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ON EXPONENTS OF CONVERGENCE OF SUBSEQUENCES

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Let

(1) $$A = \{a_k\}_{k=1}^{\infty}, \quad 0 < a_1 \leqq a_2 \leqq \ldots \leqq a_k \leqq a_{k+1} \leqq \ldots$$

be a sequence of real numbers with $\lim_{k\to\infty} a_k = +\infty$. It is well-known (cf. [6], p. 40) that there exists a unique number $\lambda = \lambda(A)$, $0 \leqq \lambda(A) \leqq +\infty$ such that for each $\sigma \in R$, $\sigma > 0$, $\sigma < \lambda$ we have $\sum_{k=1}^{\infty} a_k^{-\sigma} = +\infty$ and for each $\sigma \in R$, $\sigma > 0$, $\sigma > \lambda$ we have $\sum_{k=1}^{\infty} a_k^{-\sigma} < +\infty$. The number $\lambda = \lambda(A)$ is called the exponent of convergence of the sequence A and can be calculated by using the following well-known formula:

(2) $$\lambda(A) = \limsup_{n\to\infty} \frac{\log n}{\log a_n} \; {}^{*})$$

(cf. [6], p. 40).

Denote by S^+ the metric space of all sequences of the form (1) with Fréchet's metric. If we denote by $\lambda(x)$ (for $x \in S^+$) the exponent of convergence of the sequence x, we get a function $\lambda : S^+ \to \langle 0, +\infty \rangle$. The fundamental properties of this function are described in the papers [2] and [3].

In this paper we shall investigate the exponents of convergence of subsequences of a given sequence A of the form (1). It could be expected that the results will depend on the sequence A, but we shall show that such results can be proved which are valid for subsequences of an arbitrary sequence A of the form (1) with $\lambda(A) > 0$ (it can be $\lambda(A) = +\infty$). So e.g. we can show that in the topological as well as in the metrical sense "almost all" subsequences of an arbitrary sequence A with $\lambda(A) > 0$ have the exponent of convergence which coincides with $\lambda(A)$.

In what follows we shall use the usual method of mapping the set of all infinite subsequences of the given sequence A (see (1)) onto the interval $(0, 1\rangle$ (cf. [5], p. 17): the number

$$x = \sum_{k=1}^{\infty} 2^{-j_k} \in (0, 1\rangle$$

*) $\log x$ denotes the Napierian logarithm of x.

corresponds to the subsequence

$$(3) \qquad a_{j_1} \leqq a_{j_2} \leqq \ldots \leqq a_{j_k} \leqq \ldots \quad (j_1 < j_2 < \ldots < j_k < \ldots)$$

of the sequence A.

So we get a one-to-one mapping of the set of all infinite subsequences of the sequence A onto the interval $(0, 1\rangle$ (cf. [5], p. 17). This mapping enables us "to measure" various sets of subsequences of the sequence A. That can be done in the following way: Denote by $A(x)$ the subsequence (3) (in particular, we have $A(1) = A$). Let S be a certain set of subsequence of the sequence A. Let $M(S) \subset (0, 1\rangle$ be the set of all such $x \in (0, 1\rangle$ for which $A(x) \in S$. Then "the size" of the set S can be calculated by "the size" of the set $M(S)$ (we can investigate the Lebesgue measure of $M(S)$, the Hausdorff dimension of $M(S)$, the topological properties of $M(S)$, etc.).

The introduced mapping of the set of all subsequences of the sequence A onto the interval $(0, 1\rangle$ enables us to define the following real function λ on the interval $(0, 1\rangle$: for $x \in (0, 1\rangle$, put $\lambda(x) = \lambda(A(x))$.

If for some $\sigma \in R$, $\sigma > 0$, the series $\sum_{k=1}^{\infty} a_k^{-\sigma}$ converges, then its subseries $\sum_{k=1}^{\infty} a_{j_k}^{-\sigma}$ converges, too. This simple observation shows that for each $x \in (0, 1\rangle$ we have $\lambda(x) \leqq \lambda(A(1)) = \lambda(A)$ and hence $\lambda\colon (0, 1\rangle \to \langle 0, \lambda(A)\rangle$.

