Hindawi
Journal of Function Spaces
Volume 2021, Article ID 5510387, 12 pages
https://doi.org/10.1155/2021/5510387

Research Article

On Existence of Sequences of Weak Solutions of Fractional Systems with Lipschitz Nonlinearity

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Received 10 February 2021; Revised 26 February 2021; Accepted 2 March 2021; Published 12 March 2021

Academic Editor: Tianqing An

In this article, the variational method together with two control parameters is used for introducing the proof for the existence of infinitely many solutions for a new class of perturbed nonlinear system having p-Laplacian fractional-order differentiation.

1. Introduction

One of the main applications of fractional calculus science is the fractional-order differential equations (FDEs). Various natural phenomena are modeled mathematically through the FDEs, and this is evident in numerous areas of physics, engineering, chemistry, and other fields. The fractional-order partial differential equations have several applications in many fields such as engineering, biophysics, physics, mechanics, chemistry, and biology (see [1–7]). More and more efforts have been made in the fractional calculus field especially in FDEs (see, for instance, [2, 5, 8–14, 27–39]). Solution existence for a lot of boundary value problems and several nonlinear elementary problems is studied via a huge number of techniques and nonlinear mathematical tools (see [7, 15–23]): the theory of critical point, fixed-point theory, technique of monochromatic iterative, theory degree of coincidence, and the change methods. Motivated by multiple works involved in this domain, we concentrate in this paper on the existence of several infinite solutions to the following fractional-order differentiation system:

$$\begin{cases} tD_T^{\alpha_i}\left(\Phi_p\left({}_0D_t^{\alpha_i}u(t)\right)\right) = \lambda F_{u_i}(t, u_1(t), u_2(t)), \cdots, u_n(t)) + \mu G_{u_i}(t, u_1(t), u_2(t)), \cdots, u_n(t)) + h_i(u_i), \text{a.e.} t \in [0, T], \\ u_i(0) = u_i(T) = 0, \end{cases} \quad (1)$$

for $1 \leq i \leq n$, $\alpha_i \in (0\,;1]$, ${}_0D_t^{\alpha_i}$ and ${}_tD_T^{\alpha_i}$ are the left and right α_i fractional-order derivatives of the Riemann-Liouville operator, respectively,

$$\Phi_p(s) = |s|^{p-2}s, \quad p > 1. \tag{2}$$

λ, μ are positive and nonnegative real parameters, respectively, $(F0) F, G : [0, T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ are continuous functions according to $t \in [0, T]$ for any $(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ and are C^1 with respect to $(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ for a.e. $t \in [0, T]$, $F(t, 0, 0, \cdots, 0) = 0$ and $G(t, 0, 0, \cdots, 0) = 0$ for it is to say $t \in [0, T]$. Also, F_{u_i} and G_{u_i} indicates partial derivatives of F and G according to u_i, respectively, and $h_i : \mathbb{R} \longrightarrow \mathbb{R}$ are $(p-1)$ -order of Lipschitz continuous functions with $L_i > 0$ constants of Lipschizian, $1 \leq i \leq n$, i.e.,

$$|h_i(x_1) - h_i(x_2)| \leq L_i |x_1 - x_2|^{p-1}. \tag{3}$$

In the last few months, we treated the same area of this study, in [18], by using variational methods together with a critical point theory due to Bonano and Marano. We got at least three weak solutions for the following nonlinear dual-Laplace systems with respect to two parameters:

$$\begin{cases} {}_tD_T^{\alpha_i}\left(\dfrac{1}{w_i(t)^{p-2}}\phi_p\left(w_i(t)_0D_T^{\alpha_i}u_i(t)\right)\right) + \mu|u_i(t)|^{p-2}u_i(t) = \lambda Fu_i(t, u_1(t), u_2(t), \cdots, u_n(t)) \text{ a.e. } t \in [0, T], \\ u_i(0) = u_i(T) = 0. \end{cases} \tag{4}$$

In addition, in [24], by using the variational method and Ricceri's critical point theorems, the existence of three weak solutions has been used to investigate the following class of perturbed nonlinear fractional p-Laplacian differential systems:

$$\begin{cases} {}_tD_T^{\alpha}\left(\dfrac{1}{w_1(t)^{p-2}}\Phi_p\left(w_1(t)_0D_t^{\alpha}u(t)\right)\right) + \mu|\mu(t)|^{p-2}\mu(t) = \lambda F_u(t, u(t), v(t)) + \delta G_u(t, u(t), v(t)) \text{ a.e.} t \in [0, T], \\ {}_tD_T^{\beta}\left(\dfrac{1}{w_2(t)^{p-2}}\Phi_p\left(w_2(t)_0D_t^{\beta}v(t)\right)\right) + \mu|v(t)|^{p-2}v(t) = \lambda F_v(t, u(t), v(t)) + \delta G_v(t, u(t), v(t)) \text{ a.e.} t \in [0, T], \\ u(0) = u(T) = 0, \; v(0) = v(T) = 0, \end{cases} \tag{5}$$

where some necessary conditions on the primitive function of nonlinear terms F_u and F_v have been considered. Then, in [25], the same last methods have been used for problem (5), the existence of multiplicity of weak solutions for the following perturbed nonlinear fractional differential systems:

$$\begin{cases} {}_tD_T^{\alpha_i}\left(a_i(t)_0D_T^{\alpha_i}u_i(t)\right) = \lambda F_{u_i}(t, u_1(t), u_2(t), \cdots, u_n(t)) + \mu G_{u_i}(t, u_1(t), u_2(t), \cdots, u_n(t)) + h_1(u_i) \text{ a.e. } [0, T], \\ u_i(0) = u_i(T) = 0, \end{cases} \tag{6}$$

where Lipschitz nonlinearity order of $p-1$ has been used.

Most recently, in [26], the authors proved the existence of infinitely multiple solutions of the following perturbed nonlinear fractional p-Laplacian differential systems:

$$\begin{cases} {}_tD_T^{\alpha}\left(\Phi_p\left({}_0D_t^{\alpha}u(t)\right)\right) = \lambda F_u(t, u(t), v(t)) + h_1(u_1), \text{ a.e. } t \in [0, T], \\ {}_tD_T^{\beta}\left(\Phi_p\left({}_0D_t^{\beta}v(t)\right)\right) = \lambda F_v(t, u(t), v(t)) + h_2(u_2), \text{ a.e. } t \in [0, T], \\ u(0) = u(T) = 0, \; v(0) = v(T) = 0, \end{cases} \tag{7}$$

where one control parameter with the variational method has been used.

Motivated by recently mentioned papers, the main contribution of this article is to use two control parameters and variational method to study a class of a nonlinear perturbed fractional-order p-Laplacian differential system which is defined in (6), where we can prove that the studied system admits sequences of weak different solutions, strongly converge to zero.

2. Preliminaries

In this section, we introduce some notations, lemmas that are required for the subsequential. Then, a variational framework is constructed; then, the critical point theory is applied to explore the existence of infinite solutions for the system given in (6).

