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## ON EXTENDED CYCLIC ORDERS

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The notion of cyclically ordered set will be applied in the same sense as in the papers [5] and [6].

Let  $G$  be a nonempty set and let  $C$  be a cyclic order on  $G$ . We define a ternary relation  $C_0$  on  $G$  by putting, for any  $x, y, z \in G$ ,

$$(x, y, z) \in C_0 \text{ iff either } (x, y, z) \in C \text{ or } x = y = z.$$

The relation  $C_0$  will be said to be an extended cyclic order (corresponding to the cyclic order  $C$ ).

It is clear that  $C$  and  $C_0$  are uniquely determined by each other. Hence every result on  $C_0$  can be considered in a certain sense as a result on  $C$ .

The pair  $(G, C_0)$  will be said to be an ec-set. If, moreover,  $G$  is a group such that the group operation is compatible with the relation  $C_0$ , then  $(G, +, C_0)$  will be called an ec-group.

The present paper deals with subdirect product decompositions of ec-sets and direct product decompositions of ec-groups.

## 1. PRELIMINARIES

For the sake of completeness we recall here the basic definitions on cyclic orders.

A ternary relation  $C$  on a set  $G \neq \emptyset$  is called a cyclic order whenever the following conditions are satisfied:

- (I) If  $(x, y, z) \in C$ , then  $(z, y, x) \notin C$ .
- (II) If  $(x, y, z) \in C$ , then  $(z, x, y) \in C$ .
- (III) If  $(x, y, z) \in G$  and  $(x, z, u) \in C$ , then  $(x, y, u) \in C$ .

Under the above conditions, the pair  $\mathbf{G} = (G, C)$  is said to be a cyclically ordered set.  $\mathbf{G}$  is called a cycle if, moreover, for each  $(x, y, z) \in G^3$  such that the elements  $x, y$  and  $z$  are distinct we have either  $(x, y, z) \in C$  or  $(z, y, x) \in C$ .

We denote by  $\mathcal{C}$  the class of all cyclically ordered sets. If  $\mathbf{G} \in \mathcal{C}$ , then (I) and (II) imply that whenever  $(x, y, z) \in C$ , then  $\text{card}\{x, y, z\} = 3$ .

For  $\mathbf{G} = (G, C) \in \mathcal{C}$  let  $C_0$  be as above. The pair  $\mathbf{G}_0 = (G, C_0)$  will be said to be an ec-set. The class of all ec-sets will be denoted by  $\mathcal{C}_0$ . Next, we denote by  $\mathcal{C}^1$  the class of all cycles; let  $\mathcal{C}_0^1$  be the class of all  $(G, C_0) \in \mathcal{C}_0$  such that  $(G, C) \in \mathcal{C}^1$ .

Isomorphisms between cyclically ordered sets (or ec-sets) are defined in an obvious way. If two cyclically ordered sets  $\mathbf{G}$  and  $\mathbf{H}$  are isomorphic, then we express this fact by writing  $\mathbf{G} \cong \mathbf{H}$ ; a similar notation will be applied for elements of  $\mathcal{C}_0$ .

Let  $\mathbf{G} = (G; C_0) \in \mathcal{C}_0$ . An element  $a \in G$  will be said to be isolated (in  $\mathbf{G}$ ) if there are no elements  $b$  and  $c$  in  $G$  with  $b \neq a \neq c$  such that  $(a, b, c) \in C_0$ .

Let  $(G, \leq)$  be a partially ordered set. We define a ternary relation  $C_{\leq}$  on  $G$  as follows. For  $x, y, z \in G$  we put  $(x, y, z) \in C_{\leq}$  iff some of the following condition is valid:

$$x < y < z; \quad y < z < x; \quad z < x < y.$$

It is easy to verify that  $(G, C_{\leq})$  belongs to  $\mathcal{C}$ .

Again, let  $\mathbf{G} \in \mathcal{C}$  and let  $G_1$  be a nonempty subset of  $G$ . Put  $C^1 = C \cap G_1^3$ . Then  $\mathbf{G}_1 = (G_1, C^1)$  belongs to  $\mathcal{C}$ ; it will be called a subsystem of  $\mathbf{G}$ . Analogously,  $\mathbf{G}_{10} = (G_1, C_0^1)$  is said to be a subsystem of  $\mathbf{G}_0$ .

## 2. DIRECT AND SUBDIRECT PRODUCTS

In this section the notions of direct and subdirect product of ec-sets will be defined and it will be proved that each ec-set  $(G, C_0)$  with  $\text{card}G > 1$  can be represented as a subdirect product of ec-sets  $(G_i, C_{i0})$  ( $i \in I$ ) such that for each  $i \in I$  either  $\text{card}G_i = 2$  or  $\text{card}G_i = 3$  is valid.

Assume that  $I$  is a nonempty set and that  $\mathbf{G}_i = (G_i, C_i) \in \mathcal{C}_0$  for each  $i \in I$ . Put  $G = \prod_{i \in I} G_i$  and let  $C$  be a ternary relation on  $G$  such that for  $x, y, z \in G$  we have  $(x, y, z) \in C$  iff  $(x(i), y(i), z(i)) \in C_i$  for each  $i \in I$ . Then  $\mathbf{G} = (G, C) \in \mathcal{C}_0$  and we denote  $\mathbf{G} = \prod_{i \in I} \mathbf{G}_i$ ; we also say that  $\mathbf{G}$  is the direct product of ec-sets  $\mathbf{G}_i$ .

Let us apply the above notation and let  $G^1$  be a nonempty subset of  $G$ . Then we can construct the corresponding subsystem  $\mathbf{G}^1$  of  $\mathbf{G}$  as above. For  $i \in I$  we put  $G^1(i) = \{t \in G_i: \text{there is } g^1 \in G^1 \text{ with } g^1(i) = t\}$ . If  $G^1(i) = G_i$  for each  $i \in I$ , then  $\mathbf{G}^1$  will be said to be the subdirect product of e-cyclically ordered sets  $\mathbf{G}_i$  ( $i \in I$ ).

The direct products of cyclically ordered sets and of ordered sets are defined analogously (cf. [1] and [6]). Also, the notion of the subdirect product for these cases can be introduced in the same way as in the case of e-cyclically ordered sets above.

