20. On Exponential Semigroups. I[1)]

By Takayuki TAMURA*) and John SHAFER**)

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1. Introduction. By a semilattice-congruence ξ on a semigroup S we mean a congruence on S such that S/ξ is a semilattice, i.e., a commutative idempotent semigroup. As is well known there is a smallest semilattice-congruence ρ on S in the sense that ρ is contained in all semilattice-congruences ξ on S. If $\rho = S \times S$, then S is called semilattice-indecomposable.

A semigroup is called medial if it satisfies the identity $xyzu = xzyu$; a semigroup A is called archimedean if for every $a, b \in A$ there are elements x, y, z, u of A^1 and positive integers m, n such that $xay = b^m$ and $zbu = a^n$. Chrislock [1, 2] proved the following:

Theorem 1. *Every medial semigroup S is a semilattice of archimedean semigroups, and the congruence relation ρ on S which induces this decomposition is a smallest semilattice-congruence on S and ρ is given by $a\rho b$ if and only if $xay = b^m$ and $zbu = a^n$ for some $x, y, z, u \in S^1$ and some $m, n > 0$. A medial semigroup is semilattice-indecomposable if and only if it is archimedean.*

Theorem 2. *Let S be a medial semigroup. S is an archimedean semigroup with idempotent if and only if S is an ideal extension of the direct product I of an abelian group G and a rectangular band B by the medial nil-semigroup N.*

The above Theorems 1, 2 were generalizations of Kimura and Tamura's result for commutative semigroups [5]. The purpose of this note is to extend Chrislock's results to the exponential case. Furthermore the authors intend the development of Theorem 2 not only in the medial case, but in the exponential case. In Theorem 2 mediality is assumed in the "if" part and "only if" part. If mediality is assumed in the "only if" part, then the "only if" part is still true, but the "if" part is not true in general, that is to say, an ideal extension of $I = G \times B$ by medial N need not be medial. Part I of this paper establishes the extension of Theorem 1 and the "only if" part of Theorem 2. The "if" part will be discussed in Part II.

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1) This paper was partially obtained by J. Shafer, one of the authors, in 1969.

*) University of California, Davis, California, U. S. A.

**) University of Massachusetts, Amherst, Massachusetts, U. S. A.

2. Semilattice-decomposition of exponential semigroups. A semigroup S is called exponential if S satisfies the identities $(xy)^n = x^n y^n$ for all positive integers n. Clearly a medial semigroup is exponential.

Proposition[2] **3.** *The identity $(xy)^n = x^n y^n$ holds for all positive integers n if and only if it holds for $n=2,3$.*

We will show that if the identity holds for $n=2, m, m+1$, and $m+2$, where m is a positive integer, then it holds for $n=m+3$.

$$\begin{aligned} x^{m+3}y^{m+3} &= xx^{m+2}y^{m+2}y = x(xy)^{m+2}y = x^2(yx)^{m+1}y^2 = x^2y^{m+1}x^{m+1}y^2 \\ &= x^2y^2y^{m-1}x^{m-1}x^2y^2 = (xy)^2y^{m-1}x^{m-1}(xy)^2 = xyxy^m x^m yxy \\ &= xyx(yx)^m yxy = (xy)^{m+3}. \end{aligned}$$

Note that if $m=1$, x^{m-1} and y^{m-1} are regarded as void, but these equalities are still true. If the identity holds for $n=2,3$, then it holds for $n=2,3,4$, hence it holds for all n.

If S is exponential, then $(x_1 \cdots x_n)^m = x_1^m \cdots x_n^m$ for all $x_i \in S$ $(i=1, \cdots, n)$ and all positive integers m.

Let S be an exponential semigroup. Define ρ on S by $a\rho b$ if and only if $xay = b^m$ for some $x, y \in S^1$ and positive integer m. Then define $\tilde{\rho}$ by $\tilde{\rho} = \rho \cap \rho^{-1}$, i.e.,

$$a\tilde{\rho}b \text{ if and only if } a\rho b \text{ and } b\rho a.$$

Theorem 4. *The relation $\tilde{\rho}$ is a smallest semilattice-congruence on an exponential semigroup S and each $\tilde{\rho}$-class is an exponential archimedean semigroup. An exponential semigroup is semilattice-indecomposable if and only if it is archimedean.*

First $\tilde{\rho}$ is reflexive since $aaa = a^3$. Symmetry is by definition. To show the transitivity of ρ, let $a\rho b$ and $b\rho c$, i.e., $xay = b^m$, $zbw = c^k$ for some $x, y, z, w \in S^1$ and some $m, k > 0$. Then we get

$$z^m b^m w^m = c^{mk} \quad \text{and} \quad z^m xayw^m = c^{mk}$$

whence $a\rho c$. It follows that $\tilde{\rho}$ is transitive and hence an equivalence. To prove compatibility of $\tilde{\rho}$, we may only show that of ρ. First it is easy to see that $a^2\rho a$ and $a\rho a^2$ for all $a \in S$, and then $xy \ \rho \ yxyx = (yx)^2 \ \rho \ yx$. Hence we get $xy\rho yx$ for all $x, y \in S$. Let $a\rho b$ and $c \in S$, and $xay = b^m$. Then, by exponentiality of S, $(cb)^{2m} = c^{2m}b^{2m} = c^{2m}(xay)^2 = c^{2m-2}c^2(xa)^2y^2 = c^{2m-2}(cxa)^2y^2 = c^{2m-1}x(ac)xay^2$ [3] whence $ac\rho cb$ and hence $ac\rho cb\rho bc$. Thus we get $ac\rho bc$. Also we have $ca\rho cb$ since $ca\rho ac\rho bc\rho cb$. Accordingly, $\tilde{\rho}$ is compatible and hence a congruence on S.

