ON EXISTENCE OF EXTREMAL SOLUTIONS OF DIFFERENTIAL EQUATIONS IN BANACH SPACES

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by

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Let X be a real Banach space, $K \subset X$ a cone, $I = [0,a] \subset \mathbb{R}$ and $f: I \times K \to X$ continuous. We look for conditions on X, K and f such that the IVP

$$(1) \qquad u' = f(t,u), \quad u(0) = u_0 \in K,$$

has a maximal solution $\bar{u}$ and a minimal solution $\underline{u}$ with respect to the partial ordering induced by K. Contrary to known results, [5,6], we shall not assume that K has interior points, since the standard cones of many infinite dimensional spaces have empty interior. The second essential new feature is that f is supposed to be defined only on K and this demands that the extra conditions on f are required only with respect to points in K, and not on the whole space.

I. DEFINITIONS AND NOTATIONS

Let X^* denote the normed dual of X. The interior of a set $D \subset X$ is denoted by D^0. A closed convex subset $K \neq \{0\}$ of X is said to be a <u>cone</u> if $\lambda K \subset K$ for every $\lambda \geq 0$ and $K \cap (-K) = \{0\}$. For a cone K, we let $K^* = \{x^* \in X^*: \ x^*(x) \geq 0$ for all $x \in K\}$. We define a partial order "$\leq$" by "$x \leq y$ iff $y - x \in K$".

Let K be a cone, $D \subset X$, $f: I \times D \to X$. Then f is said to be quasimonotone (w.r. to K) if the following condition (2) is satisfied (see [1,5,6]),

(2) $t \in I$; $x, y \in D$ and $x \leq y$; $x^* \in K^*$ and

$$x^*(x-y) = 0 \Rightarrow x^*(f(t,x) - f(t,y)) \leq 0.$$

An essential tool to prove existence of solutions to problem (1) in a subset D of X is the boundary condition

(3) $$t \in I, \quad x \in \partial D \Rightarrow \liminf_{\lambda \to 0^+} \operatorname{dist}(x + \lambda f(t,x), D) = 0.$$

See [1,5]. In case D is a cone K, condition (3) is equivalent to

(4) $$t \in I, \quad x \in \partial K, \quad x^* \in K^* \text{ and } x^*(x) = 0 \Rightarrow x^*(f(t,x)) \geq 0$$

(see Example 4.1 in [1]).

Let us remark that $f: I \times K \to X$ satisfies (4) if f is quasimonotone and $f(t,0) \geq 0$ in I. In fact, $x \in \partial K$ and $x^*(x) = 0$, for some $x^* \in K^*$, imply $x^*(f(t,x)) = x^*(f(t,x) - f(t,0)) + x^*(f(t,0)) \geq 0$. A solution $\overline{u}$ (resp. $\underline{u}$) of (1) is said to be a maximal (resp. minimal) solution of (1) if $\overline{u}(t) \geq u(t)$ (resp. $\underline{u}(t) \leq u(t)$) for every solution u of (1) and all $t \in I$ such that both functions are defined at t. Since $K \cap (-K) = \{0\}$, there is at most one maximal (resp. minimal) solution on every fixed interval $[0,a] \subset I$.

II. THE FINITE DIMENSIONAL CASE

In this section we consider $X = \mathbb{R}^n$. A simple result is

Proposition 1. Let $K \subset \mathbb{R}^n$ be a cone with $K^0 \neq \emptyset$, $f\colon I \times K \to \mathbb{R}^n$ continuous, quasimonotone and such that $f(t,0) \geq 0$. Then

(i) Problem (1) has a (local) maximal solution.

(ii) Problem (1) has a minimal solution if $f(t,0) \in K^0$ in I.

Proof: (i) Consider the IVP $v' = f(t,v) + \frac{1}{p}e$, $v(0) = u_0 + \frac{1}{p}e$ for some fixed $e \in K^0$ and every $p \geq 1$. Since f satisfies the boundary condition (4), $f + \frac{1}{p}e$ satisfies this condition too. Therefore, there exists a solution v_p on some $[0,b] \subset I$, with $b > 0$ independent of p (see Theorem 4.1 in [1]). The sequence (v_p) has a uniformly convergent subsequence, the limit $\bar{v}$ of which is a solution of (1) on $[0,b]$. If u is any solution of (1) then $u(t) < v_p(t)$ as long as both exist (see Lemma 5.1 in [1]), and therefore $u(t) \leq \bar{v}(t)$ for such t, i.e. $\bar{v}$ is the maximal solution.