An ec-set  $\mathbf{G} = (G, C_0)$  will be said to be elementary if either (i)  $\text{card } G = 2$ , or (ii)  $\text{card } G = 3$  and  $C \neq \emptyset$ . It is obvious that whenever  $\mathbf{G}_i = (G_i, C_{0i})$  ( $i = 1, 2$ ) are elementary ec-sets such that  $\text{card } G_1 = \text{card } G_2$ , then  $\mathbf{G}_1$  and  $\mathbf{G}_2$  are isomorphic. Hence there are, up to isomorphism, only two elementary ec-sets.

We define  $\mathbf{A}_2$  and  $\mathbf{A}_3$  in  $\mathcal{C}$  as follows. We put  $\mathbf{A}_2 = (A_2, C_2)$ , where  $A_2 = \{0, 1\}$  and  $C_2$  is the diagonal of  $A_2^3$ . Next, let  $\mathbf{A}_3 = (A_3, C_3)$ , where  $A_3 = \{0, 1, 2\}$  and  $C_3 = (C_{\leq})_0$ , where  $\leq$  is the natural linear order on  $A_3$ .

**2.1. Theorem.** *Let  $\mathbf{G} = (G, C_0) \in \mathcal{C}_0$ ,  $\text{card } G \geq 2$ . Then  $\mathbf{G}$  is isomorphic to a subdirect product of ec-sets  $\mathbf{G}_i$  ( $i \in I$ ) where  $I$  is a nonempty set and for each  $i \in I$  either  $\mathbf{G}_i = \mathbf{A}_2$  or  $\mathbf{G}_i = \mathbf{A}_3$ .*

*Proof.* Let  $I_1$  be a set having the property that there exists a one-to-one mapping  $\varphi_1$  of  $C$  onto  $I_1$ . Next, let  $G^0$  be the set of all isolated elements of  $G$  and let  $\varphi_2$  be a one-to-one mapping of  $G^0$  onto a set  $I_2$ , where  $I_1 \cap I_2 = \emptyset$ . Put  $I = I_1 \cup I_2$ . Let us remark that  $I_1$  can be empty, and similarly for  $I_2$ .

We set  $\mathbf{G}_{i(1)} = \mathbf{A}_3$  and  $\mathbf{G}_{i(2)} = \mathbf{A}_2$  for each  $i(1) \in I_1$  and each  $i(2) \in I_2$ . Now we construct the direct product  $\prod_{i \in I} \mathbf{G}_i$  which will be denoted by  $\mathbf{H} = (H, C')$ .

Let us define a mapping  $f: G \rightarrow H$  as follows. For  $a \in G$  and  $i \in I$  we have to define  $f(a)(i)$ .

First let  $i \in I_1$ . There are distinct elements  $x, y$  and  $z$  in  $G$  such that  $(x, y, z) \in C$  and  $\varphi_1^{-1}(i) = (x, y, z)$ . We put

$$\begin{aligned} f(a)(i) &= 0 && \text{if either } x = a \text{ or } a \notin \{x, y, z\}, \\ f(a)(i) &= 1 && \text{if } y = a, \\ f(a)(i) &= 2 && \text{if } z = a. \end{aligned}$$

Next, let  $i \in I_2$ . There is  $x \in G^0$  with  $\varphi_2^{-1}(i) = x$ . We put  $f(a)(i) = 1$  if  $x = a$ , and  $f(a)(i) = 0$  otherwise.

Put  $f(G) = H'$  and let  $C''$  be the extended cyclic order on  $H'$  which is inherited from  $C'$ . Let  $a$  and  $a'$  be distinct elements of  $G$ . If  $a$  is isolated, then for  $i = \varphi_2(a)$  we have  $f(a)(i) \neq f(a')(i)$ , thus  $f(a) \neq f(a')$ . Next, assume that  $a$  fails to be isolated. Then there are  $b, c \in G$  with  $b \neq a \neq c$  such that  $(b, a, c) \in C$ . Denote  $\varphi_1((b, a, c)) = i$ . Thus  $f(a)(i) = 1$  and  $f(a')(i) \neq 1$ . Therefore  $f$  is injective.

Let  $a, b, c \in G$  and assume that  $(a, b, c) \in C_0$ . If  $a = b = c$ , then we have clearly  $(f(a), f(b), f(c)) \in C''$ . Suppose that  $a, b$  and  $c$  are distinct. Hence  $(a, b, c) \in C$  and

there is  $i \in I_1$  with  $i = \varphi_1(a, b, c)$ . Thus  $(f(a)(i), f(b)(i), f(c)(i)) \in C_3$ . Moreover, for each  $j \in I$  with  $j \neq i$  the relation  $f(a)(j) = f(b)(j) = f(c)(j) = 0$  is valid. Therefore we have again  $(f(a), f(b), f(c)) \in C''$ .

Now let us assume that  $a, b$  and  $c$  are elements of  $G$  such that  $(a, b, c) \in C_0$ . Hence at least two of the elements  $a, b$  and  $c$  are distinct. If  $a$  is isolated and  $\varphi_2(a) = i$ , then  $(f(a)(i), f(b)(i), f(c)(i)) \in C_2$ , whence  $(f(a), f(b), f(c)) \in C''$ . Suppose that the element  $a$  is not isolated. Thus there are  $b'$  and  $c'$  in  $G$  with  $b' \neq a \neq c'$  such that  $(a, b', c') \in C$ . Put  $i = \varphi_1(a, b', c')$ . Hence  $f(a)(i) = 0$ . If  $b' = b$ , then  $c' \neq c$  and thus  $f(c)(i) \neq 2$  implying that  $(f(a)(i), f(b)(i), f(c)(i)) \in C_3$ . If  $b' \neq b$ , then  $f(b)(i) \neq 1$  and hence in this case we also have  $(f(a)(i), f(b)(i), f(c)(i)) \in C_3$ . Therefore  $(f(a), f(b), f(c)) \in C''$ .

Thus we have proved that  $f$  is an isomorphism of  $\mathbf{G}$  onto  $(H', C'')$ . It remains to verify that  $(H', C'')$  is a subdirect product of ec-sets  $\mathbf{G}_i$  ( $i \in I$ ).

Let  $i \in I_1$  and  $t \in \{0, 1, 2\}$ . There is  $(a, b, c) \in C$  such that  $\varphi_1((a, b, c)) = i$ . Hence there is  $x \in \{a, b, c\}$  such that  $f(x)(i) = t$ .