We denote Y_X the class of all functionals $\phi : X \longrightarrow \mathbb{R}$, where X is real Banach space which has the following properties.

If $\{w_n\}$ is a sequence in X converge weakly to $w \in X$ and $\lim_{n\longrightarrow\infty} \inf \phi(w_n) \le \phi(w)$, thus $\{w_n\}$ has a subsequence that strongly converge to w.

As an example, suppose a uniformly convex X with $S : [0,+\infty) \longrightarrow \mathbb{R}$ is an increasing, continuous strictly function, then the functional $w \longrightarrow S(\|w\|) \in Y_X$.

Definition 1 (see [4]). Let u be a defined function on $[a, b]$. The left and right Riemann-Liouville fractional derivatives of order $\alpha > 0$, respectively, are given as

$$\begin{aligned} {}_aD_t^{\alpha}u(t) &:= \frac{d^n}{dt^n} {}_aD_t^{\alpha-n}u(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} u(s) ds, \\ {}_tD_b^{\alpha}u(t) &:= (-1)^n \frac{d^n}{dt^n} {}_tD_b^{\alpha-n}u(t) \\ &= \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^b (t-s)^{n-\alpha-1} u(s) ds, \end{aligned} \tag{8}$$

where the right-hand sides are pointwise defined over $[a, b]$, $\forall t \in [a, b]$, $n-1 \le \alpha < n$ and $n \in \mathbb{N}$.

The gamma function, $\Gamma(\alpha)$, is given by

$$\Gamma(\alpha) := \int_0^{+\infty} z^{\alpha-1} e^{-z} dz. \tag{9}$$

Setting $AC^n([a, b], \mathbb{R})$ the space of functions $u : [a, b] \longrightarrow \mathbb{R}$, where

$$u \in C^{n-1}([a, b], \mathbb{R}), u^{(n-1)} \in AC([a, b], \mathbb{R}). \tag{10}$$

As familiar, we denote $C^{n-1}([a, b], \mathbb{R})$ the mappings set indicates $(n-1)$-times continuously differentiable on $[a, b]$.

Actually, we imply

$$AC([a, b], \mathbb{R}) := AC^1([a, b], \mathbb{R}). \tag{11}$$

Definition 2 (see [22]). Let $0 < \alpha_i \le 1$ $(1 \le i \le n, 1 < p < \infty)$, we introduce the space of the fractional-order derivative as follows:

$$\begin{aligned} E_{\alpha_i}^p = \{ & u(t) \in L^p([0, T], \mathbb{R}) |_0D_t^{\alpha_i}u(t) \\ & \in L^p([0, T], \mathbb{R}), u(0) = u(T) = 0 \}, \end{aligned} \tag{12}$$

then, the norm of $E_{\alpha_i}^p$ can be defined $\forall u \in E_{\alpha_i}^p$, as the following

$$\|u\|_{\alpha_i} = \left(\int_0^T |u(t)|^p dt + \int_0^T \left| {}_0D_t^{\alpha_i}u(t) \right|^p dt \right)^{1/p}. \tag{13}$$

Lemma 3 (see [3]). *Let $0 < \alpha_I \le 1 (1 \le i \le n, 1 < p < \infty)$. For all $u_i \in E_{\alpha_i}^p$, we have*

$$\|u_i\|_{L^p} \le \frac{T^{\alpha_i}}{\Gamma(\alpha_i + 1)} \left\| {}_0D_t^{\alpha_i}u_i \right\|_{L^p}. \tag{14}$$

Also, if $\alpha_i > p$ and $1/p + 1/q = 1$, then

$$\|u_i\|_{\infty} \le \frac{T^{\alpha_i - 1/p}}{\Gamma(\alpha_i)((\alpha_i - 1)q + 1)^{1/q}} \left\| {}_0D_t^{\alpha_i}u_i \right\|_{L^p}. \tag{15}$$

Hence, the operator $E_{\alpha_i}^p$ according to the norm can be considered

$$\|u_i\|_{\alpha_i} = \left(\int_0^T \left| {}_0D_t^{\alpha_i}u(t) \right|^p dt \right)^{1/p}, \quad \forall u_i \in E_{\alpha_i}^p, \tag{16}$$

for $1 \le i \le n$, that is equivalent to (13).

Definition 4. Suppose a Cartesian product X of n spaces E_{α}^p; that is to say,

$$X = E_{\alpha_1}^p \times E_{\alpha_2}^p \times \cdots \times E_{\alpha_n}^p, \tag{17}$$

provided with the norm

$$\|u\|_X = \sum_{i=1}^n \|u_i\|_{\alpha_i}, u = (u_1, u_2, \cdots, u_n) \in X, 1 \le i \le n, \tag{18}$$

where $\|u_i\|_{\alpha_i}$ is defined in (16).

Clearly, X is embedded compact in $C^0([0, T], \mathbb{R})^n$.

Lemma 5 (see [23]). *For $0 < \alpha_i \le 1 (1 \le i \le n)$ and $1 < p < \infty$. The space of the fractional-order derivative X is a reflexive separable Banach space.*

Lemma 6 (see [16]). *Assume that $1/p < \alpha_i \le 1$ and $\{u_n\}$ be the sequence that weakly converges to u in E_{α}^p, i.e., $u_n \rightharpoonup u$. Then,*

$\{u_n\}$ have a strong convergence to u in $C([0,T],\mathbb{R})$, i.e., $\|u_n-u\|_\infty \longrightarrow 0$, as $n\longrightarrow +\infty$.

Definition 7 (see [3]). We indicate a weak solution for system (6), any $u=(u_1,u_2,\cdots,u_n)\in X$ such that

$$\begin{aligned}&\int_0^T\sum_{i=1}^n \Phi_p\left({}_0D_t^{\alpha_i}u(t)\right){}_0D_t^{\alpha_i}v_i(t)dt\\&-\lambda\sum_{i=1}^n\int_0^T F_{u_i}(t,u_1(t),u_2(t)),\cdots,u_n(t))v_i(t)dt\\&-\mu\int_0^T\sum_{i=1}^n G_{u_i}(t,u_1(t),u_2(t)),\cdots,u_n(t))v_i(t)dt\\&-\int_0^T\sum_{i=1}^n h_i(u_i(t))v_i(t)dt=0,\quad \forall v_i=(v_1,v_2,\cdots,v_n)\in X.\end{aligned}\tag{19}$$

We define

$$\begin{aligned}H_i(x)&=\int_0^x h_i(z)dz,\\ \Theta_i(x)&=\int_0^T H_i(x(s))ds, i=1,2,\\ &\forall t\in[0,T],\&x\in\mathbb{R}.\end{aligned}\tag{20}$$

Lemma 8. *Suppose that $h_i:\mathbb{R}\longrightarrow\mathbb{R}$ satisfy (3) and $H_i(x)$, $\Theta_i(x)$, $1\le i\le n$ defined through (20). Then, $\Theta(u)$: $X\longrightarrow\mathbb{R}$ be the functional that is described by*

$$\Theta(u)=\Theta(u_1,u_2,\cdots,u_n)=\sum_{i=1}^n\Theta_i(u_i)=\sum_{i=1}^n\int_0^T H_i(u_i(t))dt\tag{21}$$

is a weakly continuous sequentially Gâteaux differentiable functional on X together with a compact derivative

$$\Theta'(u_1,u_2,\cdots,u_n)(v_1,v_2,\cdots,v_n)=\sum_{i=1}^n\int_0^T h_i(u_i(t))v_i(t)dt,\tag{22}$$

for every $u=(u_1,u_2,\cdots,u_n)$,$v=(v_1,v_2,\cdots,v_n)\in X$.

