about nonlinear partial and fractional differential equations. In addition, Boussinesq equations and coupled Bousinesq equations that form the basis of the publication are explained. In the second chapter, definition and theories of comformable fractional derivative are given. In the third chapter $\left(\frac{G'}{G^2}\right)$-expansion method which is an important method for the solution is explained. In Chapter four, the application of finding exact solutions of comformable time fractional Boussinesq equation and coupled conformable time fractional Boussinesq equations with this method has been made. Finally, an explanation of the results is given in the last section.

2. Conformable fractional derivative and its properties

Various definitions have been introduced for the fractional derivative such as Riemann-Liouville, Grünwald-Letnikov and Caputo. All fractional derivatives provide linearity, but these definitions have some flaws. Khalil et al. [29] described a new, simple fractional derivative named the comformable fractional derivative that overcomes these shortcomings. The conformable derivative has the following definitions and properties as given in [29, 30, 31].

Definition: Suppose $f : [0, \infty) \to \mathbf{R}$ be a function. The αth-order conformable fractional derivative of f is defined by

$$_tD^\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \quad t > 0, \quad \alpha \in (0, 1).$$

Theorem1: Suppose $\alpha \in (0, 1]$ and f, g be α-differentiable at $t > 0$. Then,

(1) $_tD^\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g), \quad \forall a, b \in \mathbf{R}$.

(2) $_tD^\alpha(t^p) = pt^{p-\alpha}, \quad \forall p \in \mathbf{R}$.

(3) $_tD^\alpha(\lambda) = 0$ (λ is any constant).

(4) $_tD^\alpha(fg) = f(_tD^\alpha(g)) + g(_tD^\alpha(f))$.

(5) $_tD^\alpha(\frac{f}{g}) = \frac{g(_tD^\alpha(f)) - f(_tD^\alpha(g))}{g^2}$.

(6) If f is differentiable, then $_tD^\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}$.

Theorem2: Let $f, g : (0, \infty) \to \mathbf{R}$ be α-differentiable functions, when $0 < \alpha \le 1$. Then,

$$_tD^\alpha(f \circ g)(t) = t^{1-\alpha}g'(t)f'(g(t)).$$

2

3. Algorithm of $\left(\frac{G'}{G^2}\right)$-expansion method

In this section, we give an presentation of $\left(\frac{G'}{G^2}\right)$-expansion method for solving a general space time fractional differential equation as

$$H\left(u, \frac{\partial^\alpha u}{\partial t^\alpha}, \frac{\partial u}{\partial x}, \frac{\partial^{2\alpha} u}{\partial t^{2\alpha}}, \frac{\partial^2 u}{\partial x^2}...\right) = 0, \tag{3.1}$$

where u is an unknown function and H is a polynomial of u and its partial fractional derivatives [32]. It can be introduced as follows.

Step1: By applying the transformation

$$u(x,t) = U(\varepsilon),$$
$$\varepsilon = x + k\frac{t^\alpha}{\alpha}, \tag{3.2}$$

which k are nonzero constant coefficients, Eq.(3.1) can be reduced into the ordinary differential equation

$$Q\left(U, \frac{dU}{d\varepsilon}, \frac{d^2 U}{d\varepsilon^2}, \frac{d^3 U}{d\varepsilon^3}, ...\right) = 0. \tag{3.3}$$

Step2: Assume that Eq.(3.3) has the solution

$$U(\varepsilon) = a_0 + \sum_{i=1}^{M}\left[a_i\left(\frac{G'}{G^2}\right)^i + b_i\left(\frac{G'}{G^2}\right)^{-i}\right], \tag{3.4}$$

and

$$\left(\frac{G'}{G^2}\right)' = \mu + \lambda\left(\frac{G'}{G^2}\right)^2 \tag{3.5}$$

where $\lambda \neq 0$ $\mu \neq 1$ are integers and $a_0, a_i, b_i(i = 1, 2, ..., M)$ are real constants to be determined. The balancing number M is a positive integer which can be determined balancing the highest derivative terms with the highest power nonlinear terms in Eq.(3.3).

Step3: Substituting (3.4) into (3.3) using (3.5) and setting the coefficients of all powers of $\frac{G'}{G^2}$ to zero, the system of algebraic equations will be obtained. This algebraic system is solved by using programs such as Maple to obtain the values of the unknown constants a_0, a_i and $b_i(i = 1, 2, ..., M)$. More clearly, the value of M can be found from Eq. (3.3) as follows where $D(U(\varepsilon)) = M$ is degree of $U(\varepsilon)$.

$$D\left[\frac{d^q U}{d\varepsilon^q}\right] = M + q,$$
$$D\left[U^r\left(\frac{d^q U}{d\varepsilon^q}\right)^s\right] = Mr + s(q + M).$$

Step4: On the basis of the general solution of Eq. (3.5), the required exact solutions will be obtained in the following three cases:

3

Case 1. If $\lambda\mu > 0$, then

$$\frac{G'}{G^2} = \sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right), \tag{3.6}$$

Case 2. If $\lambda\mu < 0$, then

$$\frac{G'}{G^2} = -\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right), \tag{3.7}$$

Case 3. If $\lambda \neq 0$ and $\mu = 0$, then

$$\frac{G'}{G^2} = -\frac{C}{\lambda(C\varepsilon + D)}, \tag{3.8}$$

where C, D are nonzero constants.