Next, let  $i \in I_2$  and  $t \in \{0, 1\}$ . Hence there is  $a \in G^0$  such that  $\varphi_2(a) = i$ . Then  $f(a)(i) = 1$ . Since  $\text{card } G > 1$  there is  $a' \in G$  with  $a' \neq a$ . Hence  $f(a')(i) = 0$ .

Summarizing, we conclude that  $(H', C'')$  is a subdirect product of the system  $\{\mathbf{G}_i\}_{i \in I}$ ; the proof is complete.  $\square$

**2.2. Corollary.** Let  $\mathbf{G} = (G, C_0) \in \mathcal{C}_0$ ,  $\text{card } G \geq 2$ . Assume that  $\mathbf{G}$  has no isolated element. Then  $\mathbf{G}$  is isomorphic to a subdirect product of ec-sets  $\mathbf{G}_i$  ( $i \in I$ ) where  $I$  is a nonempty set and  $\mathbf{G}_i = \mathbf{A}_3$  for each  $i \in I$ .

**2.3. Remark.** The above result 2.1 can be considered to be a representation theorem for ec-sets  $\mathbf{G} = (G, C_0)$  with  $\text{card } G \geq 2$  (i.e., it gives an embedding of  $\mathbf{G}$  into a direct product of "standard" ec-sets  $\mathbf{G}_i$ ; the "standardness" of  $\mathbf{G}_i$  means that all  $\mathbf{G}_i$  are elementary ec-sets). A representation theorem for cyclically ordered sets was proved in [6]; in the corresponding theorem of [6] all direct factors under consideration are isomorphic, but a subdirect product representation is obtained.

**2.4. Remark.** If  $I$  is as in 2.1 and if we put  $\mathbf{H}_i = \mathbf{A}_2 \times \mathbf{A}_3$  for each  $i \in I$ , then by an obvious modification of the proof of 2.1 we obtain an embedding  $f'$  of  $\mathbf{G}$  into the direct product  $\prod_{i \in I} \mathbf{H}_i$ ; but in such a case  $f'(G)$  fails to be a subdirect product of the system  $\{\mathbf{H}_i\}_{i \in I}$ .

By a direct product decomposition of a cyclically ordered set  $\mathbf{G}$  we understand a triple  $(\mathbf{G}, \prod_{i \in I} \mathbf{G}_i, f)$ , where all  $\mathbf{G}_i$  are cyclically ordered sets and  $f$  is an isomorphism of  $\mathbf{G}$  onto  $\prod_{i \in I} \mathbf{G}_i$ .

In an analogous manner we can define direct product decompositions of ec-sets and of partially ordered sets.

The natural question arises whether the relations between different types of direct product decompositions are "good".

For example: let  $\mathbf{G} = (G, C) \in \mathcal{C}$  and let  $(\mathbf{G}, \prod_{i \in I} G_i, f)$  be a direct product decomposition of  $\mathbf{G}$ . Put  $\mathbf{G}_0 = (G, C_0)$ ; we can ask whether  $(\mathbf{G}_0, \prod_{i \in I} \mathbf{G}_{i0}, f)$  must be a direct decomposition of  $\mathbf{G}_0$  (where  $\mathbf{G}_i = (G_i, C_i)$  and  $\mathbf{G}_{i0} = (G_i, C_{i0})$ ).

The answers to this question and to some other analogous questions are negative in general (cf. 2.5–2.7; the proofs are routine and so they will be omitted).

Let us remark that there exist positive results for an analogous situation in the theory of directed groups (cf. [3]).

In 2.5 and 2.6 we apply the above introduced notation.

**2.5. Proposition.** *Let  $\mathbf{G} \in \mathcal{C}$  and let  $(\mathbf{G}, \prod_{i \in I} \mathbf{G}_i, f)$  be a direct product decomposition of  $\mathbf{G}$ . Assume that there is  $a \in G$  such that  $a$  fails to be isolated. Then  $(\mathbf{G}_0, \prod_{i \in I} \mathbf{G}_{i0}, f)$  is not a direct decomposition of  $\mathbf{G}_0$ .*

**2.6. Proposition.** *Let  $\mathbf{G} \in \mathcal{C}$  and let  $(\mathbf{G}_0, \prod_{i \in I} \mathbf{G}_{i0}, f)$  be a direct product decomposition of  $\mathbf{G}_0$ . Assume that there is  $a \in G$  such that  $a$  fails to be isolated. Then  $(\mathbf{G}, \prod_{i \in I} \mathbf{G}_i, f)$  is not a direct product decomposition of  $\mathbf{G}$ .*

**2.7. Proposition.** *Let  $(G, \leq)$  be a partially ordered set,  $C = C_{\leq}$ . Let  $((G, \leq), \prod_{i \in I} (G_i, \leq), f)$  be a direct product decomposition of  $(G, \leq)$ . Assume that there are  $i(1), i(2) \in I$ ,  $i(1) \neq i(2)$  and elements  $a_1, a_2 \in G_{i(1)}$ ,  $b_1, b_2 \in G_{i(2)}$  such that  $a_1 < a_2$  and  $b_1 < b_2$ . Let  $C_i$  be the cyclic order defined by means of the relation  $\leq$  on  $G_i$ . Then  $((G, C), \prod_{i \in I} (G_i, C_i), f)$  is not a direct product decomposition of  $(G, C)$ ; similarly,  $((G, C_0), \prod_{i \in I} (G_i, C_{i0}), f)$  is not a direct product decomposition of  $(G, C_0)$ .*

### 3. GROUPS ENDOWED WITH AN EXTENDED CYCLIC ORDER

In the present section we will investigate direct product decompositions of ec-groups. A sufficient condition for cancellability of direct factors will be found. This result will be applied in the next section for studying a particular type of direct product decompositions.

Cyclically ordered groups  $(G, +, C)$  such that  $(G, C)$  is a cycle were investigated by several authors; cf., e.g., the citations in [4]; the more general case where  $(G, C)$  is any cyclically ordered set was dealt with in [8], [9] and [10].

We will apply the following definition.

