International Electronic Journal of Geometry
Volume 1 No. 1 pp. 25-32 (2008) ©IEJG

ON EXISTENCE OF CANONICAL SCREENS FOR COISOTROPIC SUBMANIFOLDS

K. L. DUGGAL

(Communicated by Kazım İLARSLAN)

Abstract. In this paper we study coisotropic lightlike submanifolds of a semi-Riemannian manifold. For a large variety of this class of submanifolds, we prove two theorems on the existence of integrable canonical screen distribution and canonical null transversal bundle subject to some reasonable geometric conditions.

1. Introduction

Let $(M,\, g)$ be a m-dimensional submanifold of an $(m+n)$-dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ of a constant non-zero index q. Suppose the induced metric g on M is non-degenerate. Define

$$TM^{\perp} = \left\{ V \in \Gamma(T\bar{M}) : \; g(V,\, W) = 0, \; \forall\, W \in \Gamma(T\bar{M}) \right\} \tag{1.1}$$

the normal bundle subspace of M in $\bar{M}$. Following is the orthogonal complementary decomposition:

$$T\bar{M} = TM \oplus TM^{\perp}, \quad TM \cap TM^{\perp} = \{0\}. \tag{1.2}$$

Here, both the tangent and the normal bundle subspaces are non-degenerate and any vector field of $T\bar{M}$ splits uniquely into a component tangent to M and a component perpendicular to M. Now let g degenerate on M. Then, there exists a vector field $\xi \neq 0$ of M such that

$$g(\xi,\, X) = 0, \quad \forall\, X \in \Gamma(TM). \tag{1.3}$$

The radical space (O'Neill [17, page 53]) of T_xM, at each point $x \in M$, is a subspace $Rad\, T_xM$ defined by

$$Rad\, T_xM = \{\xi \in T_xM \; : \; g_x(\xi,\, X) = 0, \; \forall\, X \in T_xM\}\,. \tag{1.4}$$

M is called a lightlike submanifold of $\bar{M}$ [9]. We follow [9] for the notations and the results used in this paper. Suppose $dim(Rad\, T_xM) = r \neq 0$. Comparing (1.1)

2000 *Mathematics Subject Classification.* 53B25, 53C50, 53B50.
Key words and phrases. coisotropic submanifold, canonical screen distribution, screen transformation equations, null transversal bundle.

with (1.4) with respect to degenerate g and any null vector being perpendicular to itself implies that $T_xM^{\perp}$ is also null and

$$Rad\,T_xM = T_xM \cap T_xM^{\perp}. \tag{1.5}$$

Thus, for a lightlike submanifold M, (1.2) does not hold because TM and $TM^{\perp}$ have a non-trivial intersection and their sum is not the whole of tangent bundle space $T\bar{M}$. In other words, a vector of $T_x\bar{M}$ cannot be decomposed uniquely into a component tangent to T_xM and a component of $T_xM^{\perp}$. Therefore, the standard text-book definition of the second fundamental form and the Gauss-Wiengarten formulas do not work, in the usual way, for the lightlike case.

To deal with this anomaly, lightlike manifolds have been studied by several ways corresponding to their use in a given problem. Indeed, see Akivis-Goldberg [1], Bonnor [5], IIyenko[10], Israel [11, 12], Katsuno [14], Leistner [15], Nurowski-Robinson [16], Penrose [18], Perlick [19], Rosca [20], and more referred in these papers. In 1991, Bejancu-Duggal [3] introduced a general geometric technique of using a non-degenerate screen distribution $S(TM)$ to deal with the above anomaly for lightlike hypersurfaces (also applicable for a general submanifold). Based on this specific technique of using a screen distribution, we have the following:

$$TM = Rad\,TM \oplus_{orth} S(TM). \tag{1.6}$$

$$T\bar{M}|_M = TM \oplus tr(TM) \quad TM \cap tr(TM) = \{0\} \tag{1.7}$$

$$= S(TM) \perp S(TM)^{\perp} \tag{1.8}$$

where $tr(TM)$ is a complementary (but never orthogonal) transversal vector bundle to TM in $T\bar{M}|_M$, $S(TM)^{\perp}$ is non-degenerate of rank $2n$ and $Rad\,TM$ is its vector subbundle. The submanifold $(M, g, S(TM))$ is called r-lightlike or coisotropic [13] according as $Rad\,TM \subset TM^{\perp}$ or $Rad\,TM = TM^{\perp}$. In the later case, $r = n < m$. Unfortunately, due to the degenerate metric g, an arbitrary screen $S(TM)$ is not unique and, therefore, the induced objects of the submanifold depend on its choice that creates a problem. Thus, it is reasonable to look for a canonical screen in lightlike geometry. First paper on the existence of a canonical screen distribution for lightlike hypersurfaces was published by Bejancu [2] in 1993. Since then considerable work has been done on this problem and now there are a large classes of lightlike hypersurfaces of semi-Riemannian manifolds with the choice of a canonical screen distribution (see a review article [7] with extensive upto-date references and three papers [1, 4, 6] on follow up work), in some cases subject to a reasonable geometric condition.

Continuing our study in this direction, the objective of this paper is to show that there exist canonical distributions for a large variety of coisotropic submanifolds of semi-Riemannian manifolds. In section 2 we recall Gauss and Weingarten type equations and find the transformation equations with respect to a change in the screen distribution. Section 3 contains proofs of two theorems on the existence of canonical screen distributions.

2. Screen transformation equations

Let $(M, g, S(TM))$ be a coisotropic lightlike submanifold of $(\bar{M}, \bar{g})$. Then, $Rad\,TM = TM^{\perp}$ and $S(TM^{\perp}) = \{0\}$. There exists a local quasi-orthonormal field of frames of $\bar{M}$ along M:

$$\{\xi_1, ..., \xi_n,\ N_1, ..., N_n,\ W_{n+1}, ..., W_m\} \tag{2.1}$$

where $\{\xi_1, ..., \xi_n\}$ is a null basis of $\Gamma\,(Rad\,TM)$, $\{N_1, ..., N_n\}$ a null basis of the transversal bundle $tr(TM)$ and $\{W_{n+1}, ..., W_m\}$ orthonormal basis of $\Gamma\,(S(TM))|U)$ respectively. Moreover,

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \text{for any} \quad i, j \in \{1, \dots, n\}. \tag{2.2}$$

Denote by P the projection of TM on the screen distribution $S(TM)$ with respect to the decomposition (1.6). Suppose $\bar{\nabla}$ and ∇ are the Levi - Civita connection on $\bar{M}$ and a linear connection on M. Following are the Gauss and Weingarten type equations [13]:

$$\bar{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^{n} h_i\,(X,\, Y)\, N_i, \tag{2.3}$$

$$\bar{\nabla}_X N_i = -\, A_{N_i} X + \sum_{j=1}^{n} \tau_{ij}\,(X)\, N_j, \tag{2.4}$$

$$\nabla_X PY = \nabla^*_X PY + \sum_{i=1}^{n} h^*_i\,(X,\, PY)\, \xi_i, \tag{2.5}$$

$$\nabla_X \xi_i = -\, A_{\xi_i}\, X - \sum_{j=1}^{n} \tau_{ij}(X)\, \xi_j, \quad \forall X\,,\, Y \in \Gamma\,(TM), \tag{2.6}$$

for every $i = 1, \dots, n$. Here h_i are the second fundamental forms of M with respect to the normals N_i, A_{N_i} are their respective shape operators and τ_{ij} are 1-forms on M. Also h^*_i are the second fundamental forms of $S(TM)$ with respect to $Rad\,TM$, A_{ξ_i} are the respective shape operators of the screen distribution and ∇^* is the metric connection on $S(TM)$. Moreover,

$$h_i(X, \xi_i) = 0, \quad \bar{g}(h_i(X, PY), \xi_i) = g(A_{\xi_i}\, X, PY), \tag{2.7}$$

$$\bar{g}(h^*_i(X, PY), N_i) = \bar{g}(A_{N_i} X, PY), \quad \forall X,\, Y \in \Gamma\,(TM). \tag{2.8}$$