(ii) If $f(t,0) \in K^0$ in I, then we consider $v' = f(t,v) - \frac{1}{p}f(t,0)$, $v(0) = u_0$ for $p \geq 2$. Since the right hand side is quasimonotone and $f(t,0) - \frac{1}{p}f(t,0) \in K^0$ in I, Condition (4) is satisfied. If u is a solution of (1) then $u(t) > v_p(t)$ in $(0,b]$ for some $b > 0$ and again (v_p) has a uniformly convergent subsequence the limit of which is the minimal solution $\underline{v}$.

q.e.d.

In general, we do not know how to prove Proposition 1 (ii) without the condition "$f(t,0) \in K^0$ in I". However, we have

Proposition 2. Let $K \subset \mathbb{R}^n$ be the standard cone, i.e. $K = \{x: x_i \geq 0$ for $i = 1,\dots,n\}$, $f: I \times K \to \mathbb{R}^n$ quasimonotone, continuous and such that $f(t,0) \geq 0$ in I. Then (1) has a maximal and a minimal solution.

Proof: The existence of $\bar{v}$ being clear from Proposition 1, we have only to prove that $\underline{v}$ exists. Let $P: \mathbb{R}^n \to K$ be the metric projection, characterized by $|x-Px| = \mathrm{dist}(x,K)$. It is easy to see that $Px = (\max(x_1,0),\dots,\max(x_n,0))$. The function $\tilde{f}(t,x) = f(t,Px)$ defines a continuous extension of f to $I \times \mathbb{R}^n$, and $\tilde{f}$ is quasimonotone there. In fact, $x \in \mathbb{R}^n$ and $z \in K$ and $v \in K = K^*$ such that $(v,z) = 0$ imply $(v, P(x+z)-Px) = 0$, and therefore $(v,\tilde{f}(t,x) - \tilde{f}(t,x+z)) \leq 0$. Now, we fix $e \in K^0$, consider $K_\delta = \{x \in \mathbb{R}^n : \mathrm{dist}(x,K) \leq \delta\}$ for some $\delta > 0$ and the function $f_\delta(t,x) = \tilde{f}(t,x) - \eta x - \eta^2 e$, with $\eta = \delta|e|^{-1}$. We claim that f_δ satisfies the boundary condition (3) for $D = K_\delta$. To see this we notice first that this condition is equivalent to

$$(5)\quad t \in I,\ \ x \in \partial K_\delta,\ \ |x^*| = 1 \ \text{ and } \ x^*(x) = \inf_{K_\delta} x^*(y) \Rightarrow x^*(f_\delta(t,x)) \geq 0;$$

(see Lemma 4.1 in [1] with sup replaced by inf.).

Now, since the ball $B_\delta(0)$ is contained in K_δ, we have $x^*(x) \leq -\delta$. We also know, that $Px \in \partial K$ and $x - Px + y \in K_\delta$ for every $y \in K$, since $|x-Px| = \delta$. Therefore, $x^*(x) \leq x^*(x-Px+y)$ for all $y \in K$, and this implies $0 = x^*(Px) = \inf_K x^*(y)$. Hence $x^* \in K^*$, $Px \in \partial K$ and $x^*(Px) = 0$ and therefore $x^*(f(t,Px)) \geq 0$, since f satisfies the boundary condition (4). Thus, we have $x^*(f_\delta(t,x)) \geq -\eta x^*(x) - \eta^2|e| \geq \eta(\delta-\eta|e|) = 0$, that is, f_δ satisfies (5). Therefore, $v' = f_\delta(t,v)$, $v(0) = u_0 - \frac{\delta}{|e|}e$ has a solution v_δ on some $[0,b] \subset I$, where $b > 0$ depends only on a bound for f in a neighborhood of the point $(0,u_0)$, and the range of v_δ is in K_δ. Now, let u be any solution of (1) on $[0,\beta]$ with $\beta \in (0,b]$, and let $w(t) = u(t) - v_\delta(t)$. Then $w(0) \in K^0$ and

$$(6) \qquad w'(t) = \tilde{f}(t,u(t)) - \tilde{f}(t,v_\delta(t)) + \frac{\delta}{|e|}v_\delta(t) + \frac{\delta^2}{|e|^2}e.$$