Proof. Assume that $\{u_k=(u_{1k},u_{2k},\cdots,u_{nk})\}\subset X$,$1\le k\le n$, $u_k=(u_{1k},u_{2k},\cdots,u_{nk})\rightharpoonup u=(u_1,u_2,\cdots,u)$ in X as $k\longrightarrow+\infty$. It follows from Lemma 8 that $\{u_k=(u_{1k},u_{2k},\cdots,u_{nk})\}$ converges uniformly to $u=(u_1,u_2,\cdots,u)$ on $[0,T]$. Therefore, there exist constants $c_i>0$, $1\le i\le n$ such that $\|u_{ik}\|_\infty\le c_i$, $1\le i\le n$.

Then,

$$\begin{aligned}&|H_i(u_{ik}(t))-H_i(u(t))|\\&\quad\le L_i\left|\int_{u(t)}^{u_{ik}(t)}|s|^{p-1}ds\right|\le\frac{L_i}{p}\left(|u_{ik}(t)|^p+|u(t)|^p\right)\\&\quad\le\frac{L_i}{p}\left(c_i^p+\|u(t)\|_\infty^p\right),\end{aligned}\tag{23}$$

for any $n\in\mathbb{N}$, & $t\in[0,T]$. Moreover, $H_i(u_{ik}(t))\longrightarrow H_i(u_i(t))$, $1\le i\le n$,$\forall t\in[0,T]$, and via the convergence theorem in the Lebesgue sense, we have

$$\begin{aligned}&\Theta(u_{1k},u_{2k},\cdots,u_{nk})\\&\quad=\sum_{i=1}^n\int_0^T H_i\left(u_{i_k}(t)\right)dt\longrightarrow\sum_{i=1}^n\int_0^T H_i(u_i(t))dt,\text{ for }1\\&\quad\le i\le n,=\Theta(u_1,u_2,\cdots,u_n)=\Theta(u).\end{aligned}\tag{24}$$

In the following, a Gâteaux differentiability of Θ will be implemented. Let $u_1,x\in E_{\alpha_1}^p$ and $s\neq0$, then we claim

$$\begin{aligned}&\left|\frac{\Theta_1(u_1+sx)-\Theta_1(u)}{s}-\int_0^T h_1(u_1(t))x(t)dt\right|\\&\quad\le\int_0^T\left|\frac{H_1(u_1+sx)-H_1(u_1)}{s}-h_1(u_1(t))x(t)\right|dt\\&\quad=\int_0^T|h_1(u_1(t))+s\zeta(t)x(t)-h_1(u_1(t))||x(t)|dt\\&\quad\le L_1\|x\|_\infty^p|s|,\end{aligned}\tag{25}$$

where $\forall t\in[0,T]$, $0<\zeta(t)<1$. Hence, $\Theta_1:E_{\alpha_1}^p\longrightarrow\mathbb{R}$ is a Gâteaux differentiable at any $u_1\in E_{\alpha_1}^p$.

Analogously, we have that $\Theta_2:E_{\alpha_2}^p\longrightarrow\mathbb{R}$ at any $u_2\in E_{\alpha_2}^p$ is a differentiable in Gâteaux sense.

Therefore, $\Theta:X\longrightarrow\mathbb{R}$ to each $u=(u_1,u_2,\cdots,u_n)\in X$ with derivative is differentiable with a Gâteaux description

$$\begin{aligned}&\Theta'(u_1,u_2,\cdots,u_n)(v_1,v_2,\cdots,v_n)\\&\quad=\sum_{i=1}^n\int_0^T h_i(u_i(t))v_i(t)dt,\quad\forall v=(v_1,v_2,\cdots,v_n)\in X.\end{aligned}\tag{26}$$

For any three elements $(u_1,u_2,\cdots,u_n)$, $(v_1,v_2,\cdots,v_n)$, and $(w_1,w_2,\cdots,w_n)$ of X, it is easy to see that

$$\begin{aligned}&\left(\Theta'(u_1,u_2,\cdots,u_n)-\Theta'(v_1,v_2,\cdots,v_n)\right)(w_1,w_2,\cdots,w_n)\\&\quad=\sum_{i=1}^n\int_0^T(h_i(u_i)-h_i(v_i))w_i(t)dt\le\sum_{i=1}^n L_i\int_0^T|u_i-v_i|^{p-1}|w_i(t)|dt\\&\quad\le\sum_{i=1}^n\frac{L_iT^{\alpha_i-1/p}}{\Gamma(\alpha_i)((\alpha_i-1)q+1)^{1/q}}\|u_i-v_i\|_\infty^{p-1}\|w_i\|_\alpha,\end{aligned}\tag{27}$$

which implies

$$\left\|\Theta'(u_1, u_2, \cdots, u_n) - \Theta'(v_1, v_2, \cdots, v_n)\right\|_X \leq T^* \sum_{i=1}^{n} \left(\|u_i - v_i\|_\infty^{p-1} \|w_i\|_\alpha\right), \tag{28}$$

where

$$T^* \coloneqq \max\left\{\frac{L_i T^{\alpha_i - 1/p}}{\Gamma(\alpha_i)((\alpha_i - 1)q + 1)^{1/q}}, 1 \leq i \leq n\right\}. \tag{29}$$

Consequently, the operator $\Theta' : X \longrightarrow X^*$ is compact.

Below, we recall Theorem 2.5 of [23] which is an essential tool in our paper.