Suppose a screen $S(TM)$ changes to another screen $S(TM)'$, where

$$\{\xi_1,\, ...,\, \xi_n,\;\, N'_1,\, ...,\, N'_n,\;\, W'_{n+1},\, ...,\, W'_m\}$$

is another quasi-orthonormal field of frames for the same set of null sections $\{\xi_1, \dots, \xi_n\}$. Following are the transformation equations due to this change (see details in [9, pages 164-165]).

$$W'_a = \sum_{b=1}^{m-n} A^b_a \left(W_b - \epsilon_b \sum_{i=1}^{n} \mathbf{f}_{ib}\, \xi_i \right), \tag{2.9}$$

$$N'_i = N_i + \sum_{j=1}^{n} N_{ij} \xi_j + \sum_{a=1}^{m-n} \mathbf{f}_{ia} W_a, \tag{2.10}$$

with the conditions

$$2N_{ii} = -\sum_{a=1}^{m-n} \epsilon_a (\mathbf{f}_{ia})^2, \quad N_{ij} + N_{ji} + \sum_{a=1}^{m-n} \epsilon_a \mathbf{f}_{ia} \mathbf{f}_{ja} = 0, \;\forall i \neq j. \tag{2.11}$$

$$h'_i(X,Y) = h_i(X,Y), \quad \forall X\,,\, Y \in \Gamma\,(TM), \tag{2.12}$$

$$\begin{aligned} \nabla'_X PY &= \nabla_X PY - \sum_{j=1}^{n}\left(\sum_{i=1}^{n} h_i(X,PY)N_{ij}\right)\xi_j \\ & - \sum_{a=1}^{m-n}\left(\sum_{i=1}^{n} h_i(X,PY)\mathbf{f}_{ia}\right)W_a. \end{aligned} \tag{2.13}$$

Lemma 2.1. *The second fundamental forms h^*_i and h'^*_i of the screen distributions $S(TM)$ and $S(TM)'$ respectively are related as follows:*

$$\begin{aligned} h'^*_i(X,\, PY) &= h^*_i(X,\, PY) + \frac{1}{2}||\mathbf{Z}_i||^2 h_i(X,PY) + g(\nabla_X PY,\, \mathbf{Z}_i) \\ & - \sum_{j\neq i}\left\{g(\mathbf{Z}_j,\mathbf{Z}_j) - N_{ij}\right\} h_j(X,PY) \end{aligned} \tag{2.14}$$

for a fixed i and j summed from 1 to n and each $\mathbf{Z}_i = \sum_{a=1}^{m-n} \mathbf{f}_{ia}\, W_a$ are n characteristic vector fields of the screen change.

Proof. Using (2.13) and then (2.10) we obtain

$$\begin{aligned} \bar{g}(\nabla'_X PY, N'_i) &= \bar{g}(\nabla_X PY, N_i) + \bar{g}(\nabla_X PY, \sum_{a=1}^{m-n} \mathbf{f}_{ia} W_a) \\ & - \sum_{j=1}^{n}\left(\sum_{i=1}^{n} h_i(X,PY)N_{ij}\right)\bar{g}(\xi_j, N_i) \\ & - g\left(\sum_{a=1}^{m-n}\left(\sum_{i=1}^{n} h_i(X,PY)\mathbf{f}_{ia}\right)W_a, \sum_{a=1}^{m-n}\mathbf{f}_{ia}W_a\right). \end{aligned}$$

Hence, we get

$$\begin{aligned} \bar{g}(\nabla'_X PY, N'_i) &= \bar{g}(\nabla_X PY, N_i) + \bar{g}(\nabla_X PY, \mathbf{Z}_i) \\ & - h_i(X,PY)(N_{ii} + \sum_{a=1}^{m-n}\mathbf{f}^2_{ia}) \\ & - \sum_{i\neq j}^{n} h_j(X,Y)[N_{ji} + \sum_{a=1}^{m-n}\mathbf{f}_{ia}\mathbf{f}_{ja}]. \end{aligned}$$

Thus, from (2.11) we get

$$\begin{aligned} \bar{g}(\nabla'_X PY, N'_i) &= \bar{g}(\nabla_X PY, N_i) + \bar{g}(\nabla_X PY, \mathbf{Z}_i) \\ & - h_i(X,PY)(-\frac{1}{2} \parallel \mathbf{Z}i \parallel^2 + \sum_{a=1}^{m-n}\mathbf{f}^2_{ia}) \\ & - \sum_{i\neq j}^{n} h_j(X,Y)N_{ij}. \end{aligned}$$

Finally, using (2.5) we get (2.14).

