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On Existence Theorems for Some Generalized Nonlinear Functional-Integral Equations with Applications

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Abstract. In the present paper, utilizing the techniques of suitable measures of noncompactness in Banach algebra, we prove an existence theorem for nonlinear functional-integral equation which contains as particular cases several integral and functional-integral equations that appear in many branches of nonlinear analysis and its applications. We employ the fixed point theorems such as Darbo's theorem in Banach algebra concerning the estimate on the solutions. We also provide a nontrivial example that explains the generalizations and applications of our main result.

1. Introduction

Functional integral equations of various types lead as a fascinating and important branch of nonlinear analysis and find numerous applications in describing of miscellaneous real world problems. Nonlinear integral equations are often investigated in research papers and monographs (see [1, 2, 9, 14, 18, 20, 21, 25–30] and the references therein).

In this paper, we study the existence of solutions of nonlinear functional-integral equation

$$x(t) = \left(q(t) + f(t, x(t), x(\theta(t))) + F\left(t, x(t), \int_0^t u(t, s, x(\acute{a}(s)))ds, x(b(t)) \right) \right) \times \left(g(t, x(t), x(\zeta(t))) + G\left(t, x(t), \int_0^a v(t, s, x(c(s)))ds, x(d(t)) \right) \right), \quad (1)$$

for $t \in [0, a]$.
Maleknejad et al. [23, 24] investigated the existence of solutions for the nonlinear functional-integral

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equations

$$x(t) = g(t, x(t)) + f\left(t, \int_0^t u(t, s, x(s))ds, x(\alpha(t))\right), \tag{2}$$

and

$$x(t) = f(t, x(\alpha(t)) \int_0^t u(t, s, x(s))ds, \tag{3}$$

respectively, by employing the Darbo fixed-point theorem with respect to measure of noncompactness defined in [3]. Banaś and Sadarangani [8] as well as Maleknejad et al. [22] discussed the existence of solutions for nonlinear functional-integral equation

$$f\left(t, \int_0^t v(t, s, x(s))ds, x(\alpha(t))\right) \cdot g\left(t, \int_0^a u(t, s, x(s))ds, x(\beta(t))\right). \tag{4}$$

Banaś and Rzepka [6, 7] studied the existence of solutions of nonlinear functional-integral equation and nonlinear quadratic Volterra integral equation

$$x(t) = f(t, x(t)) \int_0^t u(t, s, x(s))ds, \tag{5}$$

$$x(t) = p(t) + f(t, x(t)) \int_0^t v(t, s, x(s))ds, \tag{6}$$

respectively. The well known nonlinear Volterra integral equation and Urysohn integral equation are given as follows

$$x(t) = a(t) + \int_0^t u(t, s, x(s))ds, \tag{7}$$

$$x(t) = b(t) + \int_0^a v(t, s, x(s))ds, \tag{8}$$

respectively. Dhage [15] discussed the following nonlinear integral equation

$$x(t) = a(t) \int_0^a v(t, s, x(s))ds + \left(\int_0^t u(t, s, x(s))ds\right) \cdot \left(\int_0^a v(t, s, x(s))ds\right). \tag{9}$$

Deepmala and Pathak [12, 13] examined the existence of the solutions for nonlinear functional-integral equation

$$x(t) = \left(u(t, x(t)) + f\left(t, \int_0^t p(t, s, x(s))ds, x(\alpha(t))\right)\right) \cdot g\left(t, \int_0^a q(t, s, x(s))ds, x(\beta(t))\right). \tag{10}$$

Moreover, the famous quadratic integral equation of Chandrasekhar type [10] has the form

$$x(t) = 1 + x(t) \int_0^a \frac{t}{t+s} \phi(s) x(s) ds, \tag{11}$$

which is applied in the theories of radiative transfer, neutron transport and kinetic energy of gases (see [10, 17, 19]).
The equation (1) is of interest, since it is fairly general and includes equations (2)–(11) as special cases, those were applicable in several real world problems of engineering, mechanics, physics, economics and so on, for instance (cf. [5, 10–13, 17, 19]). After all, equation (1) also covers the well known functional equation of the first order having the form $x(t) = f(t, x(t), x(\theta(t)))$. The goal here, is to study the solvability of existence of solutions of certain nonlinear functional-integral equation (1), by utilizing the fixed point theorem for the product of two operators which satisfies the Darbo condition with respect to a measure of noncompactness in the Banach algebra of continuous functions in the interval [0, a].