4. Applications

In this section, two important nonlinear fractional differential equations will be solved by $\left(\frac{G'}{G^2}\right)$-expansion method.

4.1. Conformable time fractional Boussinesq equation

Let us consider the following nonlinear equation [33]:

$$\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^4 u}{\partial x^4} = 0, \quad 0 < \alpha \leq 1. \tag{4.1}$$

Applying the transformation Eq.(3.2) to this equation, it is obtained the following ordinary differential equation

$$(k^2 - 1)\frac{d^2 U}{d\varepsilon^2} - \frac{d^2(U^2)}{d\varepsilon^2} + \frac{d^4 U}{d\varepsilon^4} = 0. \tag{4.2}$$

Integrating Eq.(4.2) twice and taking constants of integration as zero, the following equation is obtained.

$$\frac{d^2 U}{d\varepsilon^2} + (k^2 - 1)U - U^2 = 0. \tag{4.3}$$

Balancing the terms of U^2 and $\frac{d^2 U}{d\varepsilon^2}$ in Eq. (4.3), we get $M = 2$. For $M = 2$, solution will get the form

$$U(\varepsilon) = a_0 + a_1\left(\frac{G'}{G^2}\right) + a_2\left(\frac{G'}{G^2}\right)^2 + b_1\left(\frac{G'}{G^2}\right)^{-1} + b_2\left(\frac{G'}{G^2}\right)^{-2}, \tag{4.4}$$

where a_0, a_1, a_2, b_1 and b_2 are unknown constants.

Substituting (4.4) using (3.5) into (4.3), setting the coefficients of all powers of $\left(\frac{G'}{G^2}\right)$ to zero, nonlinear algebraic equations are achieved. The occuring algebraic system is solved by aid of

Maple to find the values of unknown constants.

$$\left(\frac{G'}{G^2}\right)^{-4} \quad : \quad 6\mu^2 b_2 - b_2^2 = 0,$$

$$\left(\frac{G'}{G^2}\right)^{-3} \quad : \quad 2\mu^2 b_1 - 2b_1 b_2 = 0,$$

$$\left(\frac{G'}{G^2}\right)^{-2} \quad : \quad 8\mu\lambda b_2 + (k^2 - 1)b_2 - b_1^2 - 2b_2 a_0 = 0,$$

$$\left(\frac{G'}{G^2}\right)^{-1} \quad : \quad 2\mu\lambda b_1 + (k^2 - 1)b_1 - 2b_2 a_1 - 2b_1 a_0 = 0,$$

$$\left(\frac{G'}{G^2}\right)^{0} \quad : \quad 2\lambda^2 b_2 + 2\mu^2 a_2 + (k^2 - 1)a_0 - a_0^2 - 2b_2 a_2 - 2b_1 a_1 = 0,$$

$$\left(\frac{G'}{G^2}\right)^{1} \quad : \quad 2\mu\lambda a_1 + (k^2 - 1)a_1 - 2b_1 a_2 + 2a_1 a_0 = 0,$$

$$\left(\frac{G'}{G^2}\right)^{2} \quad : \quad 8\mu\lambda a_2 + (k^2 - 1)a_2 - a_1^2 - 2a_2 a_0 = 0,$$

$$\left(\frac{G'}{G^2}\right)^{3} \quad : \quad 2\lambda^2 a_1 - 2a_1 a_2 = 0,$$

$$\left(\frac{G'}{G^2}\right)^{-4} \quad : \quad 6\lambda^2 a_2 - a_2^2 = 0.$$

Solving this system of equations through Maple, we get the following results;

Set1: $k = \mp\sqrt{16\lambda\mu + 1}$, $a_0 = 12\lambda\mu$, $a_1 = b_1 = 0$, $a_2 = 6\lambda^2$, $b_2 = 6\mu^2$,
Set2: $k = \mp\sqrt{4\lambda\mu + 1}$, $a_0 = 6\lambda\mu$, $a_1 = b_1 = b_2 = 0$, $a_2 = 6\lambda^2$,
Set3: $k = \mp\sqrt{4\lambda\mu + 1}$, $a_0 = 6\lambda\mu$, $a_1 = b_1 = a_2 = 0$, $b_2 = 6\mu^2$,
Set4: $k = \mp\sqrt{1 - 4\lambda\mu}$, $a_0 = 2\lambda\mu$, $a_1 = b_1 = b_2 = 0$, $a_2 = 6\lambda^2$,
Set5: $k = \mp\sqrt{1 - 4\lambda\mu}$, $a_0 = 2\lambda\mu$, $a_1 = b_1 = a_2 = 0$, $b_2 = 6\mu^2$,
Set6: $k = \mp\sqrt{1 - 16\lambda\mu}$, $a_0 = -4\lambda\mu$, $a_1 = b_1 = 0$, $a_2 = 6\lambda^2$, $b_2 = 6\mu^2$.

The above set of values gives the following exact solutions for conformable time fractional Boussinesq equation

Solution 1:

(i) If $\lambda\mu > 0$, the trigonometric solution is found

$$u_1(x, t) = U_{11}(\varepsilon) = 12\lambda\mu + 6\lambda^2 \left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^2$$

$$+ 6\mu^2 \left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^{-2}.$$

(ii) If $\lambda\mu < 0$, the hyperbolic solution is obtained

$$u_2(x, t) = U_{12}(\varepsilon) = 12\lambda\mu + 6\lambda^2 \left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^2$$

$$+ 6\mu^2 \left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^{-2}.$$

(iii) If $\lambda \neq 0$, $\quad \mu = 0$, the rational solution is found

$$u_3(x,t) = U_{13}(\varepsilon) = 6\left(\frac{C}{C\varepsilon + D}\right)^2$$

In (i)-(iii), $\varepsilon = x \mp \sqrt{16\lambda\mu + 1}\left(\frac{t^\alpha}{\alpha}\right)$.