3. Existence theorems for canonical screen

We show that there are two classes (labeled as **class A** and **class B**) of coisotropic submanifolds which admit integrable canonical screen distributions. Let ω_i be the respective n dual 1-forms of the characteristic vector fields $\mathbf{Z}_i$ given by

$$\omega_i(X) = g(X, \mathbf{Z}_i), \quad \forall\, X \in \Gamma(TM), \tag{3.1}$$

where $1 \leq i \leq n$. Denote by S the first derivative of a screen distribution $S(TM)$ given by

$$S(x) = span\{[X, Y]_x, \quad X_x, Y_x \in S(TM), \quad x \in M\}, \tag{3.2}$$

where $[,]$ is the Lie-bracket. $S(TM)$ integrable implies that $S \subseteq S(TM)$.

Class A[1]. Consider a complementary vector bundle F of $Rad\,TM$ in $S(TM)^{\perp}$ and choose a basis $\{V_i\}$, $i \in \{1, \ldots, n\}$ of $\Gamma(F_{|\,U})$. Thus the sections we are looking for are expressed as follows

$$N_i = \sum_{k=1}^{n} \{A_{i\,k}\,\xi_k \,+\, B_{i\,k}\,V_k\}\ , \tag{3.3}$$

where $A_{i\,k}$ and $B_{i\,k}$ are smooth functions on U. Then $\{N_i\}$ satisfy (2.2) if and only if $\sum_{k=1}^{n} B_{i\,k}\,\bar{g}_{j\,k} = \delta_{i\,j}$, where $\bar{g}_{j\,k} = \bar{g}(\xi_j,\, V_k\,)$, $j,\, k \in \{1, \ldots, n\}$. Observe that $G = det\,[\bar{g}_{j\,k}]$ is everywhere non-zero on U, otherwise $S(TM)^{\perp}$ would be degenerate at least at a point of U. Assume that F is parallel along the tangent direction.

Theorem 3.1. *Let $(M, g, S(TM), F)$ be a coisotropic submanifold of a semi-Riemannian manifold $\bar{M}$ such that the complementary vector bundle F of $Rad\,TM$ in $S(TM)^{\perp}$ is parallel along the tangent direction. Then,*

(a) *any choice of a screen distribution is integrable and*
(b) *all the n-forms ω_i in (3.1) vanish identically on the first derivative S given by (3.2).*
(c) *If S coincides with $S(TM)$, then, there exist n null sections $\{\xi_1, \ldots, \xi_n\}$ of $\Gamma(Rad\,TM)$ with respect to which $S(TM)$ is a canonical screen distribution, up to an orthogonal transformation with a canonical set $\{N_1, \ldots, N_n\}$ of null transversal vector bundles and the screen fundamental forms h^*_i are independent of a screen distribution.*

Proof. Taking covariant derivative of N_i (given by (3.3)) with respect to $X \in \Gamma(TM)$, we get

$$\bar{\nabla}_X N_i = \sum_{k=1}^{n} \{X(A_{ik})\xi_k + A_{ik}\bar{\nabla}_X \xi_k + X(B_{ik})V_k + B_{ik}\bar{\nabla}_X V_k\}.$$

Using (2.3), (2.4) and (2.6), we obtain

$$\begin{aligned} A_{N_i} X \;\; &= \;\; \sum_{k=1}^{n} \left\{ (A_{ik} A^*_{\xi_k} X - A_{ik})\xi_k + \sum_{j=1}^{n} \tau_{kj}(X)\xi_j + X(B_{ik})V_k + B_{ik}\bar{\nabla}_X V_k \right\} \\ &\quad + \sum_{j=1}^{n} \tau_{ij}(X) N_j. \end{aligned}$$