We consider the interval $(0, 1\rangle$ as a metric space (with the Euclidean metric); the same holds for the interval $\langle 0, \lambda(A)\rangle$ if $\lambda(A) < +\infty$. In the case $\lambda(A) = +\infty$ we consider the interval $\langle 0, \lambda(A)\rangle$ as a metric space with the metric d, where

$$d(x, y) = |v(x) - v(y)|, \quad v(t) = \frac{t}{1+t} \quad \text{for} \quad t \in \langle 0, +\infty) \quad \text{and} \quad v(+\infty) = 1.$$

In the first part of the paper we shall deduce some fundamental results on the function λ, in the second part we shall introduce some fundamental metrical and topological results on the sets $\{x \in (0, 1\rangle; \lambda(x) = \lambda(A)\}$, $\{x \in (0, 1\rangle; \lambda(x) < \lambda(A)\}$.

1. FUNDAMENTAL PROPERTIES OF THE FUNCTION $\lambda : (0, 1\rangle \to \langle 0, \lambda(A)\rangle$

The question arises whether the function λ maps the interval $(0, 1\rangle$ onto the whole interval $\langle 0, \lambda(A)\rangle$. We shall give the affirmative answer to this question (see Theorem 1.1). First we shall prove two auxiliary results.

Lemma 1.1. *Let*

$$g(1) \leqq g(2) \leqq \ldots \leqq g(n) \leqq \ldots, \quad g(n) \to +\infty \ (n \to \infty).$$

Let

$$b = \limsup_{n \to \infty} \frac{\log n}{g(n)} > 0$$

(*we admit* $b = +\infty$) *and let* j *be a positive integer. Then we have*

$$\limsup_{n\to\infty} \frac{\log n}{g(n+j)} = b\,.$$

Proof. According to the assumption of Lemma there exists such an $n_0 \geqq 2$ that for each $n \geqq n_0$ we have $g(n) > 0$. For such n the identity

$$\frac{\log n}{g(n+j)} = \frac{\log(n+j)}{g(n+j)} \frac{\log n}{\log(n+j)} \tag{4}$$

holds. The lim sup of the first factor on the right-hand side of (4) is b and the limit of the second factor is 1. Hence the assertion of Lemma immediately follows.

Lemma 1.2. *Let*

$$g(1) \leqq g(2) \leqq \ldots \leqq g(n) \leqq \ldots, \quad g(n) \to +\infty \ (n \to \infty)\,.$$

Let

$$b = \limsup_{n\to\infty} \frac{\log n}{g(n)} > 0$$

(*we admit* $b = +\infty$). *Let* $0 < t < b$. *Then there exists such a sequence* $j_1 < < j_2 < \ldots < j_n < \ldots$ *of positive integers that*

$$\limsup_{n\to\infty} \frac{\log n}{g(j_n)} = t\,.$$

Proof. Choose an integer $k_1 \geqq 1$ such that $g(k_1 + 2) > 0$ and

$$\frac{\log 2}{g(k_1 + 2)} < t\,.$$

According to Lemma 1.1 we have

$$\limsup_{k\to\infty} \frac{\log k}{g(k_1 + k)} = b > t\,.$$

Therefore there exists such a positive integer $k_2 \geqq 3$ that

$$\frac{\log k_2}{g(k_1 + k_2)} \geqq t$$

and for each n, $2 \leqq n < k_2$ we have

$$\frac{\log n}{g(k_1 + n)} < t\,.$$

Let us choose $k_3 \geqq 1$ such that

$$\frac{\log(k_2 + 1)}{g(k_1 + k_2 + k_3 + 1)} < t\,.$$

Such a k_3 exists for $\lim_{n\to\infty} g(n) = +\infty$. Again according to Lemma 1.1 there exists such a $k_4 \geqq 1$ that

$$\frac{\log(k_2 + k_4)}{g(k_1 + k_2 + k_3 + k_4)} \geqq t$$

and

$$\frac{\log(k_2 + n)}{g(k_1 + k_2 + k_3 + n)} < t$$

for each n, $1 \leqq n < k_4$.

