On Exponential Sums in Finite Fields

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(dedicated to E. Szemeredi)

0 Introduction

The purpose of this paper is to establish certain multilinear exponential sums in arbitrary finite fields, extending some of the results from [B] for prime fields.

Let us first recall the main result from [B].

Theorem A. *Let $1 > \delta > 0$ and $r \in \mathbb{Z}_+, r \geq 2$. There is $\delta' > \left(\frac{\delta}{r}\right)^{Cr}$ such that if p is a sufficiently large prime and $A_1, \dots, A_r \subset \mathbb{F}_p$ satisfy*

$$|A_i| > p^\delta \text{ for } 1 \leq i \leq r \tag{0.1}$$

$$\prod_{i=1}^{r} |A_i| > p^{1+\delta}. \tag{0.2}$$

Then we have the exponential sum bound

$$\Big| \sum_{x_1 \in A_1, \dots, x_r \in A_r} e_p(x_1 \dots x_r) \Big| < p^{-\delta'} |A_1| \dots |A_r|. \tag{0.3}$$

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Consider now a field $\mathbb{F}_q, q = p^n$. An obvious issue one encounters with a generalization of Theorem A is the presence of non-trivial subfields. More surprisingly perhaps, it turns out that even if $\mathbb{F}_q$ has no large non-trivial subfields, the condition (0.2) still needs to be modified.

Theorem 3 on p. 20 below implies the following statement:

Theorem B. *Let $0 < \delta, \delta_2 < 1$ and $r \in \mathbb{Z}_+, r \geq 2$. Let $q = p^n$ be sufficiently large and $A_1, \dots, A_r \subset \mathbb{F}_q$ satisfy*

$$|A_i| > q^\delta \text{ for } 1 \leq i \leq r \tag{0.4}$$

$$|A_i \cap (aG + b)| < q^{-\delta_2}|A_i| \text{ for } 3 \leq i \leq r, \tag{0.5}$$

whenever $a, b \in \mathbb{F}_q$ and G a proper subfield

$$|A_1|.|A_2|.\prod_{i=3}^{r} |A_i|^{\frac{1}{2}} > q^{1+\delta}. \tag{0.6}$$

Then, denoting $\psi(x) = e_p(Tr_{\mathbb{F}_q/\mathbb{F}_p} x)$, we have

$$\Big| \sum_{x_1 \in A_1, \dots, x_r \in A_r} \psi(x_1 \dots x_r) \Big| < q^{-\delta'} |A_1| \dots |A_r| \tag{0.7}$$

where we may take $\delta' = C^{-\frac{r}{\delta_2}} \left(\frac{\delta}{r}\right)^{Cr}$.

Remarks.

(0.8) Condition (0.5) may in fact be replaced by

$$|A_i \cap (aG + b)| < |A_i|^{1-\delta_2} \quad (3 \leq i \leq r).$$

It follows in particular that if we fix the characteristic p and let n be prime, we may take $\delta' = \left(c\frac{\delta}{r}\right)^{Cr}$.

(0.9) Assume $|A_1| = \cdots = |A_r| = q^\sigma$. Condition (0.6) becomes then

$$\sigma > \frac{2}{r+2}. \tag{0.10}$$

This condition is in some sense optimal, as seen from the obvious example $\mathbb{F}_q = \mathbb{F}_{p^n} = \mathbb{F}_p[\xi], r = n - 1, A_1 = \cdots = A_r = \mathbb{F}_p + \xi\mathbb{F}_p, \sigma = \frac{2}{n}$.

Multilinear exponential sums arise naturally if one applies Weyl's differencing scheme to Gauss sums. More precisely, consider $B \subset \mathbb{F}_q$, $r \in \mathbb{Z}_+, r \geq 2$ and

$$S = \sum_{x \in B} \psi(x^r) \tag{0.11}$$

with ψ as above.

One obtains (cf. [Sch], Lemma 3.1)

$$|S|^{2^{r-1}} \leq |B - B|^{2^{r-1}-r} \sum_{x_1 \in B-B} \cdots \sum_{x_{r-1} \in B-B} \Big| \sum_{x_r \in B(x_1, \ldots, x_{r-1})} \psi(2^r r! x_1 \cdots x_r) \Big| \tag{0.12}$$

where

$$B(x_1, \ldots, x_{r-1}) = \bigcap_{\varepsilon_1 = 0}^{1} \cdots \bigcap_{\varepsilon_{r-1}=0}^{1} (B - \varepsilon_1 x_1 - \cdots - \varepsilon_{r-1} x_{r-1}). \tag{0.13}$$

If $B \subset \mathbb{F}_q$ is a linear subspace over $\mathbb{F}_p$, we derive immediately from Theorem B and the preceding

Theorem C. *Let $r \in \mathbb{Z}_+, r \geq 2, p > r$ and V a linear subspace of $\mathbb{F}_{p^n}$ over $\mathbb{F}_p$ of dimension*

$$m = \dim V > (1 + \delta) \frac{2n}{r+2}. \tag{0.14}$$

where $0 < \delta < 1$.

Assume further that

$$|V \cap aG| < q^{-\delta_2} p^m \tag{0.15}$$

if G is a proper subfield, $a \in \mathbb{F}_{p^n}^*$.

Then (assuming q large enough)

$$\max_{a \in \mathbb{F}_q^*} \Big| \sum_{x \in V} \psi(a x^r) \Big| < q^{-\delta'} p^m \text{ with } \delta' > C^{-\frac{r}{\delta_2}} \Big(\frac{\delta}{r}\Big)^{Cr}. \tag{0.16}$$

From Remark (0.9), we see that condition (0.14) on $\dim V$ is essentially optimal.

If $e_1, \ldots, e_n$ is an (arbitrary) basis of $\mathbb{F}_q$ over $\mathbb{F}_p$, we define a 'box' as a translate of a set

$$B = \{t_1 e_1 + \cdots + t_n e_n |\ 1 \leq t_i \leq H_i, 1 \leq i \leq n\} \tag{0.17}$$

where $1 \leq H_1, \dots, H_n \leq p$. For $H_1 = \cdots = H_n = H$, denote (0.17) by B_H. One easily verifies that if G is a proper subfield of $\mathbb{F}_q$ and $a, b \in \mathbb{F}_q$,

$$|B_H \cap (aG + b)| \leq |B_H|^{\frac{1}{2}}. \tag{0.18}$$

Also, in (0.13), the set $B_H(x_1, \dots, x_{r-1})$ ia a union of at most 2^{nr} boxes (0.17) with $H_i \leq H$.