[1] suggested by Bayram Sahin, Inonu University, Turkey

Since F is parallel, $\bar{\nabla}_X V_k \in \Gamma(F)$. Thus for $Y \in \Gamma(S(TM))$, we get

$$g(A_{N_i}X, Y) = \sum_{k=1}^{n} A_{ik} g(A^*_{\xi_k}X, Y).$$

Then using (2.8) we get

$$\bar{g}(h^*_i(X,Y), N_i) = \sum_{k=1}^{r} A_{ik}\bar{g}(h_i(X,Y), \xi_k). \tag{3.4}$$

Since the right side of (3.4) is symmetric, it follows that each h^*_i is symmetric on $S(TM)$. This [9, theorem 2.5, page 161] implies that any choice of $S(TM)$ is integrable. Thus, (a) holds. Choose an integrable screen $S(TM)$. Thus, S is a subbundle of $S(TM)$. Now using (3.4) in (2.14) and $h_i = h'_i$, we obtain

$$\begin{aligned} g(\nabla_X PY, \mathbf{Z}_i) &= \sum_{i\neq j} h_j(X, PY)[N_{ji} + g(\mathbf{Z}_i, \mathbf{Z}_j)] \\ &\quad + \frac{1}{2} h_i(X, PY) \parallel \mathbf{Z}_i \parallel^2, \end{aligned} \tag{3.5}$$

$\forall\, X, Y \in \Gamma(TM_{|_U})$ and for each fixed i. Since the right hand side of (3.5) is symmetric in X and Y, we have $g([X,Y], \mathbf{Z}_i) = \omega_i([X,Y]) = 0,\ \forall X, Y \in \Gamma(S(TM)_{|_U})$, that is, ω_i vanishes on S. Similarly, repeating n-times above steps for each i we claim that each ω_i vanishes on S which proves (b). If we take $S = S(TM)$, then, each ω_i vanish on this choice of $S(TM)$ which implies that all the n characteristic vector fields $\mathbf{Z}_i$ vanish. Therefore, all the functions $\mathbf{f}_{ia}$ vanish. Finally, substituting this data in (2.10) and (2.11) it is easy to see that all the functions N_{ij} also vanish. Thus, the Eqs. (2.9), (2.10) and (2.14) become $W'_a = \sum_{b=1}^{m-n} A^b_a W_b$ $(1 \le a \le m-n)$, $N'_i = N_i$ and $h'^*_i = h^*_i$ where (A^b_a) is an orthogonal matrix of $S(TM)$ at any point $x \in M$. Therefore, $S(TM)$ is a canonical screen up to an orthogonal transformation with canonical transversal vector fields N_i and the screen fundamental forms h^*_i are independent of a screen distribution. This completes the proof.

Class B[2]. It is known that the second fundamental forms and their respective shape operators of a non-degenerate submanifold are related by means of the metric tensor. Contrary to this we see from Eqs. (2.7) and (2.8) that there are interrelations between the second fundamental forms of the lightlike M and its screen distribution and their respective shape operators. This interrelation indicates that the lightlike geometry depends on a choice of screen distribution. While we know from Eq. (2.12) that the second fundamental forms of the lightlike M are independent of a screen, the same is not true for the fundamental forms of $S(TM)$ (see Eq. (2.14), which is the root of non-uniqueness anomaly in the lightlike geometry. Since, in general, it is impossible to remove this anomaly, we consider a family of coisotropic submanifolds M such that the fundamental forms of the screen distribution $S(TM)$ are related with the fundamental forms of M as follows:

$$h^*_i(X, PY) = \varphi_i h_i(X,Y), \quad \forall X, Y, \Gamma(TM|_U), \quad i \in \{1, \ldots, n\}, \tag{3.6}$$

where each φ_i is a conformal smooth function on $\mathcal{U}$ in M. To avoid trivial ambiguities, we will consider $\mathcal{U}$ to be connected and maximal in the sense that there is no larger domain $\mathcal{U}' \supset \mathcal{U}$ on which Eq. (3.6) holds.