So by induction we construct two sequences $\{k_{2i-1}\}_{i=1}^{\infty}$, $\{k_{2i}\}_{i=1}^{\infty}$ of positive integers such that

$$\frac{\log(k_2 + k_4 + \ldots + k_{2i})}{g(k_1 + k_2 + k_3 + \ldots + k_{2i})} \geqq t \tag{5}$$

$(i = 1, 2, \ldots)$ and

$$\frac{\log(k_2 + k_4 + \ldots + k_{2i-2} + n)}{g(k_1 + k_2 + k_3 + \ldots + k_{2i-1} + n)} < t \tag{6}$$

for each n, $1 \leqq n < k_{2i}$ $(i = 2, 3, \ldots)$. Put

$$j_1 = k_1,\ j_2 = k_1 + 2,\ j_3 = k_1 + 3,\ \ldots,$$

$$j_{k_2} = k_1 + k_2,\ j_{k_2+1} = k_1 + k_2 + k_3 + 1\,,$$

$$j_{k_2+2} = k_1 + k_2 + k_3 + 2, \ldots, j_{k_2+k_4} = k_1 + k_2 + k_3 + k_4\,,$$

$$j_{k_2+k_4+1} = k_1 + k_2 + k_3 + k_4 + k_5 + 1, \ldots, j_{k_2+k_4+\cdots+k_{2i}} = k_1 + k_2 + \ldots + k_{2i}\,,$$

$$j_{k_2+k_4+\cdots+k_{2i}+1} = k_1 + k_2 + \ldots + k_{2i} + k_{2i+1} + 1\,,$$

$$j_{k_2+k_4+\cdots+k_{2i}+2} = k_1 + k_2 + \ldots + k_{2i} + k_{2i+1} + 2, \ldots.$$

By a simple estimation we get from (5), (6)

$$0 < \frac{\log(k_2 + k_4 + \ldots + k_{2i})}{g(k_1 + k_2 + k_3 + \ldots k_{2i})} - \frac{\log(k_2 + k_4 + \ldots + k_{2i-2} + k_{2i} - 1)}{g(k_1 + k_2 + k_3 + \ldots + k_{2i-1} + k_{2i} - 1)} \leqq$$

$$\leqq \frac{1}{g(k_1 + k_2 + \ldots + k_{2i})} \log \frac{k_1 + k_4 + \ldots + k_{2i}}{k_2 + k_2 + \ldots + k_{2i} - 1} \to 0$$

$(i \to \infty)$. This together with (5), (6) immediately yields the assertion of Lemma.

Theorem 1.1. *Let $0 \leqq t \leqq \lambda(A)$. Then there exists such an $x \in (0, 1\rangle$ that $\lambda(x) = t$.*

Proof. The assertion of Theorem is obvious for $t = \lambda(A)$ or $t = 0$. In the first case it suffices to put $x = 1$. In the second case we proceed as follows. Since $a_n \to +\infty$ $(n \to \infty)$, there exists such a sequence $k_1 < k_2 < k_3 < \ldots$ of positive integers that

$$a_{k_n} > n^n \quad (n = 1, 2, \ldots)\,.$$

Put $c_{k_n} = 1$ $(n = 1, 2, \ldots)$ and $c_i = 0$ for $i \neq k_n$ $(n = 1, 2, \ldots)$. Then evidently $\lambda(x) = 0$, where $x = \sum_{i=1}^{\infty} c_i 2^{-i} \in (0, 1\rangle$.

Let $0 < t < \lambda(A)$. Put $g(n) = \log a_n$ $(n = 1, 2, \ldots)$. Then the sequence $\{g(n)\}_{n=1}^{\infty}$ fulfils the assumptions of Lemma 1.2. According to this lemma there exists such a sequence $j_1 < j_2 < \ldots$ of positive integers that

$$\limsup_{n \to \infty} \frac{\log n}{\log a_{j_n}} = t .$$

Hence $t = \lambda(x)$ (see (2)), where $x = \sum_{n=1}^{\infty} 2^{-j_n} \in (0, 1\rangle$. This completes the proof.