Lemma 9 (see [23], Theorem 2.5). *Let a real reflexive Banach space X. Also, suppose two Gâteaux differentiable functionals $\phi, \Psi : X \longrightarrow \mathbb{R}$ such that ϕ is sequentially weakly lower semicontinuous, strongly continuous, and coercive where sequentially weakly upper semicontinuous achieved for Ψ. $\forall r > \inf_X \phi$, put*

$$\varphi(r) = \inf_{u\in\phi^{-1}(]-\infty,r])} \frac{\sup_{v\in\phi^{-1}(]-\infty,r])} \Psi(v) - \Psi(u)}{r - \phi(u)}, \tag{30}$$

$$\gamma =: \lim_{r\longrightarrow+\infty} \inf \varphi(r), \delta \coloneqq \lim_{r\longrightarrow\left(\inf_X \phi\right)^+} \inf \varphi(r). \tag{31}$$

Then,

(a) For every $r > \inf_X \phi$ & $\lambda \in (0, 1/\varphi(r))$, the functional constraint of $I_\lambda = \phi - \lambda\Psi$ to $\phi^{-1}(-\infty, r)$ allows a global minimum, which is a critical point (local minimum) for I_λ in the space X

(b) If $\gamma < +\infty$ & $\lambda \in]0, 1/\gamma[$, the subsequent alternative holds:

(b_1)The functional $\phi - \lambda\Psi$ has a global minimum or

(b_2)$\exists$ a sequence $\{u_n\}$ of critical points (local minima) for I_λ such that $\lim_{n\longrightarrow+\infty}\phi(u_n) = +\infty$

(c) If $\delta < +\infty$ & $\lambda \in]0, 1/\delta[$, the next alternative exists:

(c_1)$\exists$ a global minimum of ϕ that is a local minimum of I_λ or

(c_2)$\exists$ a sequence $\{u_n\}$ of pairwise disjoint critical points (local minima) for I_λ that one converges weakly to a global minimum of ϕ together with $\lim_{n\longrightarrow+\infty}\phi(u_n) = \inf_X \phi$

3. Principle Results

This section deals with stating and proving our main results. For assistance, suggest

$$\begin{aligned} k' &\coloneqq \min_{1\leq i\leq n}\left\{1 - \frac{L_i T^{p\alpha_i}}{(\Gamma(\alpha_i + 1))^p}\right\}, \\ \rho &\coloneqq \max_{1\leq i\leq n}\left\{1 + \frac{L_i T^{p\alpha_i}}{(\Gamma(\alpha_i + 1))^p}\right\}, \\ k &= \max_{1\leq i\leq n}\left\{\frac{T^{p\alpha_i - 1}}{(\Gamma(\alpha_i))^p((\alpha_i - 1)q + 1)^{p/q}}\right\}. \end{aligned} \tag{32}$$

For a given constant $\theta \in (1/p, 0)$, set

$$\begin{aligned} P(\alpha_i, \theta) &= \frac{1}{p(\theta T)^p}\left\{\int_0^{\theta T} t^{p(1-\alpha_i)} dt + \int_{\theta T}^{(1-\theta)T} \left(t^{1-\alpha_i} - (t - \theta T)^{1-\alpha_i}\right)^p dt + \int_{(1-\theta)T}^{T} \left[\left(t^{1-\alpha_i} - (t - \theta T)^{1-\alpha_i}\right) - (t - ((1-\theta)T))^{1-\alpha_i}\right]^p\right\}, \\ \Delta &\coloneqq \min_{1\leq i\leq n} P(\alpha_i, \theta), \\ \Delta' &\coloneqq \max_{1\leq i\leq n} P(\alpha_i, \theta). \end{aligned} \tag{33}$$

For any $\rho > 0$, we set

$$\Omega(\varrho) = \left\{x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} \frac{1}{p}|x_i|^p \leq \varrho\right\}. \tag{34}$$

Theorem 10. *Let $1/p < \alpha_i \leq 1$ for $1 \leq i \leq n$. Suppose that there exists $\theta \in (0, 1/p)$ such that*

(h1)$F(t, x_1, \cdots, x_n) \geq 0$ for each $(t, x_1, \cdots, x_n) \in ([0, \theta T] \cup [(1-\theta)T, T]) \times \mathbb{R}^n$

(h2)

$$\begin{aligned} &\lim_{\xi\longrightarrow+\infty} \inf \frac{\int_0^T \sup_{(x_1,x_2,\cdots,x_n)\in\Omega(\xi)} F(t, x_1, \cdots, x_n) dt}{\xi^p} \\ &< \frac{k'}{pk^p n^p \rho \Delta'} \lim_{\xi\longrightarrow+\infty} \sup \frac{\int_\theta^{(1-\theta)T} F(t, \Gamma(2-\alpha_1)\xi, \cdots, \Gamma(2-\alpha_n)\xi) dt}{\xi^p}. \end{aligned} \tag{35}$$

Then, for each $\lambda \in \Lambda \coloneqq]\lambda_1, \lambda_2[$ where

$$\lambda_1 := \frac{\rho\Delta'}{\lim_{\xi\longrightarrow+\infty} \sup \int_\theta^{(1-\theta)T} F(t, \Gamma(2-\alpha_1)\xi, \cdots, \Gamma(2-\alpha_n)\xi)dt/\xi^p},$$

$$\lambda_2 := \frac{k'/pk^p n^p}{\lim_{\xi\longrightarrow+\infty} \inf \int_0^T \sup_{(x_1,x_2,\cdots,x_n)\in\Omega(\xi)} F(t, x_1, \cdots, x_n)dt/\xi^p}, \tag{36}$$

for each nonnegative function $G : [0, T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ achieving the constrain

$$G_\infty := \lim_{\xi\longrightarrow+\infty} \sup \frac{\int_0^T \sup_{(x_1,x_2,\cdots,x_n)\in\Omega(\xi)} G(t, x_1, \cdots, x_n)dt}{\xi^p} < +\infty, \tag{37}$$

and for every $\mu \in [0, \mu_{G,\lambda}[$ where

$$\mu_{G,\lambda} := \frac{k'}{pk^p n^p G_\infty}\left(1 - \lambda\frac{pk^p n^p}{k'} \lim_{\xi\longrightarrow+\infty} \inf \frac{\int_0^T \sup_{(x_1,x_2,\cdots,x_n)\in\Omega(\xi)} F(t, x_1, \cdots, x_n)dt}{\xi^p}\right), \tag{38}$$

system (1) has an unbounded sequence of weak solutions in space X.

Proof. The main aim here is applying Lemma 6 (see [16]) over system (1). For this purpose, fix $\lambda \in \Lambda$, and let G be a function that satisfies our hypotheses. Since $\bar{\lambda} < \lambda_2$, we claim

$$\mu_{G,\bar{\lambda}} = \frac{k'}{pk^p n^p G_\infty}\left(1 - \bar{\lambda}\frac{pk^p n^p}{k'} \lim_{\xi\longrightarrow+\infty} \inf \frac{\int_0^T \sup_{(x_1,x_2,\cdots,x_n)\in\Omega(\xi)} F(t, x_1, \cdots, x_n)dt}{\xi^p}\right) > 0. \tag{39}$$

Now, fix $\bar{\mu} \in]0, \mu_{G,\bar{\lambda}}[$. Set

$$J(t, \xi_1, \cdots, \xi_n) := F(t, \xi_1, \cdots, \xi_n) + \frac{\bar{\mu}}{\bar{\lambda}} G(t, \xi_1, \cdots, \xi_n), \tag{40}$$

for every $t \in [0, T]$ and $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$. We construct the mappings $\phi, \Psi : X \longrightarrow \mathbb{R}$ as follows:

$$\phi(u) := \sum_{i=1}^n \frac{\|u_i\|_{\alpha_i}^p}{p} - \Theta(u),$$
$$\Psi(u) := \int_0^T J(t, u_1(t), \cdots, u_n(t))dt, \tag{41}$$

$\forall u = (u_1, \cdots, u_n) \in X$ and determine

$$I_{\bar{\lambda},\bar{\mu}}(u) := \phi(u) - \lambda\Psi(u), u \in X. \tag{42}$$

Let us prove that ϕ & Ψ satisfy the required constrains. Since X is compactly embedded in $(C([0, T], \mathbb{R}))^n$, it is well known that Ψ is well-defined Gateaux differentiable functional whose Gateaux derivative at $u \in X$ is the functional Ψ', given by

$$\Psi'(u)(v) = \int_0^T \sum_{i=1}^n J(t, u_1(t), \cdots, u_n(t))v(t)dt, \tag{43}$$

$\forall v = (v_1, \cdots, v_n) \in X$. Moreover, Ψ is sequentially weakly continuous.