2. Preliminaries

In this section, we gather some facts which will be needed in our further considerations.
Let E is a real Banach space with the norm $\|\cdot\|$ and zero element θ'. Denote by $B(x, r)$ the closed ball centered at x and with radius r as well as the symbol B_r stands for the ball $B(\theta', r)$. The notation $\mathcal{M}_E$ denotes the family of all nonempty and bounded subsets of E and notation $\mathcal{N}_E$ denotes its subfamily consisting of all relatively compact subsets. Moreover, if X is a nonempty subset of E we write $\bar{X}, ConvX$ in order to denote the closure and convex closure of X, respectively. We denote the standard algebraic operations on sets by the symbols λX and $X + Y$.
We use the following definition on the concept of a measure of noncompactness [3].

Definition 2.1. *Let $X \in \mathcal{M}_E$ and*

$$\mu(X) = \inf \left\{ \delta > 0 : X = \bigcup_{i=1}^{m} X_i \ \ with \ \ diam(X_i) \leq \delta, \ i = 1, 2, ...m \right\},$$

where for a fixed number $t \in [0, a]$, we denote

$$diam\ X(t) = \sup\{|x(t) - y(t)| : x, y \in X\}.$$

Clearly, $0 \leq \mu(X) < \infty$. $\mu(X)$ is called the Kuratowski measure of noncompactness.

Theorem 2.2. *Let $X, Y \in \mathcal{M}_E$ and $\lambda \in \mathbb{R}$. Then*

(i) $\mu(X) = 0$ *if and only if* $X \in \mathcal{N}_E$*;*

(ii) $X \subseteq Y \Rightarrow \mu(X) \leq \mu(Y)$*;*

(iii) $\mu(\bar{X}) = \mu(ConvX) = \mu(X)$*;*

(iv) $\mu(X \cup Y) = \max\{\mu(X), \mu(Y)\}$*;*

(v) $\mu(\lambda X) = |\lambda| \mu(X)$*, where* $\lambda X = \{\lambda x : x \in X\}$*;*

(vi) $\mu(X + Y) \leq \mu(X) + \mu(Y)$*, where* $X + Y = \{x + y : x \in X, y \in Y\}$*;*

(vii) $|\mu(X) - \mu(Y)| \leq 2d_h(X, Y)$, *where* $d_h(X, Y)$ *denotes the Hausdorff metric of* X *and* Y, *i.e.*

$$d_h(X, Y) = \max\left\{\sup_{y \in Y} d(y, X), \sup_{x \in X} d(x, Y)\right\},$$

where $d(.,.)$ *is the distance from an element of* E *to a set of* E.

Further on, every function $\mu : \mathcal{M}_E \to [0, \infty)$, satisfying conditions (i)-(vi) of Theorem 2.2, will be called a regular measure of noncompactness in the Banach space E (cf. [6]).

Now let us suppose that Ω is a nonempty subset of a Banach space E and $S : \Omega \to E$ is a continuous operator, which transforms bounded subsets of Ω to bounded ones. Moreover, let μ be a regular measure of noncompactness in E.

Definition 2.3. *(see [3]) We say that* S *satisfies the Darbo condition with a constant* K *with respect to measure* μ *provided*

$$\mu(SX) \leq K\mu(X)$$

for each $X \in \mathcal{M}_E$ *such that* $X \subset \Omega$.
If $K < 1$, *then* S *is called a contraction with respect to* μ.

In the sequel, we will work in the space $C[0, a]$ consisting of all real functions defined and continuous on the interval $[0, a]$. The space $C[0, a]$ is equipped with standard norm

$$\|x\| = \sup\{|x(t)| : t \in [0, a]\}.$$

Obviously, the space $C[0, a]$ has also the structure of Banach algebra.
In our considerations, we will use a regular measure of noncompactness defined in [4] (cf. also [3]).
In order to recall the definitions of that measure let us fix a set $X \in \mathcal{M}_{C[0,a]}$. For $x \in X$ and for a given $\epsilon > 0$ denote by $w(x, \epsilon)$ the modulus of continuity of x, i.e.,

$$w(x, \epsilon) = \sup\{|x(t) - x(s)| : t, s \in [0, a]; |t - s| \leq \epsilon\}.$$

Further, put

$$w(X, \epsilon) = \sup\{w(x, \epsilon) : x \in X\},$$

$$w_0(X) = \lim_{\epsilon \to 0} w(X, \epsilon).$$

It can be shown, as in [4] that the function $w_0(X)$ is a regular measure of noncompactness in the space $C[0, a]$.
For our purposes we will require the following lemma and theorem [4, 16].