Let  $(G, +)$  be a group and let  $(G, C)$  be a cyclically ordered set such that whenever  $(a, b, c) \in C$  and  $x, y \in G$ , then

$$(x + a + y, x + b + y, x + c + y) \in C \quad \text{and} \quad (-c, -b, -a) \in C.$$

Under these assumption  $(G, +, C)$  is said to be a cyclically ordered group. The class of all cyclically ordered groups will be denoted by  $\mathcal{G}_c$ .

Next, we denote by  $\mathcal{G}_c^0$  the class of all structures  $(G, +, C_0)$ , where  $(G, +, C) \in \mathcal{G}_c$ . The elements of  $\mathcal{G}_c^0$  will be called ec-groups. If  $(G, +, C_0) \in \mathcal{G}_c^0$ ,  $\text{card } G > 2$  and  $(G, C_0) \in \mathcal{C}_0^1$  (cf. Section 1), then  $(G, +, C_0)$  is said to be an  $\ell c$ -group.

Let  $I$  be a nonempty set and for each  $i \in I$  let  $\mathbf{G}_i = (G_i, +, C_{i0}) \in \mathcal{G}_c^0$  and  $\mathbf{G} \in \mathcal{G}_c^0$ . The direct product  $\prod_{i \in I} \mathbf{G}_i$  is defined in an obvious way. The meaning of the notation  $((\mathbf{G}, \prod_{i \in I} \mathbf{G}_i, f)$  is analogous to that introduced in Section 2.

Under the above notation let  $i(1) \in I$ . Put

$$H_{i(1)} = \{x \in G : f(x)(i) = 0 \quad \text{for each} \quad i \in I \setminus \{i(1)\}\}.$$

The corresponding ec-group (with the extended cyclic order and the group operation inherited from  $\mathbf{G}$ ) will be denoted by  $\mathbf{H}_{i(1)}$ . We will call  $\mathbf{H}_{i(1)}$  a direct factor of  $\mathbf{G}$ .

Let  $F(\mathbf{G})$  be the system of all direct factors of  $\mathbf{G}$ ; this system is partially ordered by inclusion. Then  $\mathbf{G}$  and  $\mathbf{O} = \{\{0\}, +, (0, 0, 0)\}$  are the greatest and the least elements of  $F(\mathbf{G})$ , respectively.

In an analogous way we can define the system  $S(G)$  of direct factors of a directed group  $G$ . It is well-known that  $S(G)$  is a Boolean algebra.

Returning to  $F(\mathbf{G})$  let us remark that the question whether  $F(\mathbf{G})$  is a lattice remains open. Some results concerning  $F(\mathbf{G})$  will be proved below.

Let  $\mathbf{G} = (G, +, C_0) \in \mathcal{G}_c^0$  and let  $H$  be a subgroup of  $G$ . The ec-group  $\mathbf{H}$  is defined by the inherited extended cyclic order; this will be denoted by  $C_0(H)$ . (Analogous notation are applied below.)

**3.1. Lemma.** *Let  $\mathbf{G}$  and  $\mathbf{H}$  be as above. Then the following conditions are equivalent:*

- (i)  $\mathbf{H} \in F(\mathbf{G})$ .

(ii) There exists a subgroup  $H'$  of  $G$  such that the group  $G$  is a direct product of  $H$  and  $H'$  and for each triple  $(x, y, z) \in G^3$  the relation  $(x, y, z) \in C_0$  is valid iff  $(x(H), y(H), z(H)) \in C_0(H)$  and  $(x(H'), y(H'), z(H')) \in C_0(H')$  (where  $x(H')$  is the component of  $x$  in  $H$  or in  $H'$  with respect to the direct product decomposition  $G = H \times H'$ , and similarly for  $y$  and  $z$ ).

**Proof.** This is an immediate consequence of the definition of  $F(\mathbf{G})$ . □

If the condition (ii) from 3.1 is satisfied then we write  $\mathbf{G} = \mathbf{H} \times \mathbf{H}'$ .

More generally, let  $A_1, A_2, \dots, A_n$  be subgroups of  $G$ . The corresponding ec-groups will be denoted by  $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n$ . We will write  $\mathbf{G} = \mathbf{A}_1 \times \dots \times \mathbf{A}_n$  if

- (i) the group  $G$  is a direct product of its subgroups  $A_1, A_2, \dots, A_n$ ;
- (ii) if  $g^i \in G, g^i = a_1^i + a_2^i + \dots + a_n^i$  with  $i = 1, 2, 3$ , and  $a_j^i \in A_j^i$  for  $j = 1, 2, \dots, n$  and  $i = 1, 2, 3$ , then the relation  $(g^1, g^2, g^3) \in C$  is valid iff  $(a_j^1, a_j^2, a_j^3) \in C(A_j)$  holds for each  $j \in \{1, 2, \dots, n\}$ .

It is clear that each  $\mathbf{A}_j$  belongs to  $F(\mathbf{G})$ . The above definition yields

**3.2. Lemma.** Let  $\mathbf{G} = \mathbf{A} \times \mathbf{B}, \mathbf{A} = \mathbf{C} \times \mathbf{D}$ . Then  $\mathbf{G} = \mathbf{C} \times \mathbf{D} \times \mathbf{B}$ .

**3.3. Lemma.** Let  $\mathbf{H}$  and  $\mathbf{H}_1$  be elements of  $F(\mathbf{G})$  such that  $H_1 \subseteq H$ . Let  $\mathbf{H}'$  and  $\mathbf{H}'_1$  be defined analogously as in 3.1. Then  $\mathbf{H} = \mathbf{H}_1 \times \mathbf{H}_2$ , where  $H_2 = H \cap H'_1$ .

**Proof.** The validity of the relation  $H = H_1 \times H_2$  in the group theoretical sense is obvious. The remaining part of the proof concerning the extended cyclic order on  $H$  is a consequence of the relation  $\mathbf{G} = \mathbf{H}_1 \times \mathbf{H}'_1$ . □

**3.4. Corollary.** Let  $\mathbf{H}, \mathbf{K}$  and  $\mathbf{H}_1$  be elements of  $F(\mathbf{G})$  such that  $H_1 \subseteq H, H_1 \subseteq K$  and  $H \subseteq K$ . Then  $H \cap H'_1 \subseteq K \cap H'_1$ .