Suppose that there exists a first time $t_0 > 0$ such that $w(t_0) \in \partial K$. Then we have $x^*(w(t_0)) = 0$ for some $x^* \in K^*$ with $|x^*| = 1$, and $\phi(t) = x^*(w(t))$ satisfies $\phi'(t_0) \leq 0$ since $w(t) \in K^0$ for $t < t_0$. However (6) and the quasimonotonicity of $\tilde{f}$ imply that $\phi'(t_0) \geq \frac{\delta}{|e|} x^*(u(t\ )) + \frac{\delta^2}{|e|^2}x^*(e) > 0$, a contradiction. Therefore, $u(t) - v_\delta(t) \in K^0$ in $[0,\beta]$. Now, (v_{δ_n}) with $\delta_n \to 0$ has a uniformly convergent subsequence, the limit $\underline{v}$ of which is a solution of (1), and evidently the minimal solution.

q.e.d.

The essential point in this proof has been to construct a quasimonotone extension $\tilde{f}$ of f such that $v' = \tilde{f}(t,v) - \varepsilon e$, $v(0) = u_0 - \varepsilon e$ has a solution lying near the cone and smaller than every solution of (1). The use of the metric projection $P: \mathbb{R}^n \to K$ for the definition of a quasimonotone extension is restricted to cones K with the following

Property (θ_n). There exist m vectors $e_i \in K\setminus\{0\}$, for some $1 \le m \le n$, such that $(e_i,e_j) = \delta_{ij}$ for $i,j = 1,\dots,m$ and $K \subset \operatorname{span}\{e\ ,\dots,e_m\}$.

For an arbitrary cone $K \subset \mathbb{R}^n$, it is easy to see that $(x-Px,Px) = 0$ for all $x \in \mathbb{R}^n$. Hence, if K does not enjoy Property (θ_n), it is easy to find $(x,z,v) \in \mathbb{R}^n \times K \times K^*$ such that $(v,z) = 0$ but $(v,P(x+z)-Px) \neq 0$; consider for example $K = \{x \in \mathbb{R}^2\colon\ x_2 \ge \frac{1}{2}|x_1|\}$, where $K^* = \{x \in \mathbb{R}^2\colon\ x_2 \ge 2|x_1|\}$. Therefore, we can not show that $f(t,Px)$ is quasimonotone on $I \times \mathbb{R}^n$. Suppose now, that K has (θ_n), f is continuous, quasimonotone and $f(t,0) \in K$ on I. Then we know that the IVP $u' = f(t,u)$, $u(s) = u_0 \in K$ has a local solution. Therefore, $f(s,u_0) = u'(s) \in \operatorname{span}\{e_1,\dots,e_m\}$, i.e. we have already $f\colon\ I \times K \to \operatorname{span}\{e_1,\dots,e_m\}$. In this space, K has nonempty interior and $f(t,Px)$ defines a quasimonotone extension as before. Thus, we have

Corollary 1. Proposition 2 is true for every cone $K \subset \mathbb{R}^n$ with Property (θ_n).

The interesting question, whether Proposition 2 is true for every cone K, remains open. Let us remark that following the results in [4], it is not difficult to consider the more general notion of mini-max solutions rather than maximal and minimal solutions as we have chosen to discuss.