From (0.12), one obtains therefore

$$|S|^{2^{r-1}} \leq 2^{nr}|B-B|^{2^{r-1}-r} \sum_{x_1,\dots,x_{r-1}\in B-B} \prod_{i=1}^{n} \min\Big(H, \Big\|\frac{2^r r!}{p} Tr(x_1 \dots x_{r-1}e_i)\Big\|^{-1}\Big). \tag{0.19}$$

Denote $\varphi = \varphi_H$ the function on $\mathbb{F}_p$

$$\varphi(z) = \min\Big(H, \Big\|\frac{z}{p}\Big\|^{-1}\Big), z \in \mathbb{F}_p \tag{0.20}$$

and

$$\hat{\varphi}(t) = \frac{1}{p} \sum_{0\leq z<p} \varphi(z) e_p(-tz). \tag{0.21}$$

Hence

$$\varphi(z) = \sum_{0\leq t\leq p} \hat{\varphi}(t) e_p(tz)$$

and

$$cH < \|\hat{\varphi}\|_1 < C(\log p)H \tag{0.22}$$

$$\|\hat{\varphi}\|_2 = \frac{1}{\sqrt{p}} \|\varphi\|_2 \sim c\sqrt{H}. \tag{0.23}$$

Thus

$$\begin{aligned}(0.19) &= 2^{nr}|B - B|^{2^{r-1}-r} \sum_{x_1,\dots,x_{r-1}\in B-B} \prod_{i=1}^{n} \varphi\big(2^r r!(Tr x_1 \dots x_{r-1}e_i)\big) \\ &= 2^{nr}|B - B|^{2^{r-1}-r} \sum_{x_1\dots x_{r-1}\in B-B} \sum_{x\in\mathbb{F}_q} \alpha(x)\psi(x_1 \dots x_{r-1}x) \end{aligned} \tag{0.24}$$

where

$$\alpha(x) = \prod_{i=1}^{n} \hat{\varphi}(t_i) \text{ for } x = t_1 e_1 + \cdots + t_n e_n$$

satisfies by (0.22), (0.23)

$$(cH)^n < \|\alpha\|_1 < C^n (\log p)^n H^n \tag{0.25}$$

$$\|\alpha\|_2 \sim C^n H^{n/2} \tag{0.26}$$

The double sum in (0.24) is estimated using Theorem 3 (stated on p. 20), taking $\alpha_1 = \frac{\alpha}{\|\alpha\|_1}$ and $\alpha_2 = \cdots = \alpha_r = \frac{1_{|B-B|}}{|B-B|}$.

Take $H = p^\sigma$ and σ satisfying

$$\sigma > (1+\delta)\frac{2}{r+2} \tag{0.27}$$

$(0 < \delta < 1)$. Assume $p > p(r, \delta)$. It follows from (0.18) that (8.3) holds with $\delta_2 = \frac{\sigma}{2}$. From (0.25), (0.26) and (0.27), (8.2) and (8.4) hold with $\delta = \min\left(\frac{\delta}{4}, \frac{\sigma}{2}\right)$. From (8.5), we obtain

$$|(0.24)| < C^{n2^r} |B|^{2^{r-1}-1} \|\alpha\|_1 q^{-\delta'} < C^{n2^r} (\log p)^n |B|^{2^{r-1}} q^{-\delta'} \tag{0.28}$$

with $\delta' > C^{-\frac{r}{\sigma}} \left(\frac{\delta}{r}\right)^{Cr}$.

Hence, we proved

Theorem D. *Let $q = p^n, r \in \mathbb{Z}_+, r \geq 2, 0 < \delta < 1$ and $p > p(r, \delta)$. Let $H = p^\sigma$, with*

$$\sigma > (1+\delta)\frac{2}{r+2} \tag{0.29}$$

and $B_H \subset \mathbb{F}_q$ the box as defined above. Then, with $\delta' > C^{-\frac{r}{\sigma}} \left(\frac{\delta}{r}\right)^{Cr}$

$$\max_{a \in \mathbb{F}_q^*} \Big| \sum_{x \in B_H} \psi(ax^r) \Big| < C^n (\log p)^{n2^{-r+1}} q^{-\delta'} H^n. \tag{0.30}$$

Remarks.

1. Both Theorem C and Theorem D remain of course valid if we replace ax^r by an arbitrary polynomial $f(x) = a_r x^r + a_{r-1} x^{r-1} + \cdots + a_0 \in \mathbb{F}_q[X]$ with $a_r \neq 0$, as r-fold Weyl differencing leads to the same multi-linear expression (0.12).

2. Theorem D should be compared with Theorem 2 from [Sch] on incomplete exponential sums in one and several variables (only the 1-variable result, i.e. $s = 1$ in the notation from [Sch], is of relevance here). In [Sch], Theorem 2, a nontrivial estimate on $\sum_{x\in B_H} \psi\big(f(x)\big)$ is obtained, $f(x) \in \mathbb{F}_q[X]$ as above, under the assumption

$$H = p^\sigma, \sigma > \frac{1}{r} \tag{0.31}$$

which is weaker than (0.29) (and optimal). However the result from [Sch] is not uniform in $n(q = p^n)$, in the sense that it requires $p > p(r, n)$, while (0.30) provides non-trivial bounds for $p > p(r)$ (assuming σ fixed). The method from [Sch] relies on geometry of numbers and the dependence on n results from dimensional factors in Minkowski's second theorem. Whether (or to what extent) they are avoidable in this particular application seems an interesting question.

The reminder of the paper is organized as follows:

In §1, we establish a 'sum-product' type result in a general finite field $\mathbb{F}_q$, which is the main new underlying ingredient (compared with [B]). The later sections are basically an adjustment from [B] to convert this set-theoretical property (Lemma 1 below) in bounds on convolutions and exponential sums.

1 A Sum-Product Property

The following will be the substitute for Lemma 2 in [B].

Lemma 1. *Let $X, Y \subset \mathbb{F}_q^*$ and assume Y not contained in a proper subfield of $\mathbb{F}_q$.*

There are elements $x_1, x_2, x_3, x_4 \in \pm X$ and $y_1, y_2, y_3, y_4 \in \pm Y \cup \{1\}$ such that for all $X' \subset X, Y' \subset Y$

$$|y_1X' + y_2X' + y_3X' + y_4X' + x_1Y' + x_2Y' + x_3Y' + x_4Y'| \geq \min\Big\{\frac{1}{6}|X'|\,|Y'|^{\frac{1}{2}}, q\Big(\frac{|X'|\,|Y'|}{|X|\,|Y|}\Big)^2\Big\}. \tag{1.1}$$

Remark. Assuming Y not contained in a multiplicative coset of a proper subfield, we may take y_1, y_2, y_3, y_4 above in $\pm Y$.