The functional Φ is a Gateaux differentiable functional with the differential at $u \in X$,

$$\phi'(u)(v) = \int_0^T \sum_{i=1}^n \left|{}_0D_t^{\alpha_i} u_i(t)\right|^{p-2} {}_0D_t^{\alpha_i} u_i(t) \cdot {}_0D_t^{\alpha_i} v_i(t)dt - \int_0^T \sum_{i=1}^n h_i(u_i(t))v_i(t)dt, \tag{44}$$

for every $v \in X$. Moreover, ϕ is sequentially weakly lower semicontinuous, strongly continuous, and coercive functional on X.

Obviously, the weak solutions of system (1) are precisely the critical points of the functional $I_{\bar{\lambda},\bar{\mu}}$. Furthermore, since (3) holds for every $x_1, \cdots, x_n \in \mathbb{R}$ and $h_1(0) = \cdots = h_n(0) = 0$, one has $|h_i(x)| \le L_i|x|^{p-1}$, $1 \le i \le n \forall x \in \mathbb{R}$. It obtained from (14) and (15) the following:

$$\begin{aligned}\phi(u) &\ge \frac{\sum_{i=1}^{n}\|u_i\|_{\alpha_i}^p}{p} - \left|\int_0^T \sum_{i=1}^{n} H_i(u_i(t))dt\right| \\ &\ge \frac{\sum_{i=1}^{n}\|u_i\|_{\alpha_j}^p}{p} - \sum_{i=1}^{n}\frac{L_i}{p}\int_0^T |u_i(t)|^p dt \\ &\ge \left(\frac{1}{p} - \frac{L_i T^{p\alpha_j}}{p(\Gamma(\alpha_i+1))^p}\right)\|u_i\|_{\alpha_i}^p \ge \frac{k'}{p}\sum_{i=1}^{n}\|u_i\|_{\alpha_i'}^p,\end{aligned} \tag{45}$$

$\forall u \in X$, and as a result for this, ϕ is coercive.

Now, allow us to verify that $\bar{\lambda} < 1/\gamma$. Assume $\{\xi_k\}$ is a positive number sequence such that $\xi_k \longrightarrow \infty$ as $k \longrightarrow \infty$ and

$$\begin{aligned}&\lim_{k\longrightarrow+\infty} \frac{\int_0^T \sup_{(x_1,x_2,\cdots,x_n)\in\Omega(\xi_k)} F(t,x_1,\cdots,x_n)dt}{\xi_k^p} \\ &= \lim_{\xi\longrightarrow+\infty}\inf \frac{\int_0^T \sup_{(x_1,x_2,\cdots,x_n)\in\Omega(\xi)} F(t,x_1,\cdots,x_n)dt}{\xi^2}.\end{aligned} \tag{46}$$

Put $r_k := k'\xi_k^p/pk^p n^p \forall k \in \mathbb{N}$. Since $\max_{t\in[0,T]}|u_i(t)| \le k\|u_i\|_{\alpha_i}$ for all $u_i \in E_0^{\alpha_i}([0,T])$ and $1 \le i \le n$, we have

$$\sup_{t\in[0,T]}\sum_{i=1}^{n}\left|u_i(t)\right|^p \le k^p \sum_{i=1}^{n}\left|\left|u_i\right|\right|_{\alpha_i}^p, \tag{47}$$

for each $u = (u_1,\cdots,u_n) \in X$. So, from (45) and (47), we have

$$\begin{aligned}\phi^{-1}(]-\infty, r_k[) &:= \left\{u \in X : \frac{k'}{2}\left(\sum_{i=1}^{n}\|u_i\|_{\alpha_i}^p\right) < r_k\right\} \\ &\subseteq \left\{u \in X : \sum_{i=1}^{n}|u_i(x)|^p \le \frac{pk^p}{k'} r_k \text{ for each } t \in [0,T]\right\} \\ &\subseteq \left\{u \in X : \sum_{i=1}^{n}|u_i(t)| \le \xi_k \text{ for each } t \in [0,T]\right\}.\end{aligned} \tag{48}$$

Consequently, taking into the description that $\phi(0,\cdots,0) = \Psi(0,\cdots,0) = 0$, for all k big enough, one has

$$\begin{aligned}\varphi(r_k) &= \inf_{u\in\phi^{-1}((\infty,r_k[)} \frac{\sup_{u\in\phi^{-1}((-\infty,r_k[)}\Psi(v) - \Psi(u)}{r_k - \phi(u)}, \\ &\frac{\sup_{u\in\phi^{-1}((-\infty,r_k[)}\Psi(v)}{r_k} \\ &\le \frac{\int_0^T \sup_{(x_1,\cdots,x_n)\in\Omega(\xi_k)} J(t,x_1,\cdots,x_n)dt}{k'\xi_k^p/pk^p n^p} \\ &= \frac{\int_0^T \sup_{(x_1,\cdots,x_n)\in\Omega(\xi_k)}\left[F(t,x_1,\cdots,x_n) + \bar{\mu}/\bar{\lambda}G(t,x_1,\cdots,x_n)\right]dt}{k'\xi_k^p/pk^p n^p} \\ &\le \frac{\int_0^T \sup_{(x_1,\cdots,x_n)\in\Omega(\xi_k)} F(t,x_1,\cdots,x_n)dt}{k'\xi_k^p/pk^p n^p} \\ &\quad + \frac{\bar{\mu}}{\bar{\lambda}}\frac{\int_0^T \sup_{(x_1,\cdots,x_n)\in\Omega(\xi_k)} G(t,x_1,\cdots,x_n)dt}{k'\xi_k^p/pk^p n^p}.\end{aligned} \tag{49}$$