Lemma 2.4. *Let* D *be a bounded, closed and convex subset of* E. *If operator* $S : D \to D$ *is a strict set contraction, then* S *has a fixed point in* D.

Theorem 2.5. *Assume that* Ω *is a nonempty, bounded, convex and closed subset of* $C[0, a]$ *and the operators* P *and* T *transform continuously the set* Ω *into* $C[0, a]$ *in such a way that* $P(\Omega)$ *and* $T(\Omega)$ *are bounded. Moreover, assume that the operator* $S = P \cdot T$ *transform* Ω *into itself. If the operators* P *and* T *each satisfies the Darbo condition on the set* Ω *with the constants* K_1 *and* K_2, *respectively, then the operator* S *satisfies the Darbo condition on* Ω *with the constant*

$$\|P(\Omega)\|K_2 + \|T(\Omega)\|K_1.$$

Remark 2.6. *In Theorem 2.5, if* $\|P(\Omega)\|K_2 + \|T(\Omega)\|K_1 < 1$, *then* S *is a contraction with respect to the measure* w_0 *and has at least one fixed point in the set* Ω.

This property will permit us to identify solutions of the integral equation (1).

3. Main Results

In this section, we will study the solvability of the nonlinear functional-integral equation (1) for $x \in C[0,a]$, assuming that the following hypotheses are satisfied:

(A_1) The function $q : [0,a] \to \mathbb{R}$ is continuous with

$$k = \sup_{t\in[0,a]} |q(t)| < \infty.$$

(A_2) The functions $f, g : [0,a] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $F, G : [0,a] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous.

(A_3) There exists the continuous functions $a_j : [0,a] \to [0,a]$, for $j = 1, 2, ...10$ such that

$$\begin{aligned}
|f(t,x_1,x_2) - f(t,y_1,y_2)| &\le a_1(t)|x_1 - y_1| + a_2(t)|x_2 - y_2|,\\
|g(t,x_1,x_2) - g(t,y_1,y_2)| &\le a_3(t)|x_1 - y_1| + a_4(t)|x_2 - y_2|,\\
|F(t,x_1,y_1,x_2) - F(t,x_3,y_2,x_4)| &\le a_5(t)|x_1 - x_3| + a_6(t)|y_1 - y_2| + a_7(t)|x_2 - x_4|,\\
|G(t,x_1,y_1,x_2) - G(t,x_3,y_2,x_4)| &\le a_8(t)|x_1 - x_3| + a_9(t)|y_1 - y_2| + a_{10}(t)|x_2 - x_4|,
\end{aligned}$$

for all $t \in [0,a]$ and $x_1, x_2, x_3, x_4, y_1, y_2 \in \mathbb{R}$.

(A_4) The functions $u = u(t,s,x(\acute{a}(s)))$ and $v = v(t,s,x(c(s)))$ act continuously from the set $[0,a] \times [0,a] \times \mathbb{R}$ into $\mathbb{R}$. Moreover, the functions $\theta, \zeta, \acute{a}, b, c$ and d transform continuously the interval $[0,a]$ into itself.

(A_5) There exists a nonnegative constant K such that

$$K = \max_j \{a_j(t) : t \in [0,a]\},$$

for $j = 1, 2, ...10$.

(A_6) (Sublinear condition) There exists the constants ξ and η such that

$$\begin{aligned}
|u(t,s,x(\acute{a}(s)))| &\le \xi + \eta|x|,\\
|v(t,s,x(c(s)))| &\le \xi + \eta|x|,
\end{aligned}$$

for all $t, s \in [0,a]$ and $x \in \mathbb{R}$.

(A_7) $4\sigma\tau < 1$ and $a\eta \ge 1$, for $\sigma = 4K + Ka\eta$ and $\tau = k + l + Ka\xi + m$.

Remark 3.1. *In view of the above assumptions there exist nonnegative constants l, m such that*

$$\begin{aligned}
|f(t,0,0)| &\le l,\\
|g(t,0,0)| &\le l,\\
|F(t,0,0,0)| &\le m,\\
|G(t,0,0,0)| &\le m,
\end{aligned}$$

for all $t \in [0,a]$.