Proof of Lemma 1.

We may clearly assume $|X| > 1$.

Define

$$V = \frac{X - X}{Y - Y} \neq \{0\}$$

and notice that the properties

$$YV \subset V \tag{1.2}$$

$$V + V \subset V \tag{1.3}$$

can not both hold unless

$$V = \mathbb{F}_q. \tag{1.4}$$

Indeed, if (1.2) + (1.3), then V contains any sum of products of elements of Y and hence the field generated by Y, multiplied with V.

If (1.2) fails, there are $y_1, y_2, y_3 \in Y (y_1 \neq y_2)$ and $x_1, x_2 \in X$ such that $\xi = y_3 \frac{x_1 - x_2}{y_1 - y_2} \notin V$.

Hence, if $X' \subset X, Y' \subset Y, \xi \notin \frac{X' - X'}{Y' - Y'}$ implying

$$\begin{aligned} |X'|\,|Y'| &= |X' + \xi Y'| \\ &= |(y_1 - y_2)X' + y_3(x_1 - x_2)Y'| \\ &\leq |(y_1 - y_2)X' + y_3 X'|\,|(x_1 - x_2)Y' - X'|\,|X'|^{-1}. \end{aligned}$$

Hence, either

$$|(y_1 - y_2)X' + y_3 X'| \geq |X'|\,|Y'|^{\frac{1}{2}}$$

or

$$|X' + (x_2 - x_1)Y'| \geq |X'|\,|Y'|^{\frac{1}{2}}$$

and certainly

$$|X' + y_1 X' - y_2 X' + y_3 X' - x_1 Y' + x_2 Y'| \geq |X'|\,|Y'|^{\frac{1}{2}}. \tag{1.5}$$

If (1.3) fails, there are $x_1, x_2, x_3, x_4 \in X$ and $y_1, y_2, y_3, y_4 \in Y, y_1 \neq y_2, y_3 \neq y_4$ such that

$$\xi = \frac{x_1 - x_2}{y_1 - y_2} + \frac{x_3 - x_4}{y_3 - y_4} V.$$

Let $X' \subset X, Y' \subset Y$ and define

$$Z = \frac{x_1 - x_2}{y_1 - y_2} Y' \cup \frac{x_3 - x_4}{y_3 - y_4} Y'.$$

Write

$$|X' + Z| = K|X'|. \tag{1.6}$$

Applying Corollary 1.5 from [K-S], we obtain a subset $X'' \subset X'$, $|X''| > \frac{1}{2}|X'|$ and such that

$$|X'' + Z + Z| \leq 4K^2|X'|. \tag{1.7}$$

Hence

$$\begin{aligned} 4K^2|X'| &\geq \left| X'' + \frac{x_1 - x_2}{y_1 - y_2} Y' + \frac{x_3 - x_4}{y_3 - y_4} Y' \right| \\ &\geq |X'' + \xi Y'| \\ &= |X''|\,|Y'| \\ &\geq \frac{1}{2}|X'|\,|Y'| \end{aligned}$$

and

$$K > \frac{1}{\sqrt{8}}|Y'|^{\frac{1}{2}}. \tag{1.8}$$

Returning to (1.6), we showed

$$|(y_1 - y_2)X' + (x_1 - x_2)Y'| + |(y_3 - y_4)X' + (x_3 - x_4)Y'| \geq \frac{1}{\sqrt{8}}|X'|\,|Y'|^{\frac{1}{2}}$$

and therefore

$$\begin{aligned} &|y_1 X' - y_2 X' + y_3 X' - y_4 X' + x_1 Y' - x_2 Y' + x_3 Y' - x_4 Y'| \\ &\geq \frac{1}{2\sqrt{8}}|X'|\,|Y'|^{\frac{1}{2}}. \end{aligned} \tag{1.9}$$

Finally, assume (1.4).

Take $\xi \in V, \xi = \frac{x_1 - x_2}{y_1 - y_2}$, s.t.

$$\begin{aligned} &\left| \left\{ (x, x', y, y') \in X \times X \times Y \times Y \middle| \xi = \frac{x - x'}{y - y'} \right\} \right| \\ &\leq \frac{|X|^2\,|Y|^2}{q}. \end{aligned} \tag{1.10}$$

If $X' \subset X, Y' \subset Y$, we have

$$|X' + \xi Y'| \geq \frac{|X'|^2 |Y'|^2}{E_+(X', \xi Y')} \tag{1.11}$$

where

$$E_+(A, B) = |\{(a, a', b, b') \in A^2 \times B^2; a + b = a' + b'\}|$$

is the additive energy.

Clearly

$$E_+(X', \xi Y') \leq |X'|\,|Y'| + (1.10)$$

implying

$$|(y_1 - y_2)X' + (x_1 - x_2)Y'| \geq \frac{1}{2} \min\Big(|X'|.|Y'|, q\Big(\frac{|X'|\,|Y'|}{|X|\,|Y|}\Big)^2\Big). \tag{1.12}$$

Thus (1.1) holds again.

This proves Lemma 1. □

With Lemma 1 at hand, we may follow the method from [B] almost verbatim (the main steps with details of the modifications will be given). This part of the analysis in [B] does indeed not depend on the primality of the field. Of course, in the applications of Lemma 1, one has to ensure that the set Y under consideration is not contained in a proper subfield.

Recall the following property from additive combinatorics ([B], Lemma 3), which holds in the context of an arbitrary additive group.

Lemma 2. *Let $X_i \subset \mathbb{F}_q (1 \leq i \leq j)$ and $Y \subset \mathbb{F}_q$. There is $y_0 \in Y$ such that*

$$\Big|(Y - y_0) \cap \bigcap_{i=1}^{j} (X_i - X_i)\Big| \geq \Big(\prod_{i=1}^{j} \frac{|X_i|}{|X_i - Y|}\Big)|Y|. \tag{1.13}$$

2 Preliminary Estimates (1)

Recall [B], Lemma 5, which is deduced from the Balog-Szemeredi-Gowers theorem ([B], Prop. 1). Only the additive structure is involved.