III. EXISTENCE OF MINIMAL SOLUTIONS IN CASE dim X = ∞

Consider again problem (1), where $f\colon\ I \times K \to X$ is continuous, quasimonotone and $f(t,0) \in K$ on I. Since we assume now that $\dim X = \infty$ we need extra conditions on f to guarantee local existence of solutions; see e.g. [1], [3], [5]. Some of these conditions are estimates involving measures of noncompactness, for instance the ball measure $\beta(B)$ for bounded $B \subset X$ which is defined by

$$\beta(B) = \inf\{r>0\colon\ B \text{ can be covered by finitely many balls of radius } \leq r\}.$$

Let us mention that $\beta(B) = 0$ if B is relatively compact; further properties can be found e.g. in [1], [5]. Since there are examples where the method applied in the proof to Proposition 2 works, let us state this result as

<u>*Theorem 1.*</u> Let X be a real Banach space; $K \subset X$ a cone with $K^{\circ} \neq \emptyset$; $I = [0,a] \subset \mathbb{R}$; $f\colon\ I \times K \to X$ continuous, quasimonotone and such that $f(t,0) \in K$ on I. Suppose also that the following hypotheses hold

(i) The metric projection $P\colon\ X \to K$ exists and satisfies $|Px-Py| \leq |x-y|$, $Px \leq P(x+z)$ if $z \in K$ and $x^*(P(x+z)-Px) = 0$ whenever $x^* \in K^*$, $z \in K$ and $x^*(z) = 0$.

(ii) f maps bounded sets into bounded sets and $\beta(f(t,B)) \leq \omega(t,\beta(B))$ for $t \in I$ and $B \subset K$ bounded, where $\omega\colon\ I \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, monotone increasing in the second argument and such that the *IVP* $\rho' = \omega(t,\rho)$, $(0) = 0$ admits the trivial solution only.

(iii) If X is not separable then f is also uniformly continuous on bounded sets.

Then (1) has a minimal solution.

This theorem can be proved like Proposition 2. Since P is nonexpansive, we have $\beta(PB) \geq \beta(B)$ and since w is increasing in ρ, the quasimonotone extension $\tilde{f}$, defined by $\tilde{f}(t,x) = f(t,Px)$ satisfies conditions sufficient for existence of local solutions; see e.g. [1],[3], [5].

Simple examples, where condition (i) is satisfied and $K^o \neq \emptyset$, are $X = C(\Omega)$ with $\Omega \subset \mathbb{R}^m$ compact, $X = c$, the space of all convergent sequences with the sup-norm, $X = L^\infty(\Omega)$ with $\Omega \subset \mathbb{R}^m$ measurable, $X = l^\infty$ and the corresponding standard cones of nonnegative functions; here Px is given by $(Px)(t) = max\{x(t),0\}$ and $(Px)_n = max\{x_n,0\}$ respectively.

Now, let X be a real Banach space such that there exists a <u>projectional scheme</u> $\{X_n,P_n\}$, i.e. a sequence of finite dimensional subspaces X_n of X and a sequence of continuous linear projections $P_n : X \to X_n$ such that $P_n x \to x$ as $n \to \infty$, for every $x \in X$. Let $K \subset X$ be a cone such that $P_n K \subset K$ for every $n \geq 1$.

Example. Suppose X has a (Schauder-) base $\{e_i : i \geq 1\}$. Consider $K = \{\sum_{i\geq 1} x_i e_i : x_i \geq 0$ for every $i\}$, $X_n = span\{e_1,\ldots,e_n\}$ and $P_n(\sum_{i\geq 1} x_i e_i) = \sum_{i\leq n} x_i e_i$. Then $\{X_n,P_n\}$ is a projectional scheme and $P_n K \subset K$ for every $n \geq 1$, but $K^o = \emptyset$

Let us note that $P_n K \subset K$ implies $K_n := K \cap X_n = P_n K$. In connection with (1) we then consider the finite demensional problems

$$(1_n)\quad v' = f_n(t,v)\ ,\ v(0) = P_n u_0\ ,\quad \text{where}\quad f_n(t,\cdot) = P_n f(t,\cdot)\Big|_{K_n}.$$

Evidently, $f_n : I \times K_n \to X_n$ is continuous and $f_n(t,0) \in K_n$ on I. Furthermore, f_n is quasimonotone w.r. to K_n. In fact, $x \in K_n$, $z \in K_n$, $x^* \in K_n^*$ and $x^*(z) = 0$ imply $P_n^* x^* \in K^*$ and $P_n^* x^*(z) = x^*(z) = 0$, and therefore

$$x^*(f_n(t,x) - f_n(t,x+z)) = P_n^* x^*(f(t,x) - f(t,x+z)) \leq 0.$$

Finally, if u is any solution of (1) then $w_n = P_n u$ satisfies

(7) $\quad w_n' = f_n(t,w_n) + P_n(f(t,u(t)) - f(t,P_n u(t)))\ ,\ w_n\ (0) = P_n u\ .$