Solution 2:

(i) If $\lambda\mu > 0$, the trigonometric solution is found

$$u_4(x,t) = U_{21}(\varepsilon) = 6\lambda\mu + 6\lambda^2\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^2.$$

(ii) If $\lambda\mu < 0$, the hyperbolic solution is obtained

$$u_5(x,t) = U_{22}(\varepsilon) = 6\lambda\mu + 6\lambda^2\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^2.$$

(iii) If $\lambda \neq 0$, $\quad \mu = 0$, the rational solution is found

$$u_6(x,t) = U_{33}(\varepsilon) = 6\left(\frac{C}{C\varepsilon + D}\right)^2$$

In (i)-(iii), $\varepsilon = x \mp \sqrt{4\lambda\mu + 1}\left(\frac{t^\alpha}{\alpha}\right)$.

Solution 3:

(i) If $\lambda\mu > 0$, the trigonometric solution is found

$$u_7(x,t) = U_{31}(\varepsilon) = 6\lambda\mu + 6\mu^2\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^{-2}.$$

(ii) If $\lambda\mu < 0$, the hyperbolic solution is obtained

$$u_8(x,t) = U_{32}(\varepsilon) = 6\lambda\mu + 6\mu^2\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^{-2}.$$

In (i)-(ii), $\varepsilon = x \mp \sqrt{4\lambda\mu + 1}\left(\frac{t^\alpha}{\alpha}\right)$.

Solution 4:

(i) If $\lambda\mu > 0$, the trigonometric solution is found

$$u_9(x,t) = U_{41}(\varepsilon) = 2\lambda\mu + 6\lambda^2\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^2$$

(ii) If $\lambda\mu < 0$, the hyperbolic solution is obtained

$$u_{10}(x,t) = U_{42}(\varepsilon) = 2\lambda\mu + 6\lambda^2\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^2$$

(iii) If $\lambda \neq 0, \quad \mu = 0$, the rational solution is found

$$u_{11}(x,t) = U_{43}(\varepsilon) = 6\left(\frac{C}{C\varepsilon + D}\right)^2$$

In (i)-(iii), $\varepsilon = x \mp \sqrt{1 - 4\lambda\mu}\left(\frac{t^\alpha}{\alpha}\right)$.

Solution 5:

(i) If $\lambda\mu > 0$, the trigonometric solution is found

$$u_{12}(x,t) = U_{51}(\varepsilon) = 2\lambda\mu + 6\mu^2\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^{-2}.$$

(ii) If $\lambda\mu < 0$, the hyperbolic solution is obtained

$$u_{13}(x,t) = U_{52}(\varepsilon) = 2\lambda\mu + 6\mu^2\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^{-2}.$$

In (i)-(ii), $\varepsilon = x \mp \sqrt{1 - 4\lambda\mu}\left(\frac{t^\alpha}{\alpha}\right)$.

Solution 6:

(i) If $\lambda\mu > 0$, the trigonometric solution is found

$$u_{14}(x,t) = U_{61}(\varepsilon) = -4\lambda\mu + 6\lambda^2\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^2$$

$$+ 6\mu^2\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^{-2}.$$

(ii) If $\lambda\mu < 0$, the hyperbolic solution is obtained

$$u_{15}(x,t) = U_{62}(\varepsilon) = -4\lambda\mu + 6\lambda^2\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^2$$

$$+ 6\mu^2\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(-\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^{-2}.$$

(iii) If $\lambda \neq 0, \quad \mu = 0$, the rational solution is found

$$u_{16}(x,t) = U_{63}(\varepsilon) = 6\left(\frac{C}{C\varepsilon + D}\right)^2$$

In (i)-(iii), $\varepsilon = x \mp \sqrt{1 - 16\lambda\mu}\left(\frac{t^\alpha}{\alpha}\right)$.

In Figure 1, the physical properties of Eq. (4.1) whose solutions are used in **Solution 2** have been shown for $\alpha = 0.5$ with some special values.
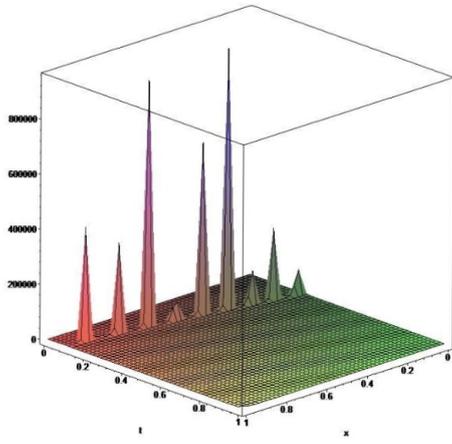
(a)
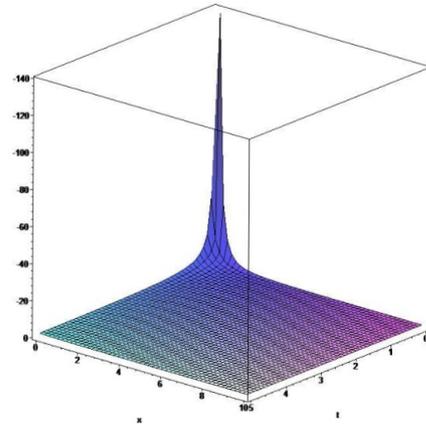
(b)
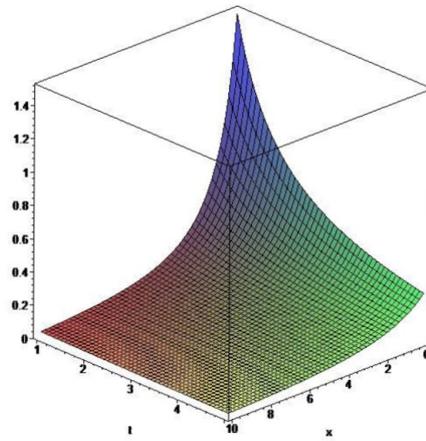
(c)