Lemma 3. *Let* $\alpha : \mathbb{F}_q \to \mathbb{R}_+$ *satisfy* $\|\alpha\|_1 = \sum_{x\in\mathbb{F}_q} |\alpha(x)| \leq 1$. *Fix* $1 < K < q$. *There are the following alternatives.*

Either

$$\|\alpha * \alpha\|_2 < \frac{1}{K}\|\alpha\|_2 \quad (* \text{ denotes additive convolution}) \tag{2.1}$$

or there is a subset $A \subset \mathbb{F}_q$ *with the following properties (we ignore multiplicative constants).*

$$\frac{2\sigma}{|A|} > \alpha|_A > \frac{\sigma}{|A|} \text{ where } 1 \geq \sigma > (\log q)^{-6} K^{-3}; \tag{2.2}$$

$$\|\alpha|_A\|_1 > (\log q)^{-6} K^{-3} \tag{2.3}$$

$$\|\alpha|_A\|_2 > (\log q)^{-4} K^{-2} \|\alpha\|_2 \tag{2.4}$$

$$|A + A| < (\log q)^{76} K^{38} |A|. \tag{2.5}$$

The argument is identical to the prime case.

Iteration of Lemma 3 gives ([B], Lemma 6).

Lemma 4. *Let* $\alpha : \mathbb{F}_q \to \mathbb{R}_+, \|\alpha\|_1 \leq 1$. *Fix* $1 \leq K \leq q$.

Then there is a decomposition (with disjointedly supported components)

$$\alpha = \sum_{j\leq J} \alpha_j + \beta \tag{2.6}$$

where each α_j *satisfies for some* $B_j \subset \mathbb{F}_q$

$$\frac{\sigma_j}{|B_j|}\mathcal{X}_{B_j} < \alpha_j < \frac{2\sigma_j}{|B_j|}\mathcal{X}_{B_j} \text{ with } 1 \geq \sigma_j > (\log q)^{-6} K^{-3} \tag{2.7}$$

and

$$\|\alpha_j\|_1 > (\log q)^{-6} K^{-3}, \tag{2.8}$$

$$\|\alpha_j\|_2 > K^{-2} (\log q)^{-4} \|\alpha\|_2. \tag{2.9}$$

$$|B_j + B_j| < K^{38} (\log q)^{76} |B_j|, \tag{2.10}$$

$$J < (\log q)^6 K^3, \tag{2.11}$$

$$\|\beta * \beta\|_2 < \frac{1}{K}\|\beta\|_2 < \frac{1}{K}\|\alpha\|_2. \tag{2.12}$$

Denote $\psi(x) = e_p(Trx), Tr = Tr_{\mathbb{F}_q/\mathbb{F}_p}$ the additive character of $\mathbb{F}_q, q = p^n$.

Lemma 5. *Let $\alpha, \beta, \gamma : \mathbb{F}_q \to \mathbb{R}_+$; $\|\alpha\|_1, \|\beta\|_1, \|\gamma\|_1 \leq 1$.*
Take $1 \leq K \leq q$.
Then

$$(2.1) \qquad |S| = \Big|\sum_{x,y,z} \alpha(x)\beta(y)\gamma(z)\psi(xyz)\Big| \leq 3|S_1| \tag{2.13}$$

$$+ 8 \max_{(*)} \frac{1}{|A|.|B|.|C|}\Big|\sum_{x\in A, y\in B, z\in C} \psi(xyz)\Big| \tag{2.14}$$

where $(*)$ refers to sets $A, B, C \subset \mathbb{F}_q$ such that

$$\frac{2\sigma}{|A|} > \alpha|_A > \frac{\sigma}{|A|} \text{ where } 1 \geq \sigma > (\log q)^{-6}K^{-3} \tag{2.15}$$

$$(\log q)^{-12}K^{-6}\|\alpha\|_2^{-2} < |A| < K^4(\log q)^8\|\alpha\|_2^{-2} \tag{2.16}$$

$$|A + A| < (\log q)^{76}K^{38}|A| \tag{2.17}$$

and similarly for B, C and

$$S_1 = \sum_{x,y,z} \alpha'(x)\beta'(y)\gamma'(z)\psi(xyz) \tag{2.18}$$

with $0 \leq \alpha' \leq \alpha, 0 \leq \beta' \leq \beta, 0 \leq \gamma' \leq \gamma$ and

$$\|\alpha' * \alpha'\|_2 < \frac{1}{K}\|\alpha'\|_2 \text{ or } \|\beta' * \beta'\|_2 < \frac{\|\beta'\|_2}{K} \text{ or } \|\gamma' * \gamma'\|_2 < \frac{1}{K}\|\gamma'\|_2. \tag{2.19}$$

Proof. Apply decomposition from Lemma 4 to each of the factors α, β, γ.

Note that in (2.7), $\sum \sigma_j \leq \|\alpha\|_1$. In order to justify the characteristic functions $\mathcal{X}_A.\mathcal{X}_B, \mathcal{X}_C$ in (2.14), we use the fact that if $\mathcal{X}_\Omega \leq f \leq 2\mathcal{X}_\Omega$, then f may be recovered as an average of $\pm\mathcal{X}_{\Omega'}$ for subset $\Omega' \subset \Omega, |\Omega'| \sim |\Omega|$. □

Recall also that by Cauchy-Schwarz, we have

$$|S_1|^2 \leq \sum_{y,z} \beta'(y)\gamma'(z)\Big|\sum_x \alpha'(x)\psi(xyz)\Big|^2$$

since $\|\beta'\|_1, \|\gamma'\|_1 \le 1$. Hence

$$|S_1| \le \Big|\sum_{x,y,z}(\alpha' * \alpha')(x)\beta'(y)\gamma'(z)\psi(xyz)\Big|^{1/2}. \tag{2.20}$$

For $a, b, c > 0$, denote $\zeta(a, b, c)$ the maximum of

$$\Big|\sum \alpha'(x)\beta'(y)\gamma'(z)\psi(xyz)\Big| \tag{2.21}$$

where $\alpha', \beta', \gamma' : \mathbb{F}_q \to \mathbb{R}_+$ satisfy

$$\|\alpha'\|_1 \le 1, \|\beta'\|_1 \le 1, \|\gamma'\|_1 \le 1 \text{ and } \|\alpha'\|_2 \le a, \|\beta'\|_2 \le b, \|\gamma'\|_2 \le c. \tag{2.22}$$

Lemma 5 implies then that

$$\zeta(\|\alpha\|_2, \|\beta\|_2, \|\gamma\|_2) \le (2.14) + 3\Big\{\zeta\Big(\frac{1}{K}\|\alpha\|_2, \|\beta\|_2, \|\gamma\|_2\Big) + {}''\Big\} \tag{2.23}$$

($''$ referring to the other 2 terms).

3 Preliminary Estimates (2)

We will use the following construction.

Let

$$S = \sum \alpha(x)\beta(y)\psi(xy)$$

with $0 \le \alpha, \beta$; $\|\alpha\|_1, \|\beta\|_1 = 1$.