Moreover, from assumption (h2) and (37), one has

$$\begin{aligned}&\lim_{k\longrightarrow\infty}\inf \frac{\int_0^T \sup_{(x_1,\ldots,x_n)\in\Omega(\xi_k)} F(t,x_1,\cdots,x_n)dt}{\left(k'/pk^p n^p\right)\xi_k^p} \\ &+ \lim_{k\longrightarrow\infty}\frac{\bar{\mu}}{\bar{\lambda}}\frac{\int_0^T \sup_{(x_1,\ldots,x_n)\in\Omega(\xi_k)} G(t,x_1,\cdots,x_n)dt}{k'\xi_k^p/pk^p n^p} < +\infty,\end{aligned} \tag{50}$$

which implies

$$\lim_{k\longrightarrow\infty}\inf \frac{\int_0^T \sup_{(x_1,\ldots,x_n)\in\Omega(\xi_k)} J(t,x_1,\cdots,x_n)dt}{\xi_k^p} < +\infty. \tag{51}$$

Therefore,

$$\begin{aligned}\gamma &\le \lim_{k\longrightarrow\infty}\inf \varphi(r_k) \\ &\le \frac{pk^p n^p}{k'}\lim_{k\longrightarrow\infty}\inf \frac{\int_0^T \sup_{(x_1,\ldots,x_n)\in\Omega(\xi_k)} J(t,x_1,\cdots,x_n)dt}{\xi_k^p} < +\infty.\end{aligned} \tag{52}$$

The assumption $\bar{\mu} \in]0, \mu_{G,\bar{\lambda}}[$ immediately yields $\bar{\lambda} < 1/\gamma$.

The succeeding step is to confirm that for a fixed $\bar{\lambda}$, the functional $I_{\bar{\lambda},\bar{\mu}}$ has no global minimum. Let us verify that $I_{\bar{\lambda},\bar{\mu}}$ is unbounded from below. Since

$$\begin{aligned}\frac{1}{\bar{\lambda}} &< \frac{1}{\rho\Delta'}\lim_{\xi\longrightarrow+\infty}\sup \frac{\int_{\theta T}^{(1-\theta)} F(t,\Gamma(2-\alpha_1)\xi,\cdots,\Gamma(2-\alpha_n)\xi)dt}{\xi^p} \\ &\le \lim_{\xi\longrightarrow+\infty}\sup \frac{\int_{\theta T}^{(1-\theta)T} J(t,\Gamma(2-\alpha_1)\xi,\cdots,\Gamma(2-\alpha_n)\xi)dt}{\xi^p},\end{aligned} \tag{53}$$

consider $\{\eta_k\}$ is a real sequence and τ is a positive constant such that $\eta_k \longrightarrow \infty$ as $k \longrightarrow \infty$, and

$$\frac{1}{\bar{\lambda}} < \tau < \frac{1}{\rho\Delta'}\frac{\int_{\theta T}^{(1-\theta)T} F\left(t, \Gamma(2-\alpha_1)\eta_k{}',\cdots{}'\Gamma(2-\alpha_n)\eta_k\right)dt}{\eta_k^p}, \tag{54}$$

$\forall k \in \mathbb{N}$ huge adequate. $\forall k \in \mathbb{N}$, and $\theta \in (0, 1/p)$ define $\{w_k = (w_{1k}, \cdots, w_{nk})\}$ by setting

$$\omega_{ik}(t) := \begin{cases} \frac{\Gamma(2-\alpha_i)\eta_k}{\theta T} t, & t \in [0, \theta T[, \\ \Gamma(2-\alpha_i)\eta_k, & t \in [\theta T, (1-\theta)T], \\ \frac{\Gamma(2-\alpha_i)\eta_k}{\theta T}(T-t), & t \in](1-\theta)T, T], \end{cases} \tag{55}$$

for $1 \le i \le n$. Clearly, $\omega_{ik}(0) = \omega_{ik}(T) = 0$ and $\omega_{ik} \in L^p[0, T]$ for $1 \le i \le n$. A direct calculation shows that

$${}_0D_t^{\alpha_i}\omega_{ik}(t) = \begin{cases} \frac{\eta_k}{\theta T} t^{1-\alpha_i}, & t \in [0, \theta T[, \\ \frac{\eta_k}{\theta T}\left(t^{1-\alpha_i} - (t-\theta T)^{1-\alpha_i}\right), & t \in [\theta T, (1-\theta)T], \\ \frac{\eta_k}{\theta T}\left(t^{1-\alpha_i} - (t-\theta T)^{1-\alpha_i} - (t-(1-\theta)T)^{1-\alpha_i}\right), & t \in](1-\theta)T, T], \end{cases} \tag{56}$$

for $1 \le i \le n$. Furthermore,

$$\begin{aligned} &\int_0^T \left| {}_0D_t^{\alpha_i}\omega_{ik}(t)\right|^p dt \\ &= \int_0^{\theta T} + \int_{\theta T}^{(1-\theta)T} + \int_{(1-\theta)T}^{T} \left| {}_0D_t^{\alpha_i}\omega_{ik}(t)\right|^p dt \\ &= \frac{\eta_k^p}{(\theta T)^p}\Bigg\{\int_0^{\theta T} t^{p(1-\alpha_i)}dt + \int_{\theta T}^{(1-\theta)T}\left(t^{1-\alpha_i} - (t-\theta T)^{1-\alpha_i}\right)^p dt \\ &+ \int_{(1-\theta)T}^{T}\left[\left(t^{1-\alpha_i} - (t-\theta T)^{1-\alpha_i}\right)\right. \\ &\left. - \left(t - ((1-\theta)T)^{1-\alpha_i}\right)\right]^p dt\Bigg\} = pP(\alpha, \theta)\eta_k^p, \end{aligned} \tag{57}$$

for $1 \le i \le n$. Thus, $\omega_k \in X$, and in particular,

$$\|\omega_{ik}\|_{\alpha_i}^p = \int_0^T \left|{}_0D_t^{\alpha_i}\omega_{ik}(t)\right|^p dt = pP(\alpha, \theta)\eta_k^p. \tag{58}$$

On the other hand, similar to (45), we have

$$\begin{aligned} \phi(\omega_k) &= \sum_{i=1}^n \frac{\|\omega_{ik}\|_{\alpha_i}^p}{p} - \Theta(\omega) \le \frac{\rho}{p}\left(\sum_{i=1}^n \|\omega_{ik}\|_{\alpha_i}^p\right) \\ &= \rho\left(\sum_{i=1}^n P_i\left(\alpha_i{}'\theta\right)\right)\eta_k^p \le \rho\Delta'\eta_k^p. \end{aligned} \tag{59}$$

Bearing in mind assumption (A) and since G is nonnegative, then using Ψ definition, we conclude that

$$\Psi(w_k) \ge \int_{\theta T}^{(1-\theta)T} F(t, \Gamma(2-\alpha_1)\eta_k, \cdots, \Gamma(2-\alpha_n)\eta_k)dt. \tag{60}$$

So, according to (54), (55), and (60),

$$\begin{aligned} I_{\bar{\lambda},\bar{\mu}}(w_k) \le \rho\Delta'\eta_k^p - \bar{\lambda}\int_{\theta T}^{(1-\theta)T} & F(t, \Gamma(2-\alpha_1)\eta_k, \cdots, \Gamma(2 \\ & - \alpha_n)\eta_k)dt < \rho\Delta'(1-\lambda\tau)\eta_k^p, \end{aligned} \tag{61}$$

for every $k \in \mathbb{N}$ large enough. Hence, $I_{\bar{\lambda},\bar{\mu}}$ is unbounded from below and so has no global minimum. Therefore, applying Lemma 6 (b), we deduce that there is a sequence $\{u_k = (u_{1k}, \cdots, u_{nk})\} \subset X$ of critical points of $I_{\bar{\lambda},\bar{\mu}}$ such that

$$\lim_{k\longrightarrow\infty} \|(u_{1k}, \cdots, u_{nk})\| = +\infty. \tag{62}$$

Here, the outcome is produced.