**3.5. Lemma.** Let  $\mathbf{H}, \mathbf{K}$  and  $\mathbf{H}_1$  be elements of  $F(\mathbf{G})$  such that  $H_1 \subseteq H, H_1 \subseteq K$  and  $H \not\subseteq K$ . Then  $H \cap H'_1 \not\subseteq K \cap H'_1$  (where  $H'_1$  is as in 3.3).

**Proof.** By way of contradiction; if  $H \cap H'_1 \subseteq K \cap H'_1$ , then in view of 3.3 we would have  $H \subseteq K$ . □

Now, 3.1, 3.4 and 3.5 yield

**3.6. Proposition.** Let  $\mathbf{G} = \mathbf{H} \times \mathbf{H}' \in \mathcal{G}_c^0$ . Put  $L = \{\mathbf{X} \in F(\mathbf{G}) : H \subseteq X\}$  and  $\varphi(\mathbf{X}) = \mathbf{Y}$ , where  $Y = H' \cap X$  for each  $\mathbf{X} \in L$ . Let  $L$  be partially ordered by inclusion. Then  $\varphi$  is an isomorphism of  $L$  onto  $F(\mathbf{H}')$ .

Next, from 3.4 we infer



**3.7. Lemma.** Let  $\mathbf{G} = \mathbf{H} \times \mathbf{H}'$ . Put  $L_1 = \{X \in F(\mathbf{G}): X \subseteq H\}$ . Then  $L_1 = F(\mathbf{H})$ .

**3.8. Corollary.** Let  $\mathbf{G} = \mathbf{H} \times \mathbf{H}'$ . Assume that  $F(\mathbf{G})$  is a lattice. Then  $F(\mathbf{H})$  is a lattice as well.

An ec-group  $\mathbf{G} = (G, C)$  will be said to be a dc-group if it satisfies the following condition:

(i) Whenever  $a$  and  $b$  are distinct elements of  $G$ , then there exists  $c \in G$  such that either  $(a, b, c) \in C$  or  $(b, a, c) \in C$ .

The condition (i) is obviously equivalent to the condition

(ii) Whenever  $a \in G$  and  $a \neq 0$ , then there exists  $b \in G$  such that either  $(0, a, b) \in C$  or  $(a, 0, b) \in C$ .

**3.9. Theorem.** Let  $\mathbf{G}$  be an ec-group,  $\mathbf{G} = \mathbf{A} \times \mathbf{B}$  and  $\mathbf{G} = \mathbf{A} \times \mathbf{D}$ . Assume that  $\mathbf{D}$  is a dc-group. Then  $\mathbf{B} = \mathbf{D}$ .

*Proof.* Since the extended cyclic orders on  $B$  and  $D$  are inherited from the cyclic order on  $G$  it suffices to verify that  $B = D$ . By way of contradiction, assume that  $B \neq D$ .

First, suppose that  $D \subset B$ . Hence in view of 3.3 there exists a direct decomposition  $\mathbf{B} = \mathbf{D} \times \mathbf{B}_1$  with  $B_1 \neq \{0\}$ . Thus according to 3.2 we have

$$(1) \mathbf{G} = \mathbf{A} \times \mathbf{D} \times \mathbf{B}_1.$$

There exists  $b_1 \in B_1$  with  $b_1 \neq 0$ . The relation (1) yields that  $b_1$  does not belong to  $A + D$ . But from  $\mathbf{G} = \mathbf{A} \times \mathbf{D}$  we infer that  $b_1$  is an element of  $A + D$ , which is a contradiction.

Next, suppose that  $D$  fails to be a subset of  $B$ . Hence there is  $g \in D \setminus B$ . Since  $D$  is a dc-group there exists  $h \in D$  such that either  $(0, g, h) \in C$  or  $(0, h, g) \in C$ .

Let  $(0, g, h) \in C$  (the case  $(0, h, g) \in C$  is analogous). Then  $g \neq 0 \neq h$  and thus  $g \bar{\in} A, h \bar{\in} A$ . Next, the relation  $h \in B$  would imply that  $g \in B$ ; therefore  $h$  does not belong to  $B$ .

There are uniquely determined elements  $g_1, h_1 \in A$  and  $g_2, h_2 \in B$  such that  $g = g_1 + g_2$  and  $h = h_1 + h_2$ . From the above mentioned relation we infer that the elements  $0, g_1, h_1$  are distinct; similarly, the elements  $0, g_2, h_2$  are distinct. Hence

$$(2) (0, g_1, h_1) \in C \quad \text{and}$$

$$(3) (0, g_2, h_2) \in C.$$

Next, from the relations

$$g_2 = -g_1 + g, \quad h_2 = -h_1 + h$$

and from  $\mathbf{G} = \mathbf{A} \times \mathbf{D}$  we obtain (by applying (3)) that  $(0, -g_1, -h_1)$  holds, which contradicts (2).  $\square$

Let  $(G, \leq, +)$  be a partially ordered group. Put  $C = C_{\leq}$ . Then  $(G, C_0, +)$  is an ec-group; it will be said to be generated by the partially ordered group  $(G, \leq, +)$ .

An ec-group  $\mathbf{G}$  is said to be directly indecomposable if, whenever  $G = \mathbf{G}_1 \times \mathbf{G}_2$ , then either  $\text{card } G_1 = 1$  or  $\text{card } G_2 = 1$ .

**3.10. Theorem.** *Let  $\mathbf{G}$  be an ec-group which is generated by a nonzero directed group  $(G, \leq, +)$ . Then  $\mathbf{G}$  is directly indecomposable.*

**Proof.** By way of contradiction, let us suppose that  $\mathbf{G} = \mathbf{A} \times \mathbf{B}$ ,  $\text{card } A > 1$ ,  $\text{card } B > 1$ .

First suppose that there exists  $a \in A$  such that the element  $a$  is isolated in  $\mathbf{A}$ . Then all elements are isolated in  $\mathbf{A}$ . Since  $\text{card } A > 1$ , there is  $a_1 \in A$  with  $a_1 \neq 0$ . Because  $G$  is directed, there are elements  $x$  and  $y$  in  $G$  such that  $0 < x < y$  and  $a_1 < x < y$ . Hence we have

$$(1) (0, x, y) \in C,$$

$$(2) (a_1, x, y) \in C.$$

There are uniquely determined elements  $a_3, a_4 \in A$  and  $b_1, b_2 \in B$  such that  $x = a_3 + b_1$  and  $y = a_4 + b_2$ .