Write

$$|S|^2 \le \Big|\sum(\alpha * \alpha)(x)\beta(y)\psi(xy)\Big|,$$

and more generally (denoting $\alpha^{(\ell)}$ the ℓ-fold additive convolution of α)

$$|S|^{2^s} \le \Big|\sum \alpha^{(2^s)}(x)\beta(y)\psi(xy)\Big|. \tag{3.1}$$

Fix $s \in \mathbb{Z}_+$ and define $L = L_s$ by

$$\|\alpha^{(2^{s+1})}\|_2 = \frac{1}{L}\|\alpha^{(2^s)}\|_2. \tag{3.2}$$

Applying Lemma 3 with α replaced by $\alpha^{(2^s)}$, it follows that there is a subset $A \subset \mathbb{F}_q$ satisfying

$$\|\alpha^{(2^s)}|_A\|_1 > (\log q)^{-6} L^{-3} \tag{3.3}$$

$$|A| < L^4 (\log q)^8 \|\alpha^{(2^s)}\|_2^{-2} \tag{3.4}$$

$$|A + A| < L^{38} (\log q)^{76} \ |A|. \tag{3.5}$$

Note that

$$\|\alpha^{(2^s)}|_A\|_1 \leq \max_{x \in \mathbb{F}} \|\alpha|_{x+A}\|_1.$$

Replacing A by a translate and denoting $\alpha_1 = \alpha^{(2^s)}$, it follows from (3.1) that

$$|S| \leq \Big| \sum \alpha_1(x) \beta(y) \psi(x.y) \Big|^{2^{-s}} \tag{3.6}$$

and there exists a set $A \subset \mathbb{F}_q$ such that

$$\|\alpha|_A\|_1 > (\log q)^{-6} L^{-3} \tag{3.7}$$

$$|A| < (\log q)^8 L^4 \|\alpha_1\|_2^{-2} \tag{3.8}$$

$$|A + A| < (\log q)^{76} L^{38} \ |A|. \tag{3.9}$$

where

$$L = \frac{\|\alpha_1\|_2}{\|\alpha_1 * \alpha_1\|_2} \tag{3.10}$$

Returning to (2.14), consider

$$S_0 = \frac{1}{|A|\,|B|\,|C|} \sum_{x \in A, y \in B, z \in C} \psi(xyz) = \frac{1}{|C|} \sum_{\substack{u \in \mathbb{F}_p \\ z \in C}} \eta(u) \psi(uz) \tag{3.11}$$

with η the image measure of $\mathcal{X}_A/|A| \otimes \mathcal{X}_B/|B|$ under the product map $(x, y) \mapsto x.y$.

Apply the considerations above with $\alpha = \eta, \beta = \frac{1}{|C|} \mathcal{X}_C$. Fix $s \in \mathbb{Z}_+$ and let $\eta_1 = \eta^{(2^s)}$.

Hence from (3.6)– (3.10)

$$|S_0| < \Big| \frac{1}{|C|} \sum_{\substack{z \in C \\ u \in \mathbb{F}}} \eta_1(u) \quad \psi(uz) \Big|^{2^{-s}} \tag{3.12}$$

and there is a set $U \subset \mathbb{F}$ satisfying $\big(U$ plays the role of A in (3.7)–(3.9)$\big)$

$$\big|\{x \in A, y \in B | xy \in U\}\big| > (\log q)^{-6} L^{-3} |A|.|B| \tag{3.13}$$

$$|U| < (\log q)^8 L^4 \|\eta_1\|_2^{-2} \tag{3.14}$$

$$|U + U| < (\log q)^{76} L^{38} |U|. \tag{3.15}$$

where

$$L = \frac{\|\eta_1\|_2}{\|\eta_1 * \eta_1\|_2} \tag{3.16}$$

4 Further Assumptions

We make the following further assumptions on $\alpha, \beta, \gamma : \mathbb{F}_q^* \to \mathbb{R}_+$

$$\|\alpha\|_2, \|\beta\|_2, \|\gamma\|_2 < q^{-\delta_0} \tag{4.1}$$

$$\|\alpha\|_2 . \|\beta\|_2^{\frac{1}{2}} \|\gamma\|_2 < q^{-\frac{1}{2} - \delta_1} \tag{4.2}$$

$$\max_{\substack{a,b \in \mathbb{F} \\ G \text{ proper subfield}}} \Big\{ \sum_{x \in G} \beta(ax + b) \Big\} < q^{-\delta_2} \tag{4.3}$$

where $\delta_0, \delta_1, \delta_2 > 0$.

In the definition of $\zeta(a, b, c)$ in §2, we make the extra hypothesis that β' satisfies (4.3). Since obviously the left side of (4.3) decreases when β' is replaced by $\beta' * \beta'$ (recall that $\|\beta'\|_1 \leq 1$), inequality (2.23) still holds in this restricted setting.

Let $B \subset \mathbb{F}_q$ be the set corresponding to β from Lemma 5. Thus by (2.15)

$$\frac{2\sigma}{|B|} > \beta|_B > \frac{\sigma}{|B|} \text{ where } (\log q)^{-6} K^{-3} < \sigma \leq 1. \tag{4.4}$$

If $B_0 \subset B$, clearly $\|\beta|_{B_0}\|_1 > (\log q)^{-6} K^{-3} \frac{|B_0|}{|B|}$ by (4.4) and (4.3) implies that B_0 is not contained in a set $aG + b$, G a proper subfield, provided

$$|B_0| > (\log q)^6 K^3 q^{-\delta_2} |B|. \tag{4.5}$$

5 Preliminary Estimates (3)

Returning to (3.11) - (3.16), we establish a lower found on $|U|$ using Lemma 1. From (3.14), this will give a bound on $\|\eta_1\|_2$.

Denote $\mathcal{G} = \{(x, y) \in A \times B | x.y \in U\}$.

Hence by (3.13)

$$|\mathcal{G}| > (\log q)^{-6} L^{-3} |A|.|B|. \tag{5.1}$$

Define

$$A_0 = \{x \in A \big| \ |\mathcal{G}(x)| > \frac{1}{2}(\log q)^{-6} L^{-3} |B|\}$$

and

$$B_0 = \{y \in B \big| \ |\mathcal{G}(y)| > \frac{1}{2}(\log q)^{-6} L^{-3} |A|\}$$

denoting $\mathcal{G}(x)$ and $\mathcal{G}(y)$ the fibers of $\mathcal{G}$.