As mentioned above $a^2\tilde{\rho}a$ and $ab\tilde{\rho}ba$ for all $a, b \in S$. Thus $\tilde{\rho}$ is a semilattice-congruence on S. To see $\tilde{\rho}$ is smallest, let ξ be a semilattice-congruence on S. Assume $a\tilde{\rho}b$; $xay = b^m$ and $zbu = a^n$. Since $x^i\xi x$ and $xy\xi yx$ for all i, we have

$$a\xi a^n = zbu\xi zb^{m+1}u = zxaybu\xi xazbuy = xa^{n+1}y\xi xay = b^m\xi b,$$

2) Originally this was discovered by Prof. Naoki Kimura.

3) If $m=1, c^{2m-2}$ is regarded as void. These equalities are still effective.

hence $a\xi b$. We have proved $\tilde{\rho}\subseteq\xi$.

Let S_α be an $\tilde{\rho}$-class of S. It is obvious that S_α is exponential. Let ρ_α be the relation ρ for S_α and $\tilde{\rho}_\alpha$ the relation $\tilde{\rho}$ for S_α. To prove that S_α is archimedean we may show that for all $a, b \in S_\alpha, a\rho_\alpha b$ and $b\rho_\alpha a$. Since $a\rho b$, $xay=b^m$ implies $(bx)a(yb)=b^{m+2}$. Then bx and yb are in S_α because $bx\rho b$ and $b\rho bx$, and the same holds for yb. It follows that a, $b\in S_\alpha$ implies $a\rho_\alpha b$. Similarly $b\rho_\alpha a$, hence $a\tilde{\rho}_\alpha b$. Thus we have proved that S_α is archimedean. As $\tilde{\rho}$ is the smallest semilattice-congruence on S it is easy to see that S is archimedean if and only if S is semilattice-indecomposable.

3. Structure of exponential archimedean semigroups with idempotents. Let L and R be sets and $B=\{(x, y): x\in L, y\in R\}$. Define an operation on B by $(x, y)(z, u)=(x, u)$. The semigroup B is called a rectangular band. By a nil-semigroup we mean a semigroup with zero 0 such that some power of every element is 0.

Theorem 5. *If S is an exponential archimedean semigroup with idempotent, then S is an ideal extension of I by N where I is the direct product of an abelian group G and a rectangular band B and N is an exponential nil-semigroup.*

Let S be an exponential archimedean semigroup with idempotents and B the set of idempotents of S. Let $e, f\in B$. Then $(ef)^2=e^2f^2=ef$, so B is a subsemigroup of S. Let $e, f\in B$. As $e=xfy$ and $f=zeu$ for some $x, y, z, u\in S^1$, we get $SeS=SfS$ for all $e, f\in B$. Let $I=SeS$. I is clearly an ideal of S, furthermore it is the smallest ideal of S. In fact let J be an ideal of S. Let $x\in J$. Since $e\in I$, $e=vxw$ for some $v, w\in S^1$. Then $I=SeS\subseteq SxS\subseteq J$. If S contains a zero 0, then $I=\{0\}$, and by archimedeaness, some power of every element is 0 and hence S itself is nil. Therefore we may assume S does not have a zero. Hence I is a simple semigroup. (See [3].) Suppose that I is not completely simple. By Anderson's theorem [3], I contains a bicyclic subsemigroup C having an idempotent as identity element. By a bicyclic semigroup we mean a semigroup with identity element 1 generated by two symbols p, q subject to a relation $pq=1$, that is, it consists of q^mp^n, m, n non-negative integers, $q^mp^n=q^{m'}p^{n'}$ if and only if $m=m'$ and $n=n'$ and an operation is defined by

$$(q^kp^l)(q^mp^n)=q^ip^j;\quad i=k+m-\min(l, m),\quad j=l+n-\min(l, m).$$

Note that $qp\neq 1$. Since the identities are preserved by subsemigroups, C must be exponential. On the other hand since $pq=1$, $qp=qpqp=(qp)^2=q^2p^2$. This is a contradiction to its unique expression. Consequently I must be completely simple. Since the set B of idempotents of I forms a subsemigroup, by Ivan [3, 4], B is a rectangular band and I is isomorphic to the direct product of B and a group G. Since I is exponential, G is exponential, and hence abelian.

Let $N=S/I$ be the Rees factor semigroup. Since archimedeaness is preserved by homomorphisms, N is an exponential, archimedean semigroup with zero 0. It is easy to see that N is nil.

Remark. Recently M. S. Putcha has determined the biggest class of semigroups S each of which is a semilattice of archimedean semigroups. The following condition characterizes S.

If $a, b \in S$, a divides b, then a^2 divides some power of b.

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