If  $a_3 \neq a_1$ , then (2) implies that  $(a_1, a_3, a_4)$  belongs to  $C$ , which is a contradiction. Therefore  $a_3 = a_1$ . Hence according to (1) the triple  $(0, a_1, a_4)$  belongs to  $C$ , which is impossible. Therefore there is no  $a \in A$  which is isolated in  $\mathbf{A}$ .

Hence there are  $a_1, a_2$  and  $a_3$  in  $A$  such that  $(a_1, a_2, a_3) \in C$ . By a routine calculation we obtain that there are  $a'_1$  and  $a'_2$  in  $A$  with  $0 < a'_1 < a'_2$ . Similarly, there are  $b'_1$  and  $b'_2$  in  $B$  such that  $0 < b'_1 < b'_2$ .

Hence

$$(3) 0 < a'_1 < a'_1 + b'_1.$$

Thus  $(0, a'_1, a'_1 + b'_1) \in C$ . From this and from the relation  $\mathbf{G} = \mathbf{A} \times \mathbf{B}$  we infer that  $(0, a'_1, a'_1) \in C$ , which is impossible.  $\square$

#### 4. DIRECT PRODUCTS OF $\ell c$ -GROUPS

In this section it will be proved that if an ec-group  $\mathbf{G}$  possesses a direct product decomposition such that all direct factors in this decomposition are  $\ell c$ -groups, then the partially ordered set  $F(\mathbf{G})$  is an atomic Boolean algebra.

In what follows we assume that  $\mathbf{G}$  is an ec-group.

**4.1. Lemma.** *Each  $\ell c$ -group is directly indecomposable.*

**Proof.** Let  $\mathbf{G}$  be an  $\ell c$ -group and suppose that  $\mathbf{G} = \mathbf{G}_1 \times \mathbf{G}_2$ , where  $\text{card } G_1 > 1$  and  $\text{card } G_2 > 1$ . Thus there are elements  $g_1 \in G_1$  and  $g_2 \in G_2$  such that

$g_1 \neq 0 \neq g_2$ . Obviously  $g_1 \neq g_2$ . Then neither  $(0, g_2, g_1) \in C$  nor  $(0, g_2, g_1)$  is valid, which is a contradiction.  $\square$

**4.2. Lemma.** *Let  $\mathbf{G} = \mathbf{H} \times \mathbf{H}'$  and let  $\mathbf{D}$  be a direct factor of  $\mathbf{G}$  such that  $\mathbf{D}$  is an  $\ell$ -group. Then either  $H \cap D = \{0\}$  or  $D \subseteq H$ .*

*Proof.* Assume that  $H \cap D \neq \{0\}$ . Hence there is  $d_1 \in D \cap H$  with  $d_1 \neq 0$ . Let  $d_2 \in D, d_2 \neq d_1, d_2 \neq 0$ . Then either  $(0, d_1, d_2) \in C$  or  $(0, d_2, d_1) \in C$ . Therefore in view of  $\mathbf{G} = \mathbf{H} \times \mathbf{H}'$  we obtain that  $d_2 \in \mathbf{H}$  and thus  $D \subseteq H$ .  $\square$

**4.3. Lemma.** *Let  $\mathbf{G}, \mathbf{H}, \mathbf{H}'$  and  $\mathbf{D}$  be as in 4.2. Then either  $D \subseteq H$  or  $D \subseteq H'$ .*

*Proof.* By way of contradiction, assume that neither  $D \subseteq H$  nor  $D \subseteq H'$  is valid. Then in view of 4.2

$$(1) \quad D \cap H = \{0\}$$

and

$$(2) \quad D \cap H' = \{0\}.$$

There exist  $d_1$  and  $d_2$  in  $D$  such that

$$(3) \quad (0, d_1, d_2) \in C.$$

Next, there are uniquely determined elements  $a_1, a_2 \in H$  and  $a'_1, a'_2 \in H'$  with

$$d_i = a_i + a'_i \quad (i = 1, 2).$$

In view of (3) we have  $d_1 \neq d_2$  and hence by applying (1) we get  $a_1 \neq a_2$ . Also,  $d_i \neq 0$  for  $i = 1, 2$  and hence  $a_1 \neq 0 \neq a_2$ . This yields that

$$(4a) \quad (0, a_1, a_2) \in C.$$

Analogously we obtain that  $a'_1 \neq a'_2, a'_1 \neq 0 \neq a'_2$  and

$$(4b) \quad (0, a'_1, a'_2) \in C.$$

There is a subgroup  $\mathbf{D}'$  of  $G$  such that  $\mathbf{G} = \mathbf{D} \times \mathbf{D}'$ . Hence for each  $t \in G$  there are uniquely determined elements  $t(D) \in D$  and  $t(D') \in D'$  such that  $t = t(D) + t(D')$ .

In particular, we have  $a_1 = a_1(D) + a_1(D')$ , whence  $d_1 - a'_1 = a_1(D) + a_1(D')$  and thus

$$(5) \quad -a_1(D) + d_1 = a_1(D') + a'_1.$$

From  $-a_1(D) + d_1 \in D$  we obtain that  $(-a_1(D) + d_1)(D') = 0$ .

Thus (5) yields that

$$(6) \quad a'_1(D') = -a_1(D').$$

Analogously we obtain

$$(7) \quad a'_2(D') = -a_2(D').$$

If  $a'_1(D') = 0$ , then according to (6) we have  $a_1(D') = 0$ , thus  $a_1 = a_1(D) \in D$ , which is a contradiction (cf. (1)). Therefore

$$(8) \quad a'_1(D') \neq 0.$$

From (4b) and from  $\mathbf{G} = \mathbf{D} \times \mathbf{D}'$  we infer that

$$(9) \quad (0, a'_1(D'), a'_2(D')) \in C_0$$

is valid. Now, (8) and (9) yield that

$$(10) \quad (0, a'_1(D'), a'_2(D')) \in C.$$

In view of (6), (7) and (10) we have

$$(11) \quad (0, -a_1(D'), -a_2(D')) \in C.$$

In particular, the elements  $0, a_1(D')$  and  $a_2(D')$  are distinct. Thus according to (4a)

$$(12) \quad (0, a_1(D'), a_2(D')) \in C,$$

which contradicts (11). □

**4.4. Lemma.** *Let  $\mathbf{G} = \prod_{i \in I} \mathbf{A}_i$  and assume that all  $A_i$  are  $\ell c$ -groups. Suppose that  $\mathbf{D}$  is a direct factor of  $\mathbf{G}$  and that  $\mathbf{D}$  is an  $\ell c$ -group. Then there is  $i(0) \in I$  such that  $\mathbf{D} = \mathbf{A}_{i(0)}$ .*