Clearly

$$|A_0| > \frac{1}{2}(\log q)^{-6} L^{-3} |A| \tag{5.2}$$

$$|B_0| > \frac{1}{2}(\log q)^{-6} L^{-3} |B|. \tag{5.3}$$

We apply Lemma 1 with $X = A_0; Y = B_0$. In view of (4.5), (5.3), the assumption

$$K.L < \frac{1}{2}(\log q)^{-4} q^{\frac{1}{3}\delta_2} \tag{5.4}$$

ensures that B_0 is not contained in a multiplicative coset of a proper subfield. From Lemma 1 and the related Remark, we obtain $a_1, a_2, a_3, a_4 \in A_0 \cup (-A_0)$ and $b_1, b_2, b_3, b_4 \in B_0 \cup (-B_0)$ such that

$$\begin{aligned}&|b_1 A' + b_2 A' + b_3 A' + b_4 A' + a_1 B' + a_2 B' + a_3 B' + a_4 B'| \geq \\ &\frac{1}{2} \min \Big\{ |A'|.|B'|^{\frac{1}{2}}, q\Big(\frac{|A'|.|B'|}{|A|.|B|}\Big)^2 \Big\}\end{aligned} \tag{5.5}$$

if $A' \subset A_0, B' \subset B_0$.

Next we apply Lemma 2.

Take $X_i = \mathcal{G}(b_i) \subset A\,(1 \leq i \leq 4)$ and $Y = A_0$. From (1.13), there is $A' \subset A_0$ and $a' \in A_0$ s.t.

$$A' - a' \subset \bigcap_{i=1}^{4} \big(\mathcal{G}(b_i) - \mathcal{G}(b_i)\big) \tag{5.6}$$

and

$$|A'| \geq \Big(\prod_{i=1}^4 \frac{|\mathcal{G}(b_i)|}{|\mathcal{G}(b_i) - A_0|}\Big)|A_0| \geq \Big(\prod_{i=1}^4 \frac{|\mathcal{G}(b_i)|}{|A-A|}\Big).|A_0|$$
$$> (\log q)^{-C} K^{-304} L^{-15} |A| \tag{5.7}$$

by (5.2), (2.17).

By (5.6) and definition of $\mathcal{G}$

$$b_1(A'-a') + b_2(A'-a') + b_3(A'-a') + b_4(A'-a') \subset 4U - 4U. \tag{5.8}$$

Similarly we obtain $B' \subset B_0$ and $b' \in B_0$ s.t.

$$|B'| > (\log q)^{-C} K^{-304} L^{-15} |B| \tag{5.9}$$

and

$$a_1(B'-b') + a_2(B'-b') + a_3(B'-b') + a_4(B'-b') \subset 4U - 4U. \tag{5.10}$$

From (5.5), (5.8), (5.10), it follows

$$|8U - 8U| \geq (\log q)^{-C} \min\{K^{-456} L^{-23} |A|.|B|^{\frac{1}{2}}, qK^{-1208} L^{-60}\}. \tag{5.11}$$

Recalling (3.14), (3.15) and the Plunnecke-Ruzsa inequality, (5.11) implies

$$|U| \geq (\log q)^{-C} \min\{K^{-456} L^{-631} |A|.|B|^{\frac{1}{2}}, qK^{-1208} L^{-668}\}$$

and

$$\|\eta_1\|_2 \leq (\log q)^C (K^{228} L^{318} |A|^{-\frac{1}{2}} |B|^{-\frac{1}{4}} + q^{-\frac{1}{2}} K^{604} L^{336}). \tag{5.12}$$

From (3.16) defining L and (5.12)

$$\|\eta_1 * \eta_1\|_2 \leq (\log q)^C K^{\frac{151}{84}} \theta^{\frac{1}{336}} \|\eta_1\|_2^{1-\frac{1}{336}} \tag{5.13}$$

where $\eta_1 = \eta^{(2^s)}$ and

$$\theta = |A|^{-\frac{1}{2}} |B|^{-\frac{1}{4}} + \frac{1}{\sqrt{q}}. \tag{5.14}$$

The validity of (5.13) is conditional to (5.4), thus

$$\frac{\|\eta^{(2^s)}\|_2}{\|\eta^{(2^{s+1})}\|_2} < (\log q)^{-4} q^{\frac{1}{3}\delta_2} K^{-1}. \tag{5.15}$$

Set

$$K = q^{\delta_3} \text{ where } \delta_3 = 10^{-4} \min(\delta_1, \delta_2). \tag{5.16}$$

From (5.13), (5.15), either

$$\|\eta^{(2^{s+1})}\|_2 < q^{-\frac{1}{4}\delta_2} \|\eta^{(2^s)}\|_2 \tag{5.17}$$

or

$$\|\eta^{(2^{s+1})}\|_2 < K^2 \theta^{\frac{1}{336}} \|\eta^{(2^s)}\|_2^{1-\frac{1}{336}}. \tag{5.18}$$

Note also that since $\eta \geq 0, \|\eta\|_1 = 1$, the sequence $\|\eta^{(2^s)}\|_2$ is monotonically decreasing in s.

Iterating (5.17)-(5.18) $s_1 = s + [\frac{5}{\delta_2}]$ times, we obtain

$$\begin{aligned} \|\eta^{(2^{s_1})}\|_2 &< \frac{1}{q} + K^{672} \theta^{1-\left(\frac{335}{336}\right)^s} \|\eta\|_2^{\left(\frac{335}{336}\right)^s} \\ &< K^{672} q^{\frac{1}{2}\left(\frac{335}{336}\right)^s} \theta. \end{aligned} \tag{5.19}$$

Choose s such that

$$\left(\frac{335}{336}\right)^s < \frac{\delta_1}{100}$$

which is possible for s_1 satisfying

$$2^{s_1} < 32^{(\delta_2^{-1})} \left(\frac{100}{\delta_1}\right)^{300}. \tag{5.20}$$

From (5.16), (5.19), we conclude that

$$\|\eta^{(2^{s_1})}\|_2 < q^{\frac{\delta_1}{10}} \Big(|A|^{-\frac{1}{2}} |B|^{-\frac{1}{4}} + \frac{1}{\sqrt{q}}\Big). \tag{5.21}$$