[2]A particular case ($n = 2$) of this class appeared in [8]

The motivation for the geometric condition (3.6) comes from the classical geometry of non-degenerate submanifolds for which there are only one type of fundamental forms with their one type of respective shape operators. Using above we prove the following existence theorem:

Theorem 3.2. *Let $(M^m, g, S(TM))$ be a coisotropic submanifold of a semi-Riemannian manifold $\bar{M}^{m+n}$ such that (3.6) holds. Then all the assertions from (a) through (c) of the theorem 3.1 will hold.*

Proof. Substituting (3.6) in (2.8) and then using (2.7) we get

$$g(A_{N_i}X, PY) = \varphi_i g(A_{\xi_i}X, PY) \quad \forall X \in \Gamma(TM_{|U}). \tag{3.7}$$

Since each A_{ξ_i} is symmetric with respect to g, Eq. (3.7) implies that each A_{N_i} is self-adjoint on $\Gamma(ST(M))$ with respect to g, which further follows from [9, theorem 2.5, page 161] that any choice of a screen distribution is integrable. Using (3.6) in (2.14) and $h'_i = h_i$ we obtain

$$\begin{aligned} g(\nabla_X PY, \mathbf{Z}_i) &= \frac{1}{2}||\mathbf{Z}_i||^2 h_i(X, PY) \\ &+ \sum_{j\neq i}\{g(\mathbf{Z}_j, \mathbf{Z}_j) - N_{ji}\}\, h_j(X, PY) \end{aligned} \tag{3.8}$$

Then the rest of the proof is similar to the proof of Theorem 3.1.

Example. Let M be a co-isotropic submanifold of $\mathbf{R}^5_2$, given by

$$x_2 = (x_3^2 + x_5^2)^{\frac{1}{2}},\ x_4 = x_1,\ x_3 > 0,\ x_5 > 0.$$

$$\begin{aligned} S(TM) &= span\{W = x_5\,\partial x_2 + x_2\,\partial x_5\} \\ Rad(TM) &= span\{\xi_1 = \partial x_1 + \partial x_4,\ \xi_2 = x_2\,\partial x_2 + x_3\,\partial x_3 + x_5\,\partial x_5\} \end{aligned}$$

$$tr(TM) = span\{N_1 = \frac{1}{2}(-\partial x_1 + \partial x_4),\ N_2 = \frac{1}{2x_3^2}\{-x_2\,\partial x_2 + x_3\,\partial x_3 - x_5\,\partial x_5\}\}.$$

Then, by direct calculations, we get

$$\bar{\nabla}_{\xi_1}W = 0, \bar{\nabla}_{\xi_2}W = W,\ \bar{\nabla}_{\xi_1}\xi_2 = 0, \quad \bar{\nabla}_W W = x_2\,\partial x_2 + x_5\,\partial x_5.$$

Using Gauss formula, we obtain

$$\begin{aligned} \nabla_W W &= \frac{1}{2}\xi_2,\ h_1^*(W,W) = 0, \quad h_1^*(\xi_1, W) = h_2^*(\xi_1, W) = 0, \\ h_1^*(\xi_2, W) &= h_2^*(\xi_2, W) = h_2(\xi, W) = h_2(\xi_2, W) = 0, \quad h_1 = 0. \end{aligned}$$

and

$$h_2(W,W) = -(x_3^2),\ h_2^*(W,W) = \frac{1}{2}.$$

Thus, M belongs to class B co-isotropic submanifolds with conformal functions φ_1 (arbitrary) and $\varphi_2 = -\frac{1}{2x_3^2}$. Therefore, theorem 3.2 holds.