Now we proceed to formulate the main result of this paper.

Theorem 3.2. *Under the assumptions $(A_1) - (A_7)$, equation (1) has at least one solution in the Banach algebra $C = C[0,a]$.*

Proof. To prove this result using Theorem 2.5, we consider the operators P and T on the Banach algebra $C[0,a]$ in the following way:

$$(Px)(t) = q(t) + f(t,x(t),x(\theta(t))) + F\left(t, x(t), \int_0^t u(t,s,x(\acute{a}(s)))ds, x(b(t))\right),$$
$$(Tx)(t) = g(t,x(t),x(\zeta(t))) + G\left(t, x(t), \int_0^a v(t,s,x(c(s)))ds, x(d(t))\right),$$

for $t \in [0,a]$.
Now, from the assumptions (A_1), (A_2) and (A_4), it follows that P and T transforms the Banach algebra $C[0,a]$ into itself.
Further, let us define the operator S on the algebra $C[0,a]$ by putting

$$Sx = (Px)\cdot(Tx).$$

Obviously, S transforms $C[0,a]$ into itself.
Now, let us fix $x \in C[0,a]$, then using our assumptions for $t \in [0,a]$, we get

$$\begin{aligned}
|(Sx)(t)| &= |(Px)(t)| \times |(Tx)(t)| \\
&= \left| q(t) + f(t,x(t),x(\theta(t))) + F\left(t, x(t), \int_0^t u(t,s,x(\acute{a}(s)))ds, x(b(t))\right)\right| \\
&\quad \times \left| g(t,x(t),x(\zeta(t))) + G\left(t, x(t), \int_0^a v(t,s,x(c(s)))ds, x(d(t))\right)\right| \\
&\le \Bigg\{ k + |f(t,x(t),x(\theta(t))) - f(t,0,0)| + |f(t,0,0)| \\
&\quad + \left| F\left(t, x(t), \int_0^t u(t,s,x(\acute{a}(s)))ds, x(b(t))\right) - F(t,0,0,0)\right| + |F(t,0,0,0)| \Bigg\} \\
&\quad \times \Bigg\{ |g(t,x(t),x(\zeta(t))) - g(t,0,0)| + |g(t,0,0)| \\
&\quad + \left| G\left(t, x(t), \int_0^a v(t,s,x(c(s)))ds, x(d(t))\right) - G(t,0,0,0)\right| + |G(t,0,0,0)| \Bigg\}
\end{aligned}$$

$$\begin{aligned}
&\le \left(k + a_1(t)|x(t)| + a_2(t)|x(\theta(t))| + l + a_5(t)|x(t)| + a_6(t)\int_0^t |u(t,s,x(\acute{a}(s)))|ds + a_7(t)|x(b(t))| + m \right) \\
&\quad \times \left(a_3(t)|x(t)| + a_4(t)|x(\zeta(t))| + l + a_8(t)|x(t)| + a_9(t)\int_0^a |v(t,s,x(c(s)))|ds + a_{10}(t)|x(d(t))| + m \right) \\
&\le \{k + 4K\|x\| + l + Ka(\xi + \eta\|x\|) + m\} \cdot \{4K\|x\| + l + Ka(\xi + \eta\|x\|) + m\} \\
&\le \{(4K + Ka\eta)\|x\| + k + l + Ka\xi + m\}^2.
\end{aligned}$$

Let $\sigma = 4K + Ka\eta$ and $\tau = k + l + ka\xi + m$, then from the above estimate, it follows that

$$\|Px\| \leq \sigma\|x\| + \tau, \tag{12}$$
$$\|Tx\| \leq \sigma\|x\| + \tau, \tag{13}$$
$$\|Sx\| \leq (\sigma\|x\| + \tau)^2, \tag{14}$$

for $x \in C[0, a]$.
From estimate (14), we deduce that the operator S maps the ball $B_r \subset C[0, a]$ into itself for $r_1 \leq r \leq r_2$, where

$$r_1 = \frac{1 - 2\sigma\tau - \sqrt{1 - 4\sigma\tau}}{2\sigma^2},$$
$$r_2 = \frac{1 - 2\sigma\tau + \sqrt{1 - 4\sigma\tau}}{2\sigma^2}.$$