6 Estimation of Trilinear Sums

Return to (3.12) with $\eta_1 = \eta^{(2^{s_1})}$. Estimate using Cauchy-Schwarz

$$\begin{aligned} \sum_{\substack{u \in \mathbb{F} \\ z \in C}} \eta_1(u) \psi(uz) &\leq \sum_u \eta_1(u) \Big| \sum_{z \in C} \psi(uz) \Big| \\ &\leq \|\eta_1\|_2 (\sqrt{q} |C|^{\frac{1}{2}}) \end{aligned} \tag{6.1}$$

and hence by (5.21)

$$\frac{1}{|C|}\Big|\sum_{\substack{u\in\mathbb{F}\\ z\in C}}\eta_1(u)\psi(uz)\Big| < q^{\frac{\delta_1}{10}}(q^{\frac{1}{2}}|A|^{-\frac{1}{2}}|B|^{-\frac{1}{4}}|C|^{-\frac{1}{2}}+|C|^{-\frac{1}{2}}). \tag{6.2}$$

From Lemma 5 and (3.12), (6.2), we obtain the following bound on (2.14).

$$(2.14)\le 8[q^{\frac{\delta_1}{9}}(q^{\frac{1}{2}}\|\alpha\|_2\ \|\beta\|_2^{\frac{1}{2}}\|\gamma\|_2+\|\gamma\|_2)]^\kappa \tag{6.3}$$

where by (5.20)

$$\kappa > \Big(\frac{1}{32}\Big)^{\frac{1}{\delta_2}}\Big(\frac{\delta_1}{100}\Big)^{300}. \tag{6.4}$$

Note that, since $\gamma(0)=0$, certainly

$$|S| = \Big|\sum_{x,y,z}\alpha(x)\beta(y)\gamma(z)\psi(xyz)\Big| \le \sqrt{q}\|\alpha\|_2\ \|\beta\|_2$$

and hence, if $\|\gamma\|_2\ge q^{-\frac{1}{2}\delta_1}$, $|S|<q^{-\frac{1}{2}\delta_1}$ by (4.2). Assuming $\|\gamma\|_2<q^{-\frac{1}{2}\delta_1}$, (6.3) and (4.2) imply

$$(2.14) < q^{-\frac{\delta_1}{2}\kappa}. \tag{6.5}$$

Hence, we proved $\big($recalling (2.23)$\big)$

$$\zeta(\|\alpha\|_2,\|\beta\|_2,\|\gamma\|_2) < q^{-\frac{\delta_1}{2}\kappa}+3\zeta(a,b,c)^{\frac{1}{2}}, \tag{6.6}$$

where certainly

$$a\le\|\alpha\|_2, b\le\|\beta\|_2, c\le\|\gamma\|_2$$

and

$$abc\le\frac{1}{K}\|\alpha\|_2\|\beta\|_2\|\gamma\|_2 = q^{-\delta_3}\|\alpha\|_2\|\beta\|_2\|\gamma\|_2. \tag{6.7}$$

Straightforward iteration of (6.6), (6.7), until reaching $abc<\frac{1}{q}$ for which $\zeta(a,b,c)<\sqrt{q}\,q^{-2/3}=q^{-\frac{1}{6}}$, gives by (5.16)

$$\zeta(\|\alpha\|_2,\|\beta\|_2,\|\gamma\|_2) < q^{-\kappa'} \tag{6.8}$$

where

$$\kappa' > C^{-\frac{1}{\delta_2}}\delta_1^C \tag{6.9}$$

(C some constant).

Hence, we obtain

Theorem 1. *Let $\alpha, \beta, \gamma : \mathbb{F}_q \to \mathbb{R}_+$ and $\delta_0, \delta_1, \delta_2 > 0$ satisfying $\|\alpha\|_1, \|\beta\|_1, \|\gamma\|_1 \leq 1$ and (4.1)-(4.3). Then*

$$\Big|\sum \alpha(x)\beta(y)\gamma(z)\psi(xyz)\Big| < q^{-\kappa'} + 3q^{-\delta_0} \tag{6.10}$$

with

$$\kappa' > C^{-\frac{1}{\delta_2}}\delta_1^C. \tag{6.11}$$

7 Convolution of Product Densities

From Theorem 1, we deduce

Theorem 2. *Let $\alpha, \beta : \mathbb{F} \to \mathbb{R}_+$ satisfying*

$$\|\alpha\|_1, \|\beta\|_1 \leq 1 \tag{7.1}$$

$$\|\alpha\|_2 < q^{-\delta} \tag{7.2}$$

$$\max_{\substack{a,b\in\mathbb{F}\\ G \text{ proper subfield}}} \Big\{\sum_{x\in G} \beta(ax+b)\Big\} < q^{-\delta_2} \tag{7.3}$$

Let η be the image measure of $\alpha \otimes \beta$ under the product map $(x, y) \mapsto x.y$.

There is $k = k(\delta, \delta_2) < \delta^{-C} C^{\frac{1}{\delta_2}}$ such that

$$\|\eta^{(k)}\|_2 < q^{\delta}\|\alpha\|_2.\|\beta\|_2^{\frac{1}{2}} + q^{\delta-\frac{1}{2}} \tag{7.4}$$

where $\eta^{(k)}$ denotes k-fold (additive) convolution.

Proof. Write

$$\|\eta^{(k)}\|_2^2 = q^{-1}\sum |\hat{\eta}(\xi)|^{2k} < q^{-2} + q^{-1}|Z| \tag{7.5}$$

with

$$Z = \{\xi \in \mathbb{F}\big|\ |\hat{\eta}(\xi)| > q^{-1/k}\}.$$

Defining $\gamma : \mathbb{F} \to \mathbb{C}$ by

$$\gamma(\xi) = \begin{cases} \frac{\overline{\hat{\eta}(\xi)}}{|Z|\,|\hat{\eta}(\xi)|} & \text{if } \xi \in Z \\ 0 & \text{otherwise} \end{cases}$$

we have $\|\gamma\|_1 = 1, \|\gamma\|_2 = |Z|^{-\frac{1}{2}}$ and

$$q^{-1/k} \leq \sum_\xi \hat{\eta}(\xi)\gamma(\xi) = \sum_{x,y,\xi} \alpha(x)\beta(y)\gamma(\xi)\psi(xy\xi). \tag{7.6}$$

Apply Theorem 1 to (7.6) with $\delta_0 = \min\big(\delta, \frac{\delta_2}{2}, \frac{2}{k}, \frac{2}{k}\big)$ and $\delta_1 = \delta$. Then (4.1) holds, unless

$$\|\gamma\|_2 \geq q^{-\delta_0} \text{ hence } |Z| \leq q^{\frac{4}{k}} \tag{7.7}$$

and (4.2), unless

$$|Z| \leq q^{1+2\delta}\|\alpha\|_2^2 \ \|\beta\|_2. \tag{7.8}$$

Clearly for $k > \frac{1}{\kappa'}, \kappa' = C^{-\frac{1}{\delta_2}}\delta^C$ given by (6.11), (6.10) and (7.6) are contradictory. Therefore, either (7.7) or (7.8) hold, i.e.

$$|Z| \leq q^{\frac{4}{k}} + q^{1+2\delta}\|\alpha\|_2^2\|\beta\|_2. \tag{7.9}$$

Substitution in (7.5) gives (7.4). □

8 The General Case

From Theorem 2, we obtain the multilinear extension of Theorem 1.