Figure 1: (a) The trigonometric solution of $u(x,t)$ when $\lambda = \mu = C = D = 1$. (b) The hyperbolic solution of $u(x,t)$ when $\lambda = 0.5, \mu = -0.3, C = D = 1$. (c) The rational solution of $u(x,t)$ when $\lambda = 1, \mu = 0, C = D = 1$.

8

4.2. Coupled conformable time fractional Boussinesq equations

Let us consider the following nonlinear equation system[33]:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial v}{\partial x} = 0,$$
$$\frac{\partial^\alpha v}{\partial t^\alpha} + \beta \frac{\partial}{\partial x}(u^2) - \gamma \frac{\partial^3 u}{\partial x^3} = 0, \tag{4.5}$$

where the time fractional derivative is αth order and $0 < \alpha \leq 1$.

Applying the transformation Eq.(3.2) to this system, it is found the following ordinary differential equation system

$$k\frac{dU}{d\varepsilon} + \frac{dV}{d\varepsilon} = 0,$$
$$k\frac{dV}{d\varepsilon} + \beta \frac{d}{d\varepsilon}(U^2) - \gamma \frac{d^3 U}{d\varepsilon^3} = 0. \tag{4.6}$$

Integrating Eq.(4.6) once and taking constant of integration as zero, the following equations are found:

$$kU + V = 0, \tag{4.7a}$$

$$kV + \beta U^2 - \gamma \frac{d^2 U}{d\varepsilon^2} = 0. \tag{4.7b}$$

From Eq.(4.7a), we get

$$V = -kU. \tag{4.8}$$

Substituting Eq.(4.8) into Eq. (4.7b), it is obtained the following differential equation

$$-k^2 U + \beta U^2 - \gamma \frac{d^2 U}{d\varepsilon^2} = 0. \tag{4.9}$$

Balancing the terms of U^2 and $\frac{d^2 U}{d\varepsilon^2}$ in Eq. (4.9), we get $M = 2$. For $M = 2$, solution will obtain the form

$$U(\varepsilon) = a_0 + a_1\left(\frac{G'}{G^2}\right) + a_2\left(\frac{G'}{G^2}\right)^2 + b_1\left(\frac{G'}{G^2}\right)^{-1} + b_2\left(\frac{G'}{G^2}\right)^{-2}, \tag{4.10}$$

where a_0, a_1, a_2, b_1 and b_2 are unknown constants.

Substituting (4.10) using (3.5) into (4.9), setting the coefficients of all powers of $\left(\frac{G'}{G^2}\right)$ to zero, nonlinear algebraic equations are obtained. The occurring algebraic system is solved by help of

9

Maple to find the values of unknown constants.

$$\left(\frac{G'}{G^2}\right)^{-4} \quad : \quad \beta b_2^2 - 6\gamma\mu^2 b_2 = 0,$$

$$\left(\frac{G'}{G^2}\right)^{-3} \quad : \quad 2\beta b_1 b_2 - 2\gamma\mu^2 b_1 = 0,$$

$$\left(\frac{G'}{G^2}\right)^{-2} \quad : \quad -k^2 b_2 + \beta b_1^2 + 2\beta b_2 a_0 - 8\gamma\mu\lambda b_2 = 0,$$

$$\left(\frac{G'}{G^2}\right)^{-1} \quad : \quad -k^2 b_1 + 2\beta b_2 a_1 + 2\beta b_1 a_0 - 2\gamma\mu\lambda b_1 = 0,$$

$$\left(\frac{G'}{G^2}\right)^{0} \quad : \quad -k^2 a_0 + \beta a_0^2 + 2\beta b_2 a_2 + 2\beta b_1 a_1 - 2\gamma\lambda^2 b_2 - 2\gamma\mu^2 a_2 = 0,$$

$$\left(\frac{G'}{G^2}\right)^{1} \quad : \quad -k^2 a_1 + 2\beta b_1 a_2 + 2\beta a_1 a_0 - 2\gamma\mu\lambda a_1 = 0,$$

$$\left(\frac{G'}{G^2}\right)^{2} \quad : \quad -k^2 a_2 + \beta a_1^2 + 2\beta a_2 a_0 - 8\gamma\mu\lambda a_2 = 0,$$

$$\left(\frac{G'}{G^2}\right)^{3} \quad : \quad 2\beta a_1 a_2 - 2\gamma\lambda^2 a_1 = 0,$$

$$\left(\frac{G'}{G^2}\right)^{4} \quad : \quad \beta a_2^2 - 6\gamma\lambda^2 a_2 = 0.$$