In the following, we will assume that $r = r_1$.
Moreover, let us observe that from estimates (12) and (13), we obtain

$$\|PB_r\| \leq \sigma r + \tau, \tag{15}$$
$$\|TB_r\| \leq \sigma r + \tau. \tag{16}$$

Now, we prove that the operator P is continuous on the ball B_r. To do this, fix $\epsilon > 0$ and take arbitrary $x, y \in B_r$ such that $\|x - y\| \leq \epsilon$. Then for $t \in [0, a]$, we have

$$\begin{aligned}
|(Px)(t) - (Py)(t)| &\leq |f(t, x(t), x(\theta(t))) - f(t, y(t), y(\theta(t)))| \\
&\quad + \left| F\left(t, x(t), \int_0^t u(t, s, x(\acute{a}(s)))ds, x(b(t))\right) - F\left(t, y(t), \int_0^t u(t, s, y(\acute{a}(s)))ds, y(b(t))\right) \right| \\
&\leq a_1(t)|x(t) - y(t)| + a_2(t)|x(\theta(t)) - y(\theta(t))| \\
&\quad + \left| F\left(t, x(t), \int_0^t u(t, s, x(\acute{a}(s)))ds, x(b(t))\right) - F\left(t, y(t), \int_0^t u(t, s, x(\acute{a}(s)))ds, y(b(t))\right) \right| \\
&\quad + \left| F\left(t, y(t), \int_0^t u(t, s, x(\acute{a}(s)))ds, y(b(t))\right) - F\left(t, y(t), \int_0^t u(t, s, y(\acute{a}(s)))ds, y(b(t))\right) \right| \\
&\leq a_1(t)|x(t) - y(t)| + a_2(t)|x(\theta(t)) - y(\theta(t))| + a_5(t)|x(t) - y(t)| + a_7(t)|x(\theta(t)) - y(\theta(t))| \\
&\quad + a_6(t) \int_0^t |u(t, s, x(\acute{a}(s))) - u(t, s, y(\acute{a}(s)))|ds \\
&\leq 4K\|x - y\| + Ka\, w(u, \epsilon) \\
&\leq 4K\epsilon + Ka\, w(u, \epsilon),
\end{aligned}$$

where

$$w(u, \epsilon) = \sup\{|u(t, s, x) - u(t, s, y)| : t, s \in [0, a]; x, y \in [-r, r]; \|x - y\| \leq \epsilon\}.$$

Since, we know that the function $u = u(t, s, x)$ is uniformly continuous on the bounded subset $[0, a] \times [0, a] \times [-r, r]$, we conclude that $w(u, \epsilon) \to 0$ as $\epsilon \to 0$. Thus, the above estimate shows that the operator P is continuous on B_r. Similarly, one can easily show that the operator T is continuous on B_r. Hence we conclude that

S is a continuous operator on B_r.
Next, we show that the operators P and T satisfies the Darbo condition with respect to the measure w_0, defined in Section 2, in the ball B_r. To do this, we take a nonempty subset X of B_r and $x \in X$, Let $\epsilon > 0$ be fixed and $t_1, t_2 \in [0, a]$ such that $t_1 \leq t_2$ and $t_2 - t_1 \leq \epsilon$. Then, in view of imposed assumptions, we have