Theorem 3. *Let $r \geq 2$ and $\alpha_1, \dots, \alpha_r : \mathbb{F} \to \mathbb{R}_+$ satisfy*

$$\|\alpha_i\|_1 \leq 1 \quad (1 \leq i \leq r) \tag{8.1}$$

$$\|\alpha_i\|_2 < q^{-\delta} \quad (1 \leq i \leq r) \tag{8.2}$$

$$\max_{\substack{a,b\in\mathbb{F} \\ G \text{ proper subfield}}} \alpha_i(aG+b) < q^{-\delta_2} \quad (3 \leq i \leq r) \tag{8.3}$$

$$\|\alpha_1\|_2.\|\alpha_2\|_2 \prod_{i=3}^r \|\alpha_i\|_2^{\frac{1}{2}} < q^{-\frac{1}{2}-\delta}. \tag{8.4}$$

Then there is the exponential sum bound

$$\Big|\sum_{x_1,\dots,x_r\in\mathbb{F}} \prod_{i=1}^r \alpha_i(x_i)\psi\Big(\prod_{i=1}^r x_i\Big)\Big| < q^{-\delta'} \text{ with } \delta' > C^{-\frac{r}{\delta_2}}\Big(\frac{\delta}{r}\Big)^{Cr}. \tag{8.5}$$

Proof. By induction on r.

For $r=2$, there is the obvious bound $q^{\frac{1}{2}}\|\alpha_1\|_2\|\alpha_2\|_2 < q^{-\delta}$.

The case $r=3$ is given by Theorem 1.

For the inductive step, we will use Theorem 2. Let $r\geq 4$. Denote S the exponential sum on the left of (8.5) and let η be the image density of $\alpha_1\otimes\alpha_3$ under the product map $(x_1,x_3)\mapsto x_1x_3$. Thus

$$S=\sum_{x,x_2,x_4,\ldots,x_r}\eta(x)\prod_{i\neq 1,3}\alpha_i(x_i)\psi\Big(x\prod_{i\neq 1,3}x_i\Big) \tag{8.6}$$

and estimate using Hölder's inequality

$$\begin{aligned}|S| &\leq \Big\{\sum_{x_2,x_4,\ldots,x_r}\prod_{i\neq 1,3}\alpha_i(x_i)\Big|\sum_{x\in\mathbb{F}}\eta(x)\psi\Big(x.\prod_{i\neq 1,3}x_i\Big)\Big|^{2k}\Big\}^{\frac{1}{2k}}\\ &=\Big\{\sum_{x,x_2,x_4,\ldots,x_r}\eta^{(2k)}(x)\cdot\prod_{i\neq 1,3}\alpha_i(x_i)\cdot\psi\Big(x.\prod_{i\neq 1,3}x_i\Big)\Big\}^{\frac{1}{2k}}. \end{aligned}\tag{8.7}$$

Theorem 2 is then applied to bound $\|\eta^{(2k)}\|_2$. Replacing δ by $\frac{\delta}{2r}$ in Theorem 2, it follows from (7.4)

$$\|\eta^{(2k)}\|_2 < q^{\frac{\delta}{2r}}\|\alpha_1\|_2\ \|\alpha_3\|_2^{\frac{1}{2}} + q^{\frac{\delta}{2r}-\frac{1}{2}} \tag{8.8}$$

for

$$k\sim\Big(\frac{2r}{\delta}\Big)^C C^{\frac{1}{\delta_2}}. \tag{8.9}$$

Hence

$$\|\eta^{(2k)}\|_2 < q^{-\delta(\frac{3}{2}-\frac{1}{2r})} + q^{\frac{\delta}{2r}-\frac{1}{2}} < q^{-\delta}$$

and since $r\geq 4$, from (8.8), (8.4)

$$\|\eta^{(2k)}\|_2\|\alpha_2\|_2\prod_{i=4}\|\alpha_i\|_2^{\frac{1}{2}} < q^{\frac{\delta}{2r}-\frac{1}{2}-\delta} + q^{\frac{\delta}{2r}-\frac{1}{2}-\frac{r-1}{2}\delta} \leq 2q^{-\frac{1}{2}-(1-\frac{1}{2r})\delta}. \tag{8.10}$$

At this point, invoke the induction hypothesis with r replaced by $r-1$ and δ by $(1-\frac{1}{2r})\delta$. Recalling (8.7), it follows that

$$|S| < q^{-\delta_r'} \tag{8.11}$$

where

$$\delta_r' = \delta_r'(\delta) = \frac{1}{2k}\delta_{r-1}'\Big(\Big(1-\frac{1}{2r}\Big)\delta\Big) \underset{(8.9)}{>} \Big(\frac{\delta}{r}\Big)^C C^{-\frac{1}{\delta_2}}\delta_{r-1}'\Big(\Big(1-\frac{1}{2r}\Big)\delta\Big). \tag{8.12}$$

It remains to iterate (8.12). □

Following the argument to derive Theorem 2 from Theorem 1, Theorem 3 implies also

Theorem 4. *Let* $\alpha_1, \ldots, \alpha_r : \mathbb{F}_q \to \mathbb{R}_+$ *satisfy*

$$\|\alpha_i\|_1 \leq 1 \quad (1 \leq i \leq r) \tag{8.13}$$

$$\|\alpha_i\|_2 < q^{-\delta} \quad (1 \leq i \leq r) \tag{8.14}$$

$$\max_{\substack{a,b\in\mathbb{F} \\ G \text{ proper subfield}}} \alpha_i(aG+b) < q^{-\delta_2} \quad (3 \leq i \leq r). \tag{8.15}$$

Denote η *the image density of* $\alpha_1 \otimes \cdots \otimes \alpha_r$ *under the product map* $(x_1, \ldots, x_r) \mapsto x_1 \cdots x_r$*. Then*

$$\|\eta^{(k)}\|_2 < q^{\delta} \|\alpha_1\|_2 . \|\alpha_2\|_2 \prod_{i=3}^{r} \|\alpha_i\|_2^{\frac{1}{2}} + q^{\delta - \frac{1}{2}} \tag{8.16}$$

provided $k > C^{r/\delta_2} \left(\frac{r}{\delta}\right)^{Cr}$.

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