Remark 11. Under the conditions

$$\begin{aligned} &\lim_{\xi\longrightarrow+\infty}\inf \frac{\int_0^T \sup_{(x_1,\cdots,x_n)\in\Omega(\xi)} F(t, x_1, \cdots, x_n)dt}{\xi^p} = 0, \\ &\lim_{\xi\longrightarrow+\infty}\sup \frac{\int_{\theta T}^{(1-\theta)} F(t, \Gamma(2-\alpha_1)\xi, \cdots, \Gamma(2-\alpha_n)\xi)}{\xi^p} = +\infty, \end{aligned} \tag{63}$$

from Theorem 10, we claimed $\forall\lambda > 0$ and $\mu \in [0, k'/pk^p n^p G_\infty[$ system (1) admits infinitely many weak solutions in the space X. Also, if $G_\infty = 0$, then the result holds $\forall\lambda > 0$ and $\mu \ge 0$.

Here, we point out the following conclusion of Theorem 10 beside $\mu = 0$.

Corollary 12. *Suppose $\theta \in (0, 1/p)$ such that hypothesis (h1) holds. Assume*

(B1)- $\lim_{\xi \longrightarrow +\infty} \inf \left(\int \sup_{(x_1,\cdots,x_n)\in\Omega(\xi)} F(t, x_1, \cdots, x_n)dt/\xi^p\right) < k'/pk^p n^p$

(B2)- $\lim_{\xi \longrightarrow +\infty} \sup \left(\int_{\theta T}^{(1-\theta)T} F(t, \Gamma(2-\alpha_1)\xi, \cdots, \Gamma(2-\alpha_n)\xi)dt/\xi^p\right) > \rho\Delta'$

Then, the system

$$\begin{cases} tD_T^{\alpha_i}\left(\Phi_p\left({}_0D_t^{\alpha_i}u_i(t)\right)\right) = Fui(t, u1(t), \cdots, un(t)) + hi(ui)a.e.t \in [0, T], \\ ui(0) = ui(T) = 0, \end{cases} \quad (64)$$

for $1 \le i \le n$, has an unbounded sequence of weak solutions in X.

Follow the same steps of Theorem 10 proving but alternatively of (b) of Lemma 6 applying conclusion (c), the following result will be obtained.

Theorem 13. *Suppose that all of Theorem 10 assumptions hold except for hypothesis (h2). Assume thatassumptions hold except for hypothesis (h2). Assume that*

$$\lim_{\xi \longrightarrow 0^+} \inf \frac{\int \sup_{(x_1,\cdots,x_n)\in\Omega(\xi)} F(t, x_1, \cdots, x_n)dt}{\xi^P} < \frac{k'}{pk^p n^p \rho\Delta'} \lim_{\xi \longrightarrow 0^+} \sup \cdot \frac{\int_{\theta T}^{(1-\theta)T} F\left(t, \Gamma(2-\alpha_1)\xi, \cdots' \Gamma(2-\alpha_n)\xi\right)dt}{\xi^p}. \quad (65)$$

Then, for each $\lambda \in]\lambda_3, \lambda_4[$, where

$$\lambda_3 := \frac{\rho\Delta'}{\lim_{\xi \longrightarrow 0^+} \sup \left(\int_{\theta T}^{(1-\theta)T} F(t, \Gamma(2-\alpha_1)\xi, \cdots, \Gamma(2-\alpha_n)\xi)dt/\xi^p\right)},$$
$$\lambda_4 := \frac{k'/pk^p n^p}{\lim_{\xi \longrightarrow 0^+} \inf \left(\int_0^T \sup_{(x_1,\cdots,x_n)\in\Omega(\xi)} F(t, x_1, \cdots, x_n)dt/\xi^p\right)}, \quad (66)$$

for every nonnegative function $G : [0, T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfying the condition

$$G_0 := \lim_{\xi \longrightarrow 0^+} \sup \frac{\int_0^T \sup_{(x_1,\cdots,x_n)\in\Omega(\xi)} G(t, x_1, \cdots, x_n)dt}{\xi^p} < +\infty, \quad (67)$$

and for every $\mu \in [0, \mu'_{G,\lambda}[$ where

$$\mu'_{G,\lambda} := \frac{k'}{pk^p n^p G_0}\left(1 - \lambda \frac{pk^p n^p}{k'} \lim_{\xi \longrightarrow 0^+} \inf \frac{\int_0^T \sup_{(x_1,\cdots,x_n)\in\Omega(\xi)} F(t, x_1, \cdots, x_n)dt}{\xi^p}\right), \quad (68)$$

a sequence of weak solutions for system (system (1) exists, and it strongly converges to zero in the space X.

Proof. Fix $\lambda \in]\lambda_3, \lambda_4[$ and let G be a function satisfying (67). Since $\lambda < \lambda_4$, one has

$$\mu'_{G,\bar{\lambda}} = \frac{k'}{2k^2 n^2 G_0}\left(1 - \bar{\lambda} \frac{2k^2 n^2}{k'} \lim_{\xi \longrightarrow 0^+} \inf \frac{\int_0^T \sup_{(x_1,\cdots,x_n)\in\Omega(\xi)} F(t, x_1, \cdots, x_n)dt}{\xi^p}\right) > 0. \quad (69)$$

Fix $\bar{\mu} \in]0, \mu'_{G,\bar{\lambda}}[$ and put

$$J(t, \xi_1, \cdots, \xi_n) := F(t, \xi_1, \cdots, \xi_n) + \frac{\bar{\mu}}{\bar{\lambda}} G(t, \xi_1, \cdots, \xi_n), \quad (70)$$

for every $[t \in 0, T]$ and $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$. We take Φ, Ψ, and $I_{\bar{\lambda},\bar{\mu}}$ as Theorem 10 proof. We verify that $\lambda < 1/\gamma$. For this, let $\{\xi_k\}$ be a sequence of positive number such that $\xi_k \longrightarrow 0^+$ as $k \longrightarrow \infty$ and

$$\lim_{k \longrightarrow +\infty} \frac{\int_0^T \sup_{(x_1,\cdots,x_n)\in\Omega(\xi_K)} F(t, x_1, \cdots, x_n)dt}{\xi_k^p} = \lim_{\xi \longrightarrow 0^+} \inf \frac{\int_0^T \sup_{(x_1,\cdots,x_n)\in\Omega(\xi)} F(t, x_1, \cdots, x_n)dt}{\xi^p}. \quad (71)$$

Through the evidence $\inf_X \Phi = 0$ in addition to δ definition, we claim $\delta = \lim_{r \longrightarrow 0^+} \inf \varphi(r)$. Then, like in display (52) of Theorem 10 proof, $\delta < +\infty$ can be proven, and hence, $\bar{\lambda} < 1/\delta$.