Solving this system of equations through Maple, we get the following results;

Set1: $k = \mp 2\sqrt{\lambda\gamma\mu}, \quad a_0 = \frac{6\lambda\gamma\mu}{\beta}, \quad a_1 = 0, \quad a_2 = \frac{6\lambda^2\gamma}{\beta}, \quad b_1 = 0, \quad b_2 = 0,$

Set2: $k = \mp 2\sqrt{\lambda\gamma\mu}, \quad a_0 = \frac{6\lambda\gamma\mu}{\beta}, \quad a_1 = 0, \quad a_2 = 0, \quad b_1 = 0, \quad b_2 = \frac{6\mu^2\gamma}{\beta},$

Set3: $k = \mp 2\sqrt{-\lambda\gamma\mu}, \quad a_0 = \frac{2\lambda\gamma\mu}{\beta}, \quad a_1 = 0, \quad a_2 = \frac{6\lambda^2\gamma}{\beta}, \quad b_1 = 0, \quad b_2 = 0,$

Set4: $k = \mp 2\sqrt{-\lambda\gamma\mu}, \quad a_0 = \frac{2\lambda\gamma\mu}{\beta}, \quad a_1 = 0, \quad a_2 = 0, \quad b_1 = 0, \quad b_2 = \frac{6\mu^2\gamma}{\beta},$

Set5: $k = \mp 4\sqrt{\lambda\gamma\mu}, \quad a_0 = \frac{12\lambda\gamma\mu}{\beta}, \quad a_1 = 0, \quad a_2 = \frac{6\lambda^2\gamma}{\beta}, \quad b_1 = 0, \quad b_2 = \frac{6\mu^2\gamma}{\beta},$

Set6: $k = \mp 4\sqrt{-\lambda\gamma\mu}, \quad a_0 = \frac{-4\lambda\gamma\mu}{\beta}, \quad a_1 = 0, \quad a_2 = \frac{6\lambda^2\gamma}{\beta}, \quad b_1 = 0, \quad b_2 = \frac{6\mu^2\gamma}{\beta}.$

The above set of values gives the following exact solutions for conformable time fractional coupled Boussinesq equation.

Solution 1:

(i) If $\lambda\mu > 0$, the trigonometric solution is found

$$u_1(x,t) = U_{11}(\varepsilon) = \frac{6\lambda\gamma\mu}{\beta} + \frac{6\lambda^2\gamma}{\beta}\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^2.$$

From Eq.(4.8),

$$v_1(x,t) = -k\left[\frac{6\lambda\gamma\mu}{\beta} + \frac{6\lambda^2\gamma}{\beta}\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^2\right].$$

(ii) If $\lambda\mu < 0$, the hyperbolic solution is obtained

$$u_2(x,t) = U_{12}(\varepsilon) = \frac{6\lambda\gamma\mu}{\beta} + \frac{6\lambda^2\gamma}{\beta}\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^2.$$

From Eq.(4.8),

$$v_2(x,t) = -k\left[\frac{6\lambda\gamma\mu}{\beta} + \frac{6\lambda^2\gamma}{\beta}\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^2\right].$$

(iii) If $\lambda \neq 0, \quad \mu = 0$, the rational solution is found

$$u_3(x,t) = U_{13}(\varepsilon) = \frac{6\gamma}{\beta}\left(\frac{C}{C\varepsilon + D}\right)^2.$$

From Eq.(4.8),

$$v_3(x,t) = -k\left[\frac{6\gamma}{\beta}\left(\frac{C}{C\varepsilon + D}\right)^2\right],$$

where $\varepsilon = x \mp 2\sqrt{\lambda\gamma\mu}\left(\frac{t^\alpha}{\alpha}\right)$.

Solution 2:

(i) If $\lambda\mu > 0$, the trigonometric solution is found

$$u_4(x,t) = U_{21}(\varepsilon) = \frac{6\lambda\gamma\mu}{\beta} + \frac{6\mu^2\gamma}{\beta}\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^{-2}.$$

From Eq.(4.8),

$$v_4(x,t) = -k\left[\frac{6\lambda\gamma\mu}{\beta} + \frac{6\mu^2\gamma}{\beta}\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^{-2}\right].$$

(ii) If $\lambda\mu < 0$, the hyperbolic solution is obtained

$$u_5(x,t) = U_{22}(\varepsilon) = \frac{6\lambda\gamma\mu}{\beta} + \frac{6\mu^2\gamma}{\beta}\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^{-2}.$$

From Eq.(4.8),

$$v_5(x,t) = -k\left[\frac{6\lambda\gamma\mu}{\beta} + \frac{6\mu^2\gamma}{\beta}\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^{-2}\right],$$

where $\varepsilon = x \mp 2\sqrt{\lambda\gamma\mu}\left(\frac{t^\alpha}{\alpha}\right)$.