Remark. It follows from above two theorems that the geometric condition of parallel F or Eq. (3.6) provide an integrable screen distribution, which is the root requirement for a coisotropic submanifold to admit a canonical screen. We also know that, in general, the induced Ricci tensor of any lightlike submanifold is not symmetric. Since a symmetric induced Ricci tensor is also a desirable property, fortunately, it is known [13] that

"A totally umbilical coisotropic submanifold M of a semi- Riemannian manifold $(\bar{M}(c), \bar{g})$ of a constant curvature c admits an integrable screen distribution if and only if the Ricci tensor on M is symmetric".

Consequently, a large class of totally umbilical coisotropic lightlike submanifolds of $(\bar{M}(c), \bar{g})$ are candidates for the existence of a canonical screen distribution and an induced symmetric Ricci tensor.

Based on above, the results of this paper can be used to introduce the concept of an induced scalar curvature for coisotropic submanifolds. For a similar study on scalar curvature for lightlike hypersurfaces of Lorentzian manifolds, see [6].

Acknowledgment. This research was supported by the Natural Sciences and Engineering Research Council (NSERC) of Canada.

References

[1] Akivis, M. A. and Goldberg, V. V., On some methods of construction of invariant normalizations of lightlike hypersurfaces, Differential Geometry and its Applications, 12(2000), no. 2, 121-143.

[2] Bejancu, A., A canonical screen distribution on a degenerate hypersurface, Scientific Bulletin, Series A, Applied Math. and Physics, 55(1993), 55-61.

[3] Bejancu, A. and Duggal, K. L., Degenerate hypersurfaces of semi-Riemannian manifolds, Bull. Inst. Politehnie Iasi, (S.1), 37(1991), 13-22.

[4] Bejancu, A., Ferrández, A. and Lucas P., A new viewpoint on geometry of a lightlike hypersurface in a semi-Euclidean space, Saitama Math. J., (1998), 31-38.

[5] Bonnor, W. B., Null hypersurfaces in Minkowski spacetime, Tensor (N.S.), 24(1972), 329-245.

[6] Duggal, K. L., On scalar curvature in lightlike geometry, J. Geom. Phys., 57(2007), 473-481.

[7] Duggal, K. L., A Report on Canonical Null Curves and Screen Distributions for Lightlike Geometry, Acta Appli Math., 95(2007), 135-149.

[8] Duggal, K. L., On canonical screen for lightlike submanifolds of codimension two. Central European Journal of Mathematics, CEJM, 5(4)(2007), 710-719.

[9] Duggal, K. L., Bejancu, A., Lightlike submanifolds of semi-Riemannian manifolds and applications, Kluwer, Dordrecht, 364, 1996.

[10] IIyenko, K., Twistor representation of null 2-surfaces, J. Math. Phys., 10(2002), 4770-4789.

[11] Israel, W., Event horizons in static vacuum spacetimes, Phys. Rev., 164(1967), 1776-1779.

[12] Israel, W., Event horizons in static electrovac spacetimes, Comm. Math. Phys., 8(1968), 245-260.

[13] Jin, D. H., Geometry of coisotropic submanifolds, , J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math.,8(2001), no. 1, 33-46.

[14] Katsuno, K., Null hypersurfaces in Lorentzian manifolds, Math. Proc. Cab. Phil. Soc., 88(1980), 175-182.

[15] Leistner, T., Screen bundles of Lorentzian manifolds and some generalizations of pp-waves, J. Geom. Phys., 56(2006), no. 10, 2117-2134.

[16] Nurowski, P. and Robinsom, D., Intrinsic geometry of a null hypersurface, Class. Quantum Grav., 17(2000), 4065-4084.

[17] O'Neill, B., Semi-Riemannian geometry with applications to relativity, Academic Press, New York, 1983.

[18] Penrose, R., The twistor geometry of light rays. Geometry and physics, Classical Quantum Gravity, 14(1A), 1997, A299-A323.

[19] Perlick, V., On totally umbilical submanifolds of semi-Riemannian manifolds, Nonlinear Analysis, 63(2005), 511-518.

[20] Rosca, R., On null hypersurfaces of a Lorentzian manifold, Tensor (N.S.), 23(1972), 66-74.

University of Windsor, Department of Mathematics and Statistics, Windsor, Ontario N9B3P4, Canada

E-mail address: yq8@uwindsor.ca