Let $\bar{\lambda}$ be fixed. We conclude that at zero, there is no local minimum for the functional $I_{\bar{\lambda},\bar{\mu}}$. For this purpose, a sequence of positive numbers $\{\eta_k\}$ is supposed such that $\eta_k \longrightarrow 0^+$ when $k \longrightarrow \infty$ and choosing $\tau > 0$ such that

$$\frac{1}{\bar{\lambda}} < \tau < \frac{1}{\rho\Delta'} \frac{\int_{\theta T}^{(1-\theta)T} F\left(t, \Gamma(2-\alpha_1)\eta_k{}'^{\cdots\prime}\Gamma(2-\alpha_n)\eta_k\right)dt}{\eta_k^p}, \quad (72)$$

for every large enough $k \in \mathbb{N}$. Assume a sequence $\{\omega_k = (\omega_{1k}, \cdots, \omega_{nk})\}$ in the space X together with ω_{ik} given in (55). Remark that $\lambda\tau > 1$. Hence, like in (62) show, the following can be obtained:

$$I_{\bar{\lambda},\bar{\mu}}(\omega_k) \le \rho\Delta' \eta_k^p - \bar{\lambda}\int_{\theta T}^{(1-\theta)T} F(t, \Gamma(2-\alpha_1)\eta_k, \cdots, \Gamma(2 - \alpha_n)\eta_k)dt < \left(1 - \bar{\lambda}\tau\right)\eta_k^p \rho\Delta' < 0, \quad (73)$$

$\forall$ large enough $k \in \mathbb{N}$. Where $I_{\bar{\lambda},\bar{\mu}}(0) = 0$, this means at the point zero there is no local minimum for the functional $I_{\bar{\lambda},\bar{\mu}}$.

Therefore, part (c) of Lemma 6 ensures that there exists a sequence $\{u_k = (u_{1k}, \cdots, u_{nk})\}$ in the space X of critical points for $I_{\bar{\lambda},\bar{\mu}}$ that is convergent weakly to the point zero. According to the established truth $X - \longrightarrow (C([0, T], \mathbb{R}))^n$ is compact, we conclude that the critical points strongly converge to zero, and the proof is performed.

Remark 14. According to the conditions

$$\lim_{\xi \longrightarrow 0^+} \inf \frac{\int_0^T \sup_{(x_1,\cdots,x_n)\in\Omega(\xi)} F(t, x_1, \cdots, x_n)dt}{\xi^p} = 0,$$
$$\lim_{\xi \longrightarrow 0^+} \sup \frac{\int_{\theta T}^{(1-\theta)T} F\left(t, \Gamma(2-\alpha_1)\xi'^{\cdots'}\Gamma(2-\alpha_n)\xi\right)dt}{\xi^p} = +\infty, \quad (74)$$

ensures that $\forall\lambda > 0$ and $\mu \in [0, k'/pk^p n^p G_0[$ system $(1, 1)$ admits infinitely many weak solutions in the sapce X. Moreover, if $G_0 = 0$, the conclusion exists $\forall\lambda > 0$ and $\mu \ge 0$.

Now, the following example will be presented for illustrating the above result.

Example 15. Consider the system

$$\begin{cases} {}_tD_1^{0.75}\left(\Phi_3\left({}_0D_t^{0.75}x_1(t)\right)\right) = \lambda F_{x_1}(t, x_1(t), x_2(t)) + \mu G_{x_1}\left(t,_{x_1}(t),_{x_2}(t)\right) + h_1(x_1(t)) \text{ a.e. } t \in [0, 1], \\ {}_tD_1^{0.8}\left(\Phi_3\left({}_0D_t^{0.8}x_2(t)\right)\right) = \lambda F_{x_2}(t, x_1(t), x_2(t)) + \mu G_{x_2}\left(t,_{x_1}(t),_{x_2}(t)\right) + h_2(x_2(t)) \text{ a.e. } t \in [0, 1], \\ x_1(0) = x_2(0) = x_1(1) = x_2(1) = 0, \end{cases} \quad (75)$$

where $h_1(x_1) = 1/4(\sin x_1)^2$ and $h_2(x_2) = 1/9x_2^2$. Moreover, for all $(t, x_1, x_2) \in [0, 1] \times \mathbb{R}^2$, let $F : [0, 1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined as

$$F\left(t, x_1{}'x_2\right) := \begin{cases} 0, & \text{for all } (t, x_1, x_2) \in [0, 1] \times \{0\}^2, \\ a(t)x_1^2(1 - \sin(\ln(|x_1|))) + b(t)x_2^2(1 - \cos(\ln(x))), & \text{for all } (t, x_1, x_2) \in [0, 1] \times (\mathbb{R} - \{0\})^2, \end{cases} \quad (76)$$

where $a, b : [0, 1] \longrightarrow \mathbb{R}$ are nonnegative continuous functions. Let $\theta = 1/4$. We observe that

$$\lim_{\xi \longrightarrow 0^+} \inf \frac{\int_0^1 \sup_{1/3\left(|x_1|^3+|x_2|^3\right)\le\xi} F\left(t, x_1{}'x_2\right)dt}{\xi^3} = 0,$$
$$\lim_{\xi \longrightarrow 0^+} \sup \frac{\int_{1/4}^{3/4} F(t, \Gamma(1.25)\xi, \Gamma(0.2)\xi)dt}{\xi^3} = +\infty. \quad (77)$$

Now, let G: $[0, T] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function defined by

$$G\left(t, x_1{}'x_2\right) = 1 - \cos(x_1x_2). \quad (78)$$

By definition, $G \in C^1(\mathbb{R}^2)$ and

$$\lim_{\xi \longrightarrow 0^+} \sup \frac{\int_0^1 \sup_{1/3\left(|x_1|^3+|x_2|^3\right)\le\xi} G\left(t, x_1{}'x_2\right)dt}{\xi^3} = 0 < \infty. \quad (79)$$

All hypotheses of Remark 14 are satisfied. Then, for all $(\lambda, \mu) \in]0,+\infty[\times [0,+\infty[$, system (75) admits a sequence of weak solutions which strongly converges to 0 in $E_{0.75}^3 \times E_{0.8}^3$.

Data Availability

No data were used to support the study.

Conflicts of Interest

This work does not have any conflicts of interest.

Acknowledgments

The fourth author extends their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through a research group program under grant (R.G.P-2/1/42).

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