Solution 3:

(i) If $\lambda\mu > 0$, the trigonometric solution is found

$$u_6(x,t) = U_{31}(\varepsilon) = \frac{2\lambda\gamma\mu}{\beta} + \frac{6\lambda^2\gamma}{\beta}\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^2.$$

From Eq.(4.8),

$$v_6(x,t) = -k\left[\frac{2\lambda\gamma\mu}{\beta} + \frac{6\lambda^2\gamma}{\beta}\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^2\right].$$

(ii) If $\lambda\mu < 0$, the hyperbolic solution is obtained

$$u_7(x,t) = U_{32}(\varepsilon) = \frac{2\lambda\gamma\mu}{\beta} + \frac{6\lambda^2\gamma}{\beta}\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^2.$$

From Eq.(4.8),

$$v_7(x,t) = -k\left[\frac{2\lambda\gamma\mu}{\beta} + \frac{6\lambda^2\gamma}{\beta}\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^2\right].$$

(iii) If $\lambda \neq 0, \quad \mu = 0$, the rational solution is found

$$u_8(x,t) = U_{33}(\varepsilon) = \frac{6\gamma}{\beta}\left(\frac{C}{C\varepsilon + D}\right)^2.$$

From Eq.(4.8),

$$v_8(x,t) = -k\left[\frac{6\gamma}{\beta}\left(\frac{C}{C\varepsilon + D}\right)^2\right],$$

where $\varepsilon = x \mp 2\sqrt{-\lambda\gamma\mu}(\frac{t^\alpha}{\alpha})$.

Solution 4:

(i) If $\lambda\mu > 0$, the trigonometric solution is found

$$u_9(x,t) = U_{41}(\varepsilon) = \frac{2\lambda\gamma\mu}{\beta} + \frac{6\mu^2\gamma}{\beta}\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^{-2}.$$

From Eq.(4.8),

$$v_9(x,t) = -k\left[\frac{2\lambda\gamma\mu}{\beta} + \frac{6\mu^2\gamma}{\beta}\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^{-2}\right].$$

(ii) If $\lambda\mu < 0$, the hyperbolic solution is obtained

$$u_{10}(x,t) = U_{42}(\varepsilon) = \frac{2\lambda\gamma\mu}{\beta} + \frac{6\mu^2\gamma}{\beta}\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^{-2}.$$

From Eq.(4.8),

$$v_{10}(x,t) = -k\left[\frac{2\lambda\gamma\mu}{\beta} + \frac{6\mu^2\gamma}{\beta}\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^{-2}\right],$$

where $\varepsilon = x \mp 2\sqrt{-\lambda\gamma\mu}\left(\frac{t^\alpha}{\alpha}\right)$.

Solution 5:

(i) If $\lambda\mu > 0$, the trigonometric solution is found

$$u_{11}(x,t) = U_{51}(\varepsilon) = \frac{12\lambda\gamma\mu}{\beta} + \frac{6\lambda^2\gamma}{\beta}\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^2$$
$$+ \frac{6\mu^2\gamma}{\beta}\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^{-2}.$$

From Eq.(4.8),

$$v_{11}(x,t) = -k\left[\frac{12\lambda\gamma\mu}{\beta} + \frac{6\lambda^2\gamma}{\beta}\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^2\right.$$
$$\left. + \frac{6\mu^2\gamma}{\beta}\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^{-2}\right].$$

(ii) If $\lambda\mu < 0$, the hyperbolic solution is obtained

$$u_{12}(x,t) = U_{52}(\varepsilon) = \frac{12\lambda\gamma\mu}{\beta} + \frac{6\lambda^2\gamma}{\beta}\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^2$$
$$+ \frac{6\mu^2\gamma}{\beta}\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(-\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^{-2}.$$

From Eq.(4.8),

$$v_{12}(x,t) = -k\left[\frac{12\lambda\gamma\mu}{\beta} + \frac{6\lambda^2\gamma}{\beta}\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^2\right.$$
$$\left. + \frac{6\mu^2\gamma}{\beta}\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(-\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^{-2}\right].$$

(iii) If $\lambda \neq 0, \quad \mu = 0$, the rational solution is found

$$u_{13}(x,t) = U_{53}(\varepsilon) = \frac{6\gamma}{\beta}\left(\frac{C}{C\varepsilon + D}\right)^2.$$

From Eq.(4.8),

$$v_{13}(x,t) = -k\left[\frac{6\gamma}{\beta}\left(\frac{C}{C\varepsilon + D}\right)^2\right],$$

where $\varepsilon = x \mp 4\sqrt{\lambda\gamma\mu}\left(\frac{t^\alpha}{\alpha}\right)$.

Solution 6:

(i) If $\lambda\mu > 0$, the trigonometric solution is found

$$u_{14}(x,t) = U_{61}(\varepsilon) = \frac{-4\lambda\gamma\mu}{\beta} + \frac{6\lambda^2\gamma}{\beta}\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^2$$
$$+ \frac{6\mu^2\gamma}{\beta}\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^{-2}.$$

From Eq.(4.8),

$$v_{14}(x,t) = -k\left[\frac{-4\lambda\gamma\mu}{\beta} + \frac{6\lambda^2\gamma}{\beta}\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^2\right.$$
$$\left.+ \frac{6\mu^2\gamma}{\beta}\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^{-2}\right].$$

(ii) If $\lambda\mu < 0$, the hyperbolic solution is obtained

$$u_{15}(x,t) = U_{62}(\varepsilon) = \frac{-4\lambda\gamma\mu}{\beta} + \frac{6\lambda^2\gamma}{\beta}\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^2$$
$$+ \frac{6\mu^2\gamma}{\beta}\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(-\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^{-2}.$$

From Eq.(4.8),

$$v_{15}(x,t) = -k\left[\frac{-4\lambda\gamma\mu}{\beta} + \frac{6\lambda^2\gamma}{\beta}\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^2\right.$$
$$\left.+ \frac{6\mu^2\gamma}{\beta}\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(-\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^{-2}\right].$$

(iii) If $\lambda \neq 0, \quad \mu = 0$, the rational solution is found

$$u_{16}(x,t) = U_{63}(\varepsilon) = \frac{6\gamma}{\beta}\left(\frac{C}{C\varepsilon + D}\right)^2.$$

From Eq.(4.8),

$$v_{16}(x,t) = -k\left[\frac{6\gamma}{\beta}\left(\frac{C}{C\varepsilon + D}\right)^2\right],$$

where $\varepsilon = x \mp 4\sqrt{-\lambda\gamma\mu}\left(\frac{t^\alpha}{\alpha}\right)$.