*Proof.* By way of contradiction, assume that  $\mathbf{D} \neq \mathbf{A}_{i(0)}$  for each  $i(0) \in I$ . Thus  $D \neq A_{i(0)}$  for each  $i(0) \in I$ . Let  $i \in I$ . If  $D \cap A_i \neq \{0\}$ , then from 4.3 we infer that  $D \subseteq A_i$  and, at the same time,  $A_i \subseteq D$ . Therefore  $D = A_i$ , which is a contradiction. Hence  $D \cap A_i = \{0\}$  for each  $i \in I$ . Put  $\mathbf{G}'_i = \prod_{j \in I \setminus \{i\}} \mathbf{A}_j$  for each  $i \in I$ . According to 4.3 the relation  $D \subseteq G'_i$  is valid for each  $i \in I$ . But  $\bigcap_{i \in I} G'_i = \{0\}$  and thus  $D = \{0\}$ , which is a contradiction. □

**4.5. Theorem.** *Let  $\mathbf{G}$  be an ec-group. If  $\mathbf{G}$  can be represented as a direct product of  $\ell c$ -groups, then this representation is unique.*

*Proof.* This is a corollary of 4.4. □

**4.6. Lemma.** *Let  $\mathbf{G} = \mathbf{A} \times \mathbf{B}$  and let  $\mathbf{D}$  be a direct factor of  $\mathbf{G}$  such that  $\mathbf{D}$  is an  $\ell c$ -group,  $\mathbf{G} = \mathbf{D} \times \mathbf{D}'$ . Assume that there is  $0 \neq a \in A$  with  $a = d + d', d \in D, d' \in D', d \neq 0$ . Then  $D \subseteq A$ .*

*Proof.* By way of contradiction, suppose that  $D$  fails to be a subset of  $A$ . Then in view of 4.3 we have  $D \subseteq B$ . Next, according to 3.3 there exists a direct decomposition  $\mathbf{B} = \mathbf{B}_1 \times \mathbf{D}$ . Hence  $\mathbf{G} = \mathbf{A} \times \mathbf{B}_1 \times \mathbf{D}$ . Now, 3.9 yields that  $\mathbf{A} \times \mathbf{B}_1 = \mathbf{D}'$ . Thus the component of the element  $a \in G$  in  $\mathbf{D}$  (with respect to the direct decomposition  $\mathbf{G} = \mathbf{D} \times \mathbf{D}'$ ) is the same as the component of  $a$  in  $\mathbf{D}$  with respect to the direct decomposition  $\mathbf{G} = \mathbf{A} \times \mathbf{B}_1 \times \mathbf{D}$ , whence  $d = 0$ , which is a contradiction. □

**4.7. Lemma.** *Let  $\mathbf{G} = \prod_{i \in I} \mathbf{G}_i$ , where all  $\mathbf{G}_i$  are  $\ell c$ -groups. Let  $\mathbf{A}$  be a nonzero direct factor of  $\mathbf{G}$ . Then there is a nonzero subset  $I(1)$  of  $I$  such that  $\mathbf{A} = \prod_{i \in I(1)} \mathbf{G}_i$ .*

*Proof.* There exists a direct decomposition  $\mathbf{G} = \mathbf{A} \times \mathbf{B}$ . Put  $I(1) = \{i \in I : G_i \subseteq A\}$  and  $I(2) = \{i \in I : G_i \subseteq B\}$ . Then  $I(1) \cap I(2) = \emptyset$ ; next, according to 4.3 we have  $I(1) \cup I(2) = I$ . If  $I(1) = \emptyset$ , then according to 4.2 the relation  $A \cap G_i = \{0\}$  is valid for each  $i \in I(1)$ . Thus by applying the same notation as in the proof of 4.4 we infer that  $A \subseteq G'_i$  for each  $i \in I$  and hence  $A = \{0\}$ , which is a contradiction. Therefore  $I(1) \neq \emptyset$ . Put

$$\mathbf{P} = \prod_{i \in I(1)} \mathbf{G}_i, \quad \mathbf{Q} = \prod_{i \in I(2)} \mathbf{G}_i.$$

Hence  $\mathbf{G} = \mathbf{P} \times \mathbf{Q}$ .

Let  $a \in A, a \neq 0$ . Thus there exists  $i \in I$  with  $a(i) \neq 0$ . In view of 4.6, each such  $i$  belongs to  $I(1)$  and hence  $a \in P$ . Therefore  $A \subseteq P$ . Analogously we can verify that  $B \subseteq Q$ .

Let  $p \in P$ . There are uniquely determined elements  $a_1 \in A$  and  $b_1 \in B$  such that  $p = a_1 + b_1$ . Because of  $A \subseteq P$  and  $B \subseteq Q$  we infer that  $b_1 = 0$  and  $a_1 = p$ . Hence  $P \subseteq A$ . Summarizing, we conclude that  $\mathbf{A} = \mathbf{P}$ . □

**4.8. Theorem.** *Let  $\mathbf{G}$  be an ec-group possessing a direct product decomposition  $\mathbf{G} = \prod_{i \in I} \mathbf{G}_i$ , where all  $\mathbf{G}_i$  are  $\ell c$ -groups. Then  $F(\mathbf{G})$  is an atomic Boolean algebra.*

*Proof.* In view of 4.7,  $F(\mathbf{G})$  is a Boolean algebra. Then according to 4.1,  $F(\mathbf{G})$  is atomic. □

## 5. EXAMPLES OF EC-GROUPS

**5.1.** Let  $\mathbb{R}$  be the additive group of all reals with the natural linear order  $\leq$ . Next, let  $C_{\leq}$  be the cyclic order on  $\mathbb{R}$  defined by means of the linear order  $\leq$ . We denote by  $G$  the set of all triples  $(x, y, z)$  with  $x, y, z \in \mathbb{R}$ . The operation  $+$  on  $G$  is defined componentwise. Let us define a ternary relation  $C$  on  $G$  as follows. Let  $a_i = (x_i, y_i, z_i) \in G$ ,  $i = 1, 2, 3$ . We put  $(a_1, a_2, a_3) \in C$  if (i)  $(x_1, x_2, x_3) \in C_{\leq}$ , and (ii)  $y_1 = y_2 = y_3$ ,  $z_1 = z_2 = z_3$ . Then  $(G, +, C)$  is a cyclically ordered group, whence  $(G, +, C_0)$  is a ec-group.  $(G, +, C)$  fails to be a dc-group (e.g., if  $a_1 = (0, 0, 0)$ ,  $a_2 = (0, 1, 0)$ ,  $a_3 \in G$ , then neither  $(a_1, a_2, a_3) \in C$  nor  $(a_1, a_3, a_2) \in C$  is valid).