Now, if we look at the example, we see that K_n is the standard cone in $\mathbb{R}^n$. Therefore (1_n) has a minimal solution $\underline{v}_n$. We also have that the defects $y_n(t) = P_n(\ldots)$ in (7) are in K_n. In fact, let $x_0^* \in K_n^*$; then x^*, defined by $x^*(e_i) = x_0^*(e_i)$ for $i \leq n$ and $= 0$ for $i > n$, is in K^* and $x^*(x - P_n x) = 0$; we also have $P_n x \leq x$ and therefore

(8) $\quad 0 \leq x^*(f(t,x) - f(t,Px)) = x_0^*(P_n f(t,x) - P_n f(t,P_n x))$;

since $x_0^* \in K_n^*$ has been arbitrary, we have $P_n f(t,x) \geq P_n f(t,P_n x)$. Therefore, $w_n(t) \geq \underline{v}_n(t)$. In order to prove convergence of $(\underline{v}_n)$, we need the following proposition which is special case of Theorem 1 in [3].

Proposition 3. Let X be a separable Banach space and (x_n) a sequence of continuously differentiable functions $x_n : I \to X$ such that $|x'_n(t)| \leq c$ in I for each n. Then $\psi(t) = \beta(\{X_n(t) : n \geq 1\})$ is absolutely continuous and $\psi'(t) \leq \beta(\{x_n'(t) : n \geq 1\})$ *a.e.* in I.

Theorem 2. Let X be a real Banach space with a projectional scheme $\{X_n, P_n\}$; $K \subset X$ a cone, $I = [0,a] \subset \mathbb{R}$; $f : I \times K \to X$ continuous, quasimonotone and such that $f(t,0) \in K$ on I. Suppose also that the following hypotheses are fulfilled:

(H_1) For each $n \geq 1$, $P_n K \subset K$, $K_n = P_n K$ has nonempty interior in X_n and property $(\theta_{m(n)})$, considered as a cone in $\mathbb{R}^{m(n)}$, where $m(n) = \dim X_n$.

(H_2) $P_n f(t,x) - P_n f(t,P_n x) \in K_n$, whenever $t \in I$, $x \in K$ and $n \geq 1$.

(H_3) f maps bounded sets into bounded sets and $\beta(f(t,B)) \leq w(t, \beta(B))$ for $t \in I$ and $B \subset K$ bounded, where $w: I \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and such that the IVP $\rho' = \lambda w(t,\rho)$, $\rho(0) = 0$ admits only the trivial solution $\rho(t) \equiv 0$; here $\lambda = \sup_n |P_n|$. Then problem (1) has a minimal solution.

Proof: (i) From the previous considerations and (H_1), we know that problem (1_n) has a minimal solution $\underline{v}_n$ on some interval $I_0 = [0,b] \subset I$, where $b > 0$ depends only on a bound for $|f(t,x)|$ in a neighborhood of $(0,u_0)$, provided that n is sufficiently large, say $n \geq n_0$, since $P_n u_0 \to u_0$ and $\lambda = \sup_n |P_n| < \infty$. If u is any solution of (1) on I, then $w_n = P_n u$ satisfies (7), and by (H_2) we have $P_n u(t) \geq \underline{v}_n(t)$ on I_0.