In Figure 2, the physical characteristics of Eq.(4.5) whose solutions are used in **Solution 3** have been shown for $\alpha = 0.5$ with some special values.
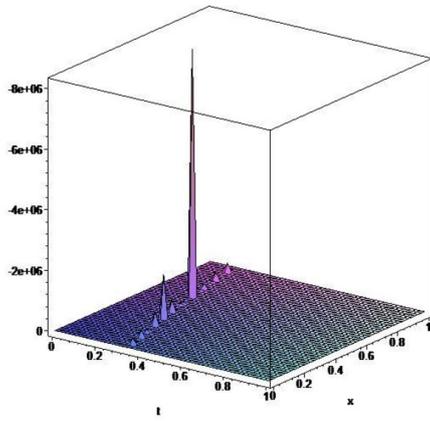
(a)
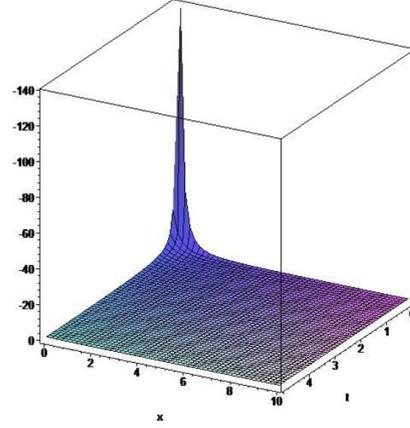
(b)
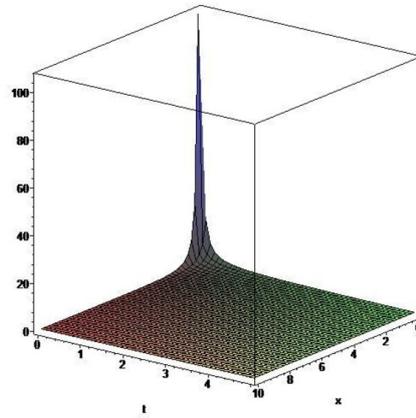
(c)

Figure 2: (a) The trigonometric solution of $u(x,t)$ when $\lambda = \mu = \beta = C = D = 1, \gamma = -1$. (b) The hyperbolic solution of $u(x,t)$ when $\lambda = 0.5, \mu = -0.3, C = D = \beta = \gamma = 1$. (c) The rational solution of $u(x,t)$ when $\lambda = 1, \mu = 0, C = D = \beta = 1, \gamma = -1$.

5. Conlusion

In this paper, analytical solutions of conformable time-fractional nonlinear Boussinesq equations which known as an important class of fractional differential equations in mathematical physics were found by $\left(\frac{G'}{G^2}\right)$-expansion method. Different types of solutions were officially found, including trigonometric, hyperbolic and rational function solutions. The obtained solutions were not the same and they were much recent than previously results. It was ensured that the solutions found were correct by replacing the original equations. Since $\left(\frac{G'}{G^2}\right)$-expansion method is efficient and not complicated, it can also be used to get solutions for other fractional differential equations encountered in mathematical physics.

References

[1] Inan, I. E., Ugurlu, Y. and Bulut, H., Auto-Bäcklund transformation for some nonlinear partial differential equation, *Optik - Int. J. Light Electr. Opt. 127(22)*, (2016), pp. 10780-10785.

[2] Yildirim, O.and Uzun, M., Weak solvability of the unconditionally stable difference scheme for the coupled sine-Gordon system, *Nonlinear Analysis: Modelling and Control 25(6)*, (2020), pp. 997–1014.

[3] Yildirim, O.and Uzun, M., On the numerical solutions of high order stable difference schemes for the hyperbolic multipoint nonlocal boundary value problems, *Applied Mathematics and Computation 254*, (2015), pp. 210–218.

[4] Yildirim, O.and Caglak, S., Lie point symmetries of difference equation for nonlinear sine-Gordon equation, *Physica Scripta 94*, (2019), 085219(8).

[5] Abdou, M. and Soliman, A., New applications of variational iteration method, *Physica D 211(1)*, (2005), pp. 1-8.

[6] Elwakil, S, El-Labany, S., Zahran, M. and Sabry, R., Modified extended tanh-function method for solving nonlinear partial differential equations, *Phys Lett A 299(2)*, (2002), pp. 179–188.

[7] Wazwaz, A-M., Multiple-soliton solutions for the KP equation by Hirota's bilinear method and by the tanh–coth method, *Appl Math Comput. 190(1)*, (2007), pp. 633–640.

[8] Zayed, E.M.E. and Alurrfi, K.A.E. , A new Jacobi elliptic function expansion method for solving a nonlinear PDE describing the nonlinear low-pass electrical lines, *Chaos Solitons Fractals 78*, (2015), pp. 148-155.