**5.2.** If  $(G, \leq, +)$  is a linearly ordered group, then  $(G, C_{\leq}, +)$  is an  $lc$ -group. It is well-known that there exist  $lc$ -groups which cannot be constructed in this way (cf. e.g. [2]). Each  $lc$ -group is a dc-group.

**5.3.** Let  $G$  be the set of all pairs  $(x, y)$  with  $x, y \in \mathbb{R}$ . The operation  $+$  on  $G$  is defined componentwise. The ternary relation  $C$  on  $G$  is defined as follows. Let  $a_i = (x_i, y_i)$ ,  $i = 1, 2, 3$ . We put  $(a_1, a_2, a_3) \in C$  if the following conditions are satisfied (we can consider  $a_i$  to be points in a plane; the relation  $C_{\leq}$  has the same meaning as in 5.1):

- (i)  $a_1, a_2$  and  $a_3$  are distinct and situated on a line;
- (ii) either  $(x_1, x_2, x_3) \in C_{\leq}$ , or  $x_1 = x_2 = x_3$  and  $(y_1, y_2, y_3) \in C_{\leq}$ .

Then  $(G, +, C_0)$  is a dc-group which fails to be an  $lc$ -group.

**5.4.** Let  $G_0$  be the set of all real functions defined on  $\mathbb{R}$ . Next, let  $G$  be the set of all  $f \in G_0$  having the property that the set of all points in which  $f$  fails to be continuous is finite. The operation  $+$  on  $G$  is defined componentwise. Let  $C_{\leq}$  be as in 5.1. Next, let  $C$  be the set of all triples  $(f_1, f_2, f_3) \in G^3$  such that (i)  $f_1, f_2$  and  $f_3$  are distinct, and (ii) for each  $i \in \mathbb{R}$  the relation  $(f_1(i), f_2(i), f_3(i)) \in (C_{\leq})_0$  is valid. Then  $\mathbf{G} = (G, C_0, +)$  is an ec-group. The system  $F(\mathbf{G})$  of all direct factors of  $\mathbf{G}$  is infinite and has no atom.

**5.5.** Let  $(G, +, C_0) = \mathbf{G}$  be as in 5.1. Put

$$A = \{(x, y, 0) : x, y \in \mathbb{R}\}, \quad B = \{(0, 0, z) : z \in \mathbb{R}\},$$

$$D = \{(0, y, z) : y, z \in \mathbb{R} \text{ and } y = z\}.$$

Next, let  $\mathbf{A}, \mathbf{B}$  and  $\mathbf{D}$  be the corresponding ec-groups (with the extended cyclic order inherited from  $\mathbf{G}$ ).

Then we have

$$(1) \quad \mathbf{G} = \mathbf{A} \times \mathbf{B}$$

and

$$(2) \quad \mathbf{G} = \mathbf{A} \times \mathbf{D},$$

but  $\mathbf{B} \neq \mathbf{D}$ . Hence the cancellation law for direct products does not hold in general. Next, if  $g \in G$ , then the component of  $g$  in  $A$  with respect to the direct decomposition (1) need not be equal to the component of  $g$  in  $A$  with respect to the direct decomposition (2).

Let us consider the partially ordered set  $F(\mathbf{G})$ . For  $\mathbf{X}, \mathbf{Y}$  and  $\mathbf{Z}$  the notation  $\mathbf{X} \wedge \mathbf{Y} = \mathbf{Z}$  will mean that  $\mathbf{Z}$  is the greatest lower bound of the system  $\{\mathbf{X}, \mathbf{Y}\}$  (and dually for  $\vee$ ). We have

$$\mathbf{A} \wedge \mathbf{B} = \mathbf{A} \wedge \mathbf{D} = \mathbf{O},$$

$$\mathbf{A} \vee \mathbf{B} = \mathbf{A} \vee \mathbf{B} = \mathbf{G}.$$

Hence  $F(\mathbf{G})$  fails to be a distributive lattice.

**5.6.** Let  $(G, C_0)$  be an ec-set and suppose that  $(G, +)$  is a group such that

(i) whenever  $a, b, c, x, y \in G$  and  $(a, b, c) \in C$ , then

$$(x + a + y, x + b + y, x + c + y) \in C.$$

If, moreover,  $(G, C)$  is a cycle, then from the well-known representation theorem (cf. [7]) we easily obtain that also the following condition is valid:

(ii) whenever  $(a, b, c) \in C_0$  then  $(-c, -b, -a) \in C_0$ .

If we do not assume that  $(G, C)$  is a cycle, then the condition (ii) need not hold. Indeed, let  $(G, +)$  be as in 5.1. Let us now define a ternary relation  $C$  on  $G$  as follows. Put  $a_1 = (1, 0, 0)$ ,  $a_2 = (0, 1, 0)$ ,  $a_3 = (0, 0, 1)$ . For  $b_1, b_2, b_3 \in G$  we put  $(b_1, b_2, b_3) \in C$  if there exist  $z \in G$  and a cyclic permutation  $(j(1), j(2), j(3))$  of  $(1, 2, 3)$  such that  $b_i = a_{j(i)} + z$  is valid for  $i = 1, 2, 3$ .

Then  $(G, C)$  is a cyclically ordered set and for the ec-set  $(G, C_0)$  the condition (i) is satisfied. We have  $(b_1, b_2, b_3) \in C_0$ , but  $(-b_3, -b_2, -b_1)$  does not belong to  $C_0$ .

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