$$\begin{aligned}
|(Px)(t_2) - (Px)(t_1)| &\leq |q(t_2) - q(t_1)| + |f(t_2, x(t_2), x(\theta(t_2))) - f(t_1, x(t_1), x(\theta(t_1)))| \\
&\quad + \left| F\left(t_2, x(t_2), \int_0^{t_2} u(t_2, s, x(\acute{a}(s)))ds, x(b(t_2)) \right) \right. \\
&\quad \left. - F\left(t_1, x(t_1), \int_0^{t_1} u(t_1, s, x(\acute{a}(s)))ds, x(b(t_1)) \right) \right| \\
&\leq w(q, \epsilon) + |f(t_2, x(t_2), x(\theta(t_2))) - f(t_2, x(t_1), x(\theta(t_1)))| + |f(t_2, x(t_1), x(\theta(t_1))) \\
&\quad - f(t_1, x(t_1), x(\theta(t_1)))| + \left| F\left(t_2, x(t_2), \int_0^{t_2} u(t_2, s, x(\acute{a}(s)))ds, x(b(t_2)) \right) \right. \\
&\quad \left. - F\left(t_2, x(t_1), \int_0^{t_1} u(t_1, s, x(\acute{a}(s)))ds, x(b(t_1)) \right) \right| \\
&\quad + \left| F\left(t_2, x(t_1), \int_0^{t_1} u(t_1, s, x(\acute{a}(s)))ds, x(b(t_1)) \right) \right. \\
&\quad \left. - F\left(t_1, x(t_1), \int_0^{t_1} u(t_1, s, x(\acute{a}(s)))ds, x(b(t_1)) \right) \right| \\
&\leq w(q, \epsilon) + a_1(t)|x(t_2) - x(t_1)| + a_2(t)|x(\theta(t_2)) - x(\theta(t_1))| + w_1(f, \epsilon) + a_5(t)|x(t_2) - x(t_1)| \\
&\quad + a_6(t) \left| \int_0^{t_2} u(t_2, s, x(\acute{a}(s)))ds - \int_0^{t_1} u(t_1, s, x(\acute{a}(s)))ds \right| + a_7(t)|x(b(t_2)) - x(b(t_1))| \\
&\quad + w_1(F, \epsilon) \\
&\leq w(q, \epsilon) + 2Kw(x, \epsilon) + Kw(x, w(\theta, \epsilon)) + w_1(f, \epsilon) \\
&\quad + K\left\{ \int_0^{t_1} |u(t_2, s, x(\acute{a}(s))) - u(t_1, s, x(\acute{a}(s)))|ds + \int_{t_1}^{t_2} |u(t_2, s, x(\acute{a}(s)))|ds \right\} \\
&\quad + Kw(x, w(b, \epsilon)) + w_1(F, \epsilon) \\
w(Px, \epsilon) &\leq w(q, \epsilon) + 2Kw(x, \epsilon) + Kw(x, w(\theta, \epsilon)) + w_1(f, \epsilon) + K\{w_1(u, \epsilon)a + K'\epsilon\} \\
&\quad + Kw(x, w(b, \epsilon)) + w_1(F, \epsilon),
\end{aligned} \tag{17}$$

where

$$\begin{aligned}
w_1(f, \epsilon) &= \sup\{|f(t, x_1, x_2) - f(t', x_1, x_2)| : t, t' \in [0, a]; |t - t'| \leq \epsilon; x_1, x_2 \in [-r, r]\}, \\
w_1(u, \epsilon) &= \sup\{|u(t, s, x) - u(t', s, x)| : t, t', s \in [0, a]; |t - t'| \leq \epsilon; x \in [-r, r]\}, \\
w_1(F, \epsilon) &= \sup\{|F(t, x_1, y_1, x_2) - F(t', x_1, y_1, x_2)| : t, t' \in [0, a]; |t - t'| \leq \epsilon; x_1, x_2 \in [-r, r]; y_1 \in [-K'a, K'a]\}, \\
K' &= \sup\{|u(t, s, x)| : t, s \in [0, a];\ x \in [-r, r]\}.
\end{aligned}$$

Observe that invoking the uniform continuity of the functions $q = q(t)$, $f = f(t, x_1, x_2)$ and $F = F(t, x_1, y_1, x_2)$ on the set $[0, a], [0, a] \times \mathbb{R} \times \mathbb{R}$ and $[0, a] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, respectively, and the function $u = u(t, s, x)$ is uniformly continuous on the set $[0, a] \times [0, a] \times \mathbb{R}$. Hence, we deduce that $w(q, \epsilon) \to 0, w_1(f, \epsilon) \to 0, w_1(u, \epsilon) \to 0$ and $w_1(F, \epsilon) \to 0$ as $\epsilon \to 0$. Thus, from the above estimate (17) we conclude

$$w_0(PX) \leq 4Kw_0(X). \tag{18}$$

Similarly, we can show that

$$w_0(TX) \leq 4Kw_0(X). \tag{19}$$

Finally, in view of the estimates (15), (16), (18), (19) and keeping in mind Theorem 2.5, we infer that the operator S satisfies the Darbo condition on B_r with respect to the measure w_0 with constant $4K(\sigma r + \tau) + 4K(\sigma r + \tau)$. Thus, we have

$$\begin{aligned} 8K(\sigma r + \tau) &= 8K(\sigma r_1 + \tau) \\ &= 8K\left\{\sigma\left(\frac{(1 - 2\sigma\tau) - \sqrt{1 - 4\sigma\tau}}{2\sigma^2}\right) + \tau\right\} \\ &= \frac{4K}{\sigma}(1 - \sqrt{1 - 4\sigma\tau}). \end{aligned}$$