The following auxiliary result will be useful for the investigation of the continuity and Darboux property of the function λ.

In what follows $\sum_{k=1}^{\infty} \varepsilon_k(x) 2^{-k}$ denotes the non-terminating dyadic development of the number $x \in (0, 1\rangle$ (i.e. $x = \sum_{k=1}^{\infty} \varepsilon_k(x) 2^{-k}$, $\varepsilon_k(x) = 0$ or $\varepsilon_k(x) = 1$ $(k = 1, 2, \ldots)$ and for an infinite number of k's we have $\varepsilon_k(x) = 1$).

Lemma 1.3. *Let $\lambda(A) > 0$, $0 \leqq t \leqq \lambda(A)$. Then the set $\{x \in (0, 1\rangle;\ \lambda(x) = t\}$ is a dense set in $(0, 1\rangle$.*

Proof. In virtue of Theorem 1.1 there exists such an $x_0 \in (0, 1\rangle$ that $\lambda(x_0) = t$. Denote by $M(x_0)$ the set of all such $x \in (0, 1\rangle$ for which there exists $n_0 = n_0(x) \in N$ such that for each $n \geqq n_0$ we have $\varepsilon_n(x) = \varepsilon_n(x_0)$. The set $M(x_0)$ is evidently dense in $(0, 1\rangle$ and the series $\sum_{k=1}^{\infty} \varepsilon_k(x) a_k^{-\sigma}$ converges if and only if the series $\sum_{k=1}^{\infty} \varepsilon_k(x_0) a_k^{-\sigma}$ converges. This yields $\lambda(x) = \lambda(x_0)$ for each $x \in M(x_0)$. This completes the proof.

The following result is an immediate consequence of the previous Lemma.

Theorem 1.2. *Let $\lambda(A) > 0$. The function $\lambda : (0, 1\rangle \to \langle 0, \lambda(A)\rangle$ is everywhere discontinuous.*

By Lemma 1.3, for each interval $I \subset (0, 1\rangle$ we get $\lambda(I) = \langle 0, \lambda(A)\rangle$. Hence we have:

Theorem 1.3. *The function $\lambda : (0, 1\rangle \to \langle 0, \lambda(A)\rangle$ has the Darboux property.*

We show that λ is a measurable function. This can be obtained from the following more exact result.

Theorem 1.4. *Let $\lambda(A) > 0$. The function $\lambda : (0, 1\rangle \to \langle 0, \lambda(A)\rangle$ belongs exactly to the second Baire class.*

Proof. It is a well-known fact that the set of discontinuity points of a function belonging to the first Baire class is a set of the first Baire category (cf. [4], p. 301). Since the function λ is everywhere discontinuous (see Theorem 1.2), it cannot belong to the first Baire class. Therefore it suffices to show that for each real d each of the sets

$$M^d = \{x \in (0, 1\rangle;\ \lambda(x) \leqq d\}\,, \quad M_d = \{x \in (0, 1\rangle;\ \lambda(x) \geqq d\}$$

belongs to the second multiplicative Borel class.

Investigate first the set M^d. If $d < 0$, then $M^d = \emptyset$ and the above assertion is evident. Let $d \geqq 0$. Then we have

$$(7) \qquad M^d = \bigcap_{m=1}^{\infty} \{x \in (0, 1\rangle;\ \sum_{k=1}^{\infty} \varepsilon_k(x)\, a_k^{-(d+1/m)} < +\infty\} = \bigcap_{m=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{p=1}^{\infty} H(m, k, n, p)\,,$$

where

$$H(m, k, n, p) = \left\{x \in (0, 1\rangle;\ \sum_{j=n+1}^{n+p} \varepsilon_j(x)\, a_j^{-(d+1/m)} < \frac{1}{k}\right\}.$$

Denote by X the metric space (with the Euclidean metric) of all irrational numbers of the interval $(0, 1\rangle$. Then it is evident that for fixed m, k, n, p the set $H(m, k, n, p) \cap X$ is closed in X (it can be expressed as a finite union of sets of the form $(j \,.\, 2^{-(n+p)}, (j + 1)\, 2^{-(n+p)}\rangle \cap X$). But then it follows from (7) that $M^d \cap X$ is an $F_{\sigma\delta}$-set in X and so an $F_{\sigma\delta}$-set in $(0, 1\rangle$ as well.