[9] Manafian, J. and Lakestani, M., Application of tan(//2)-expansion method for solvingthe Biswas-Milovic equation for Kerr law nonlinearity, *Optik - Int. J. Light Electr.Opt. 127(4)*, (2012), pp. 2040-2054.

[10] Kudryashov, N. A., One method for finding exact solutions of nonlinear differential equations *Commun. Nonlinear Sci. Numer. Simulat. 17(6)*, (2012), pp. 2048-2053.

[11] Akbulut, A., Kaplan, M. and Tascan, F., The investigation of exact solutions of nonlinear partial differential equations by using exp(-$\Phi(\varepsilon)$) method, *Optik - Int. J. Light Electr.Opt. 132*, (2017), pp. 382-387.

[12] Bekir, A. and Güner, Ö., Exact solutions of nonlinear fractional differential equations by (G'/ G)-expansion method, *Chin Phys B. 22(11)*, (2013), pp. 110202-1 - 110202-6.

[13] Gómez, S. and Cesar, A. A nonlinear fractional Sharma–Tasso–Olver equation, *App. Math. Comput. 266*, (2015), pp. 385–389.

[14] Chen, C.and Jiang,Y.L. Lie group analysismethod for two classes of fractional partial differential equations, *Commun. Nonlinear Sci. Numer. Simul. 26(1)*, (2015), pp. 24–35.

[15] Lu, B., The first integral method for some time fractional differential equations. *J. Math. Anal. Appl. 395(2)*, (2012), pp. 684–693.

[16] Zheng, B., Exp-function method for solving fractional partial differential equations, *Sci. World J. 2013*, (2013), 465723.

[17] Aksoy, E., Çevikel A. C. and Bekir, Soliton solutions of (2+1)-dimensional time-fractional Zoomeron equation, *Optik - Int. J. Light Electr.Opt. 127*,(2016), pp. 6933–6942.

[18] Matinfar, M., Eslami, M. and Kordy, M. The functional variable method for solving thefractional Korteweg de Vries equations and the coupled Korteweg de Vries equations, *Pramana J Phys. 85*,(2015), pp. 583-592.

[19] Madsen, P. A., Murray, R. and Sorensen, O.R., A new form of the Boussinesq equations with improved linear dispersion characteristics, *Coast English Journal 15(4)*, (1991), pp. 371-388

[20] Sahadevan, R. and Prakash, P. Exact solutions and maximal dimension of invariant subspaces of time fractional coupled nonlinear partial differential equations, *Commun Nonlinear Sci Numer Simulat 42*, (2017), pp. 158-177

[21] Hosseini, K., Bekir, A., Ansari, R., Exact solutions of nonlinear conformable time-fractional Boussinesq equations using the exp$(-\phi(\varepsilon))$-expansion method, *Opt Quant Electron 49*, (2017), 131

[22] Yaslan, H.Ç. and Girgin, A., Exp-function method for the conformable space-time fractional STO, ZKBBM and coupled Boussinesq equations, *Arab Journal of Basic and Applied Sciences 26*, (2019), pp. 163-170

[23] Triki, H., Kara, A. H. and Biswas, A. Domain walls to Boussinesq-type equations in $(2 + 1)$-dimensions, *Indian J Phys 88(7)* , (2014), pp. 751–755

[24] Abazari, R., Jamshidzadeh, S. and Biswas, A. Solitary wave solutions of coupled Boussinesq equation, *Complexity 21(52)* , (2016), pp. 151–155

[25] Abazari, R., Jamshidzadeh, S. and Biswas, A. Multi soliton solutions based on interactions of basic traveling waves with an application to the nonlocal Boussinesq equation, *Acta Physica Polonica B 47* , (2016), pp. 1101-1112

[26] Biswas, A., Kara, A. H., Moraru, L., Triki, H., and Moshokoa, S. P. Shallow water waves modeled by the Boussinesq equation having logarithmic non linearity, *Proceedings of the Romanian academy Series A, 18(2)* , (2017), pp. 144–149

[27] Biswas, A., Ekici, M. and Sonmezoglu, A. Gaussian solitary waves to Boussinesq equation with dual dispersion and logarithmic non linearity, *Nonlinear Analysis: Modelling and Control, 23(6)* , (2018), pp. 942–950

[28] Jawad, A.J.M, Petković,.M.D., Laketa, P. and Biswas, A., Dynamics of shallow water waves with Boussinesq equation, *Scientia Iranica B 20(1)*, (2013), pp. 179–184

[29] Khalil, R. , Al Horani, M. , Yousef, A. and Sababheh, M., A new definition of fractional derivative, *J. Comput. Appl. Math. 264*, (2014), 65, pp. 65-70

[30] Abdeljawad, T., On conformable fractional calculus *J. Comput. Appl. Math. 279*, (2015), pp. 57-66

[31] Eslami, M., Rezazadeh, H., The first integral method for Wu–Zhang system with conformable time-fractional derivative, *Calcolo 53*, (2016), pp. 475-485

[32] Chen, J and Chen, H. The (G'/G^2) method and its application to coupled nonlinear Klein-Gordon equation, *Journal of South China normal University (Natural Science Edition) 2*, (2012), 013.

[33] Hosseini, K.and Ansari, R. New exact solutions of nonlinear conformable time-fractional Boussinesq equations using the modified Kudryashov method *Waves in random and complex media 27(4)*, (2017), pp. 628–636