Under the assumption (A_7), we know that $1 - \sqrt{1 - 4\sigma\tau} < 1$ and $\dfrac{4K}{\sigma} = \dfrac{4K}{4K + Ka\eta} < 1$. Hence, the operator S is a contraction on B_r with respect to measure w_0. Thus, by applying Theorem 2.5 and Remark 2.6 we get that S has at least one fixed point in B_r. Consequently, the nonlinear functional-integral equation (1) has at least one solution in B_r. □

4. An Example

Now, we present an example of a functional-integral equation and consequently, see the existence of its solutions by using Theorem 3.2.

Example 4.1. *Consider the following nonlinear functional integral equation:*

$$\begin{aligned} x(t) = &\left[\frac{t}{4}e^{-t^2/2} + \frac{t^2}{8(1+t^2)}\sin x(t) + \frac{t}{9}\arctan\left|x\left(\frac{1}{2+t}\right)\right| + \frac{t}{7}\int_0^t \left\{\frac{t\sin x(\sqrt{s})}{2} + (2+t)\ln(1 + |x(\sqrt{s})|)\right\}ds\right] \\ &\times\left[\frac{t^2}{12}\arctan|x(t)| + \frac{t}{4(1+t)}\ln(1 + |x(\sqrt[3]{t})|) + \frac{1}{14}\int_0^1 \left\{\frac{\cos(x(1-s))}{2} + 3t^2\arctan\left(\frac{|x(1-s)|}{1 + |x(1-s)|}\right)\right\}ds\right], \end{aligned} \tag{20}$$

where $t \in [0, 1]$.

Observe that equation (20) *is a particular case of equation* (1). *Let us take* $q : [0, 1] \to \mathbb{R}; f, g : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}; F, G : [0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *and* $u, v : [0, 1] \times [0, 1] \times \mathbb{R} \to \mathbb{R}$ *and comparing* (20) *with equation* (1), *we get*

$$q(t) = \frac{t}{4}e^{-t^2/2},$$

$$f(t, x_1, x_2) = \frac{t^2}{8(1+t^2)}\sin x_1 + \frac{t}{9}\arctan|x_2|,$$

$$g(t, x_1, x_2) = \frac{t^2}{12} \arctan|x_1| + \frac{t}{4(1+t)} \ln(1 + |x_2|),$$

$$F(t, x_1, y_1, x_2) = \frac{t}{7} y_1,\ G(t, x_1, y_1, x_2) = \frac{1}{14} y_1,$$

$$u(t, s, x) = \frac{t \sin x}{2} + (2 + t) \ln(1 + |x|),$$

$$v(t, s, x) = \frac{\cos x}{2} + 3t^2 \arctan\left(\frac{|x|}{1 + |x|}\right),$$

then we can easily check that the assumptions of Theorem 3.2 are satisfied. In fact, we have that the function $q(t)$ *is continuous and bounded on* $[0, 1]$ *with* $k = \frac{e^{-1/2}}{4} = 0.15163\ldots$ *. Thus, the assumption* (A_1) *is satisfied. Moreover, these functions are continuous and satisfies the assumption* (A_3) *with* $a_1 = \frac{1}{16}, a_2 = \frac{1}{9}, a_3 = \frac{1}{12}, a_4 = \frac{1}{8}, a_5 = a_7 = a_8 = a_{10} = 0, a_6 = \frac{1}{7}, a_9 = \frac{1}{14}$.
In this case, we have

$$K = \max\left\{\frac{1}{16}, \frac{1}{9}, \frac{1}{12}, \frac{1}{8}, 0, \frac{1}{7}, \frac{1}{14}\right\} = \frac{1}{7}.$$

Further,

$$|u(t, s, x)| \le \frac{1}{2} + 3|x|,$$

$$|v(t, s, x)| \le \frac{1}{2} + 3|x|.$$

It is observed that $l = m = 0, \xi = \frac{1}{2}, \eta = 3$ *and* $a = 1$.
Finally, we see that

$$4\sigma\tau = 4(4K + Ka\eta)(k + l + Ka\xi + m) < 1.$$

Hence, all the assumptions from (A_1) *to* (A_7) *are satisfied. Now, based on result obtained in Theorem 3.2, we deduce that equation* (20) *has at least one solution in Banach algebra* $C[0, 1]$.

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