Denote by Q the set of all rational numbers of the interval $(0, 1\rangle$. Then $M^d \cap Q$ is a countable set and therefore an $F_{\sigma\delta}$-set in $(0, 1\rangle$. Hence the set $M^d = (M^d \cap X) \cup (M^d \cap Q)$ as a union of two $F_{\sigma\delta}$-sets in $(0, 1\rangle$ is an $F_{\sigma\delta}$-set in $(0, 1\rangle$ as well.

Investigate the set M_d. If $d > \lambda(A)$, then $M_d = \emptyset$ and all is clear. Let $d \leqq \lambda(A)$. On the basis of the definition of the exponent of convergence we get

$$(8) \qquad M_d = \bigcap_{m=1}^{\infty} \{x \in (0, 1\rangle;\ \sum_{k=1}^{\infty} \varepsilon_k(x)\, a_k^{-(d-1/m)} = +\infty\} = \bigcap_{m=1}^{\infty} \bigcap_{p=1}^{\infty} \bigcup_{n=1}^{\infty} L(m, p, n)\,,$$

where

$$L(m, p, n) = \{x \in (0, 1\rangle;\ \sum_{k=1}^{\infty} \varepsilon_k(x)\, a_k^{-(d-1/m)} > p\}\,.$$

Analogously as in the foregoing part of the proof we can show that $L(m, p, n) \cap X$ is a closed set in X and from (8) we deduce that $M_d \cap X$ is an $F_{\sigma\delta}$-set in $(0, 1\rangle$. Using analogous considerations as in the previous part of the proof we can show that M_d is an $F_{\sigma\delta}$-set in $(0, 1\rangle$. The theorem follows.

2. METRICAL AND TOPOLOGICAL PROPERTIES OF THE SETS

$T = \{x \in (0, 1\rangle;\ \lambda(x) = \lambda(A)\}\,, \quad T' = \{x \in (0, 1\rangle;\ \lambda(x) < \lambda(A)\}$

In this part of our paper we shall prove some metrical and topological results on the sets T, T'.

In what follows $|M|$ denotes the Lebesgue measure of the set M and $\dim M$ the Hausdorff dimension of M (cf. [5], p. 190).

The definition of the sets T, T' yields

$$T \cup T' = (0, 1\rangle\,, \quad T \cap T' = \emptyset\,. \tag{9}$$

Since λ is a measurable function (Theorem 1.4), the sets T, T' are measurable (more precisely, it follows from the previous part of the paper that T is an $F_{\sigma\delta}$-set and T' is a $G_{\delta\sigma}$-set in $(0, 1\rangle$).

The following theorem shows that "almost all" subsequences of the given sequence (1) have an exponent of convergence which is equal to the exponent of convergence of the sequence (1).

Theorem 2.1. *We have* $|T| = 1$.

Corollary. *We have* $|T'| = 0$ (see (9)).

We need not prove Theorem 2.1 since it follows immediately from the following stronger result.

Theorem 2.2. *We have* $\dim T' = 0$.

Proof. Denote by H_0 the set of all $x \in (0, 1\rangle$, $x = \sum_{k=1}^{\infty} \varepsilon_k(x) \,.\, 2^{-k}$, which satisfy

$$\liminf_{n \to \infty} \frac{p(n, x)}{n} = 0\,, \tag{10}$$

where $p(n, x) = \sum_{k=1}^{n} \varepsilon_k(x)$ $(n = 1, 2, \ldots)$. It is well-known that

$$\dim H_0 = 0 \tag{11}$$

(cf. [1]; [5], p. 194).