(ii) Let $\psi(t) = \beta(\{\underline{v}_n(t) : n \geq n_0\})$. Since $\underline{v}_n(0) = P_n u_0 \to u_0$, we have $\psi(0) = 0$. Since $(\underline{v}_n'(t))$ is uniformly bounded on I_0, and $y_n(t) \to 0$ as $n \to \infty$, Proposition 3 implies $\psi'(t) \leq \beta(P_n f(t,\underline{v}_n(t))\ ;\ n \geq n_0)$ a.e. on I_0. By Proposition 7.2 in [1] and (H_3), we therefore have

$$\psi'(t) \leq \lambda\beta(f(t,\{\underline{v}_n(t);\ n \geq n_0\})) \leq \lambda w(t,\psi(t)) \quad \text{a.e. in } I_0, \quad \psi(0) = 0.$$

Since $\lambda\omega$ is a uniqueness function, this implies $\psi(t) \equiv 0$ in I_0, and since $(\underline{v}_n)$ is equicontinuous, we find a uniformly convergent subsequence, by the Lemma of Ascoli-Arzela, the limit $\underline{v}$ of which is a solution of (1), and evidently the minimal solution of (1).

q.e.d.

It is obvious that the whole sequence $(\underline{v}_n)$ converges to $\underline{v}$ since there is only one minimal solution.

IV. EXISTENCE OF MAXIMAL SOLUTIONS IN CASE dim $X = \infty$

From the proof to Proposition 1 (i) it is obvious that (1) has a maximal solution if $K^o \neq \emptyset$ and one of the conditions sufficient for local existence of solutions, mentioned in the preceding section, is satisfied. However, if $K^o = \emptyset$ then the problem is difficult.

Suppose again that the hypotheses of Theorem 2 except (H_2) are fulfilled, let $I_0 = [0,b]$ be such that every solution of (1) exists on I_0, and let S be the set of all solutions. Then S is compact in $C_K(I_0)$. In fact, let $(u_n) \subset S$ and $\psi(t) = \beta(\{u_n(t) : n \geq 1\})$. Then $\psi(0) = 0$ and $\psi'(t) \leq \omega(t,\psi(t)) \leq \lambda\omega(t,\psi(t))$ a.e. in I_0, since $\lambda \geq 1$ and ω is nonnegative. Therefore $\psi(t) \equiv 0$. The compactness of S implies that the defects

$$y_n(t,u) = P_n(f(t,u(t)) - f(t,P_n u(t)))$$

in (7) tend to zero as $n \to \infty$, uniformly on I_0 and uniformly with respect to $u \in S$. In general, this does not seem to be enough to prove existence of a maximal solution by this approach. We should find $e_n \in K_n$ such that $e_n \geq y_n(t,u)$ for all $u \in S$ and $|e_n| \to 0$ as $n \to \infty$;

then we could consider the maximal solution $\overline{v}_n$ of $v' = f_n(t,v) + e_n$, $v(0) = P_n u_0$ and we would obtain $\overline{u}$ in this way. But in general, we can expect only $|e_n| \leq C\sqrt{n}\delta_n$ if $|y_n(t,u)| \leq \delta_n$; consider e.g. $X = \ell^2$.

However, in case $X = c_0$, the space of all sequences tending to zero (with the sup-norm), we have an upper bound e_n with $|e_n| = \delta_n$, namely e_n defined by $e_{n_i} = \delta_n$ for $i \leq n$. Let us state this curiosity as

Theorem 3. Let $X = c_0$, $K \subset X$ the standard cone of X. Let f be continuous, quasimonotone and such that $f(t,0) \in K$ on I. Suppose also that f satisfies (H_3) in Theorem 2. Then (1) has a maximal solution.

REFERENCES

[1] K. Deimling: "Ordinary differential equations in Banach spaces", *Lect. Notes Vol. 596*, Springer-Verlag, Berlin-Heidelberg-New York (1977).

[2] K. Deimling: "Open problems for ordinary differential equations in Banach spaces", *Comm. Convegno Equa-Diff. 78*, pp. 127-137, Centro 2P-Firenze (1978).

[3] H. Mönch, G. v. Harten: "On the Cauchy problem for ordinary diffirential equations in Banach spaces", (submitted).

[4] V. Lakshmikantham and S. Leela: "Differential and Integral Inequalities, Vol. I and II", Academic Press, New York (1969).

[5] V. Lakshmikantham and S. Leela: "An Introduction to Nonlinear Differential Equations in Abstract Spaces", (Manuscript to be published).

[6] V. Lakshmikantham, A. R. Mitchell, and R. W. Mitchell: "Maximal and minimal solutions and comparison results for differential equations in abstract cones", *Ann. Polon. Math. 34* (1977), pp. 317-324.