Choose $z \in (0, 1\rangle - H_0$. Hence

$$\liminf_{n \to \infty} \frac{p(n, z)}{n} > 0\,. \tag{12}$$

It is proved in the paper [7] that if

$$d_1 \geqq d_2 \geqq \ldots \geqq d_k \geqq d_{k+1} \geqq \ldots$$

is a sequence of positive real numbers, $\sum_{k=1}^{\infty} d_k = +\infty$ and for a certain $x \in (0, 1\rangle$ we have $\sum_{k=1}^{\infty} \varepsilon_k(x)\, d_k < +\infty$, then this x fulfils the condition (10).

Let $\sigma \in R$, $\sigma > 0$, $\sum_{k=1}^{\infty} a_k^{-\sigma} = +\infty$. Put $d_k = a_k^{-\sigma}$ $(k = 1, 2, \ldots)$. Then on the basis of (12) it follows from the quoted results of [7] that $\sum_{k=1}^{\infty} \varepsilon_k(z)\, a_k^{-\sigma} = +\infty$. Hence we get $\lambda(z) \geqq \lambda(A)$. Since $\lambda(z) \leqq \lambda(A)$, we have $\lambda(z) = \lambda(A)$. Thus we have proved the inclusion $(0, 1\rangle - H_0 \subset T$. Taking complements of the above sets we get $T' \subset H_0$ and in virtue of (11) we get $\dim T' = 0$. This completes the proof.

The following theorem shows that also from the topological viewpoint "almost all" subsequences of the sequence (1) have exponents of convergence that are equal to the exponent of convergence of the sequence (1).

Theorem 2.3. *The set T is residual of the second Baire category in* $(0, 1\rangle$.

Corollary. *The set T' is a set of the first Baire category in* $(0, 1\rangle$.

Proof. First let $0 < \lambda(A) < +\infty$. It is easy to check that

$$T = \bigcap_{k=1}^{\infty} T_k, \tag{13}$$

where

$$T_k = \{x \in (0, 1\rangle;\ \sum_{i=1}^{\infty} \varepsilon_i(x)\, a_i^{\lambda(A) - 1/k} = +\infty\}$$

$(k = 1, 2, \ldots)$. Further, we have

$$T_k = \bigcap_{j=1}^{\infty} \bigcup_{s=1}^{\infty} P(k, j, s), \tag{14}$$

where

$$P(k, j, s) = \{x \in (0, 1\rangle;\ \sum_{i=1}^{s} \varepsilon_i(x)\, a_i^{\lambda(A) - 1/k} > j\}.$$

Let X have the same meaning as in the proof of Theorem 1.4. Then it is easy to verify that $P(k, j, s) \cap X$ is an open set in X and therefore according to (13), (14) the set $T \cap X$ is a G_δ-set in X and so a G_δ-set in $(0, 1\rangle$ as well. Since $|T| = 1$ (Theorem 2.1), the set $T \cap X$ is dense in $(0, 1\rangle$. Therefore the set $T \cap X$ is a residual set in $(0, 1\rangle$ (cf. [4], p. 49). But then also the set $T \supset T \cap X$ is residual in $(0, 1\rangle$. Since $(0, 1\rangle$ is a set of the second Baire category in itself, the set T is a set of the second category in $(0, 1\rangle$.

Now let $\lambda(A) = +\infty$. Then

$$T = \bigcap_{k=1}^{\infty} V_k, \tag{15}$$

where

$$V_k = \{x \in (0, 1\rangle;\ \sum_{i=1}^{\infty} \varepsilon_i(x)\, a_i^{-k} = +\infty\} \quad (k = 1, 2, \ldots).$$

Further,

$$V_k = \bigcap_{j=1}^{\infty} \bigcup_{s=1}^{\infty} B(k, j, s), \tag{16}$$

where

$$B(k, j, s) = \{x \in (0, 1\rangle;\ \sum_{i=1}^{s} \varepsilon_i(x)\, a_i^{-k} > j\}.$$

Let X have the same meaning as above. Then it is easy to verify that $B(k, j, s) \cap X$

is an open set in X. Using (15), (16), an analogous consideration as that used in the previous part of the proof shows that T is a residual set of the second Baire category in $(0, 1\rangle$. The theorem follows.

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