

On extensions of wavelet systems to dual pairs of frames

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Abstract

It is an open problem whether any pair of Bessel sequences with wavelet structure can be extended to a pair of dual frames by adding a pair of singly generated wavelet systems. We consider the particular case where the given wavelet systems are generated by the multiscale setup with trigonometric masks and provide a positive answer under extra assumptions. We also identify a number of conditions that are necessary for the extension to dual multiscale wavelet frames with any number of generators, and show that they imply that an extension with two pairs of wavelet systems is possible. Along the way we provide examples showing that extensions to dual frame pairs are attractive because they often allow better properties than the more popular extensions to tight frames.

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1 Introduction

Extension problems exist in a large variety in the frame literature. In its basic version the question is whether a given sequence of elements in a Hilbert space can be extended to a frame with prescribed properties. For example, it is natural to ask for extensions such that the resulting frame is computationally convenient, e.g., a tight frame or a frame for which a dual frame can be found easily. A natural generalization of this idea is to start with *two* sequences and ask for extension of these sequences to dual frame pairs.

It is known that any pair of Bessel sequences in a separable Hilbert space can be extended to a pair of dual frames by adding appropriate collections of vectors. But if we

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require the added sequences to have a special structure or to satisfy certain constraints, many open problems appear. A key question is whether any given pair of Bessel sequences with wavelet structure can be extended to a pair of dual frames by adding a pair of wavelet systems, each with a single generator (see the paper [16] by D. Han for a discussion and conjecture about the corresponding tight case).

In this paper we will analyze this problem under the extra assumption that the given wavelet systems are generated by the MRA-setup as considered, e.g., in [18, 9, 12]. Given two scaling functions, we will consider the associated wavelet systems generated by letting the masks be trigonometric polynomials, and ask for extensions to dual pairs of frames by adding wavelet systems of the same type. We will first identify a condition on the refinement masks that is necessary for this extension to be possible, and then show that this condition is also sufficient for the possibility to extend to dual pairs using two generators. A stronger condition characterizes the possibility to extend the given wavelet systems to a pair of dual wavelet frames by adding a pair of wavelet systems, each with a single generator.

Note that Daubechies and B. Han already in [11] showed that for any given pair of scaling functions one can construct dual pairs of wavelet frames, each with two generators. Our setup is different from the one in [11]: in our extension of Bessel systems to dual pairs with two generators, we consider one of the pairs of Bessel sequences to be given, i.e., we only have freedom with respect to one pair of wavelet systems.

The rest of this introduction gives a short introduction to the key ingredients of the paper and connects our results to the literature. In Section 2 we present a few preliminary results on Bessel sequences and the mixed extension principle. The results about extensions with singly generated systems are stated in Section 3, while the extension by systems with two generators is treated in Section 4.

A sequence $\{f_i\}_{i \in I}$ in a separable Hilbert space \mathcal{H} is called a *frame* if there exist constants $A, B > 0$ such that

$$A \|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B \|f\|^2, \quad \forall f \in \mathcal{H}.$$

A frame is *tight* if we can choose $A = B$; and if at least the upper frame condition is satisfied, $\{f_i\}_{i \in I}$ is called a *Bessel sequence*. For any frame $\{f_i\}_{i \in I}$, there exist at least one *dual frame*, i.e., a frame $\{g_i\}_{i \in I}$ such that $f = \sum_{i \in I} \langle f, g_i \rangle f_i$, $\forall f \in \mathcal{H}$. A tight frame with $A = B = 1$ leads to an expansion of arbitrary elements $f \in \mathcal{H}$ of exactly the same type as we know for orthonormal bases, i.e., $f = \sum_{i \in I} \langle f, f_i \rangle f_i$, $\forall f \in \mathcal{H}$. For more information on frames we refer to the books [10, 4].

In this paper we will exclusively consider systems of functions in $L^2(\mathbb{R})$ with *wavelet structure*, that is, collections of functions of the type $\{2^{j/2}\psi(2^jx - k)\}_{j,k \in \mathbb{Z}}$ for a fixed function ψ . Considering the operators on $L^2(\mathbb{R})$ given by $T_k f(x) := f(x - k)$ and $Df(x) := 2^{1/2}f(2x)$, the wavelet system can be written as $\{D^j T_k \psi\}_{j,k \in \mathbb{Z}}$.

Let \mathbb{T} denote the unit circle which will be identified with $[-1/2, 1/2]$. Also, for $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ we denote the Fourier transform by $\mathcal{F}f(\gamma) = \hat{f}(\gamma) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \gamma} dx$. As usual, the Fourier transform is extended to a unitary operator on $L^2(\mathbb{R})$.

In the entire paper we will use the following setup that appeared, e.g., in [12], except that we restrict our attention to trigonometric masks.

General setup: Consider a function $\varphi \in L^2(\mathbb{R})$ such that

- (i) $\widehat{\varphi}$ is continuous at the origin and $\widehat{\varphi}(0) = 1$;
- (ii) There exists a 1-periodic trigonometric polynomial m_0 (called a *refinement mask*) such that

$$\widehat{\varphi}(2\gamma) = m_0(\gamma)\widehat{\varphi}(\gamma), \quad \text{a.e. } \gamma \in \mathbb{R}. \quad (1.1)$$

Given 1-periodic trigonometric polynomials m_1, m_2, \dots, m_n , consider the functions $\psi_\ell \in L^2(\mathbb{R})$ defined by

$$\widehat{\psi}_\ell(2\gamma) = m_\ell(\gamma)\widehat{\varphi}(\gamma), \quad \ell = 1, \dots, n. \quad (1.2)$$

Note that the technical condition

$$\sum_{k \in \mathbb{Z}} |\widehat{\varphi}(\gamma + k)|^2 \leq K < \infty,$$

which is used in [12], automatically is satisfied in our setting. In fact, it is well known that φ has compact support whenever the scaling equation (1.1) holds for a trigonometric polynomial m_0 . Thus,

$$\sum_{k \in \mathbb{Z}} |\widehat{\varphi}(\gamma + k)|^2 = \sum_{k \in \mathbb{Z}} \langle \varphi, \varphi(\cdot - k) \rangle e^{-2\pi i k \gamma}$$

is a trigonometric polynomial.

We will base the analysis on the *mixed extension principle* (MEP) by Ron and Shen [18] which is formulated in terms of the $(n+1) \times 2$ matrix-valued functions M and \widetilde{M} defined by

$$M(\gamma) = \begin{pmatrix} m_0(\gamma) & m_0(\gamma + \frac{1}{2}) \\ m_1(\gamma) & m_1(\gamma + \frac{1}{2}) \\ \vdots & \vdots \\ m_n(\gamma) & m_n(\gamma + \frac{1}{2}) \end{pmatrix}, \quad \widetilde{M}(\gamma) = \begin{pmatrix} \widetilde{m}_0(\gamma) & \widetilde{m}_0(\gamma + \frac{1}{2}) \\ \widetilde{m}_1(\gamma) & \widetilde{m}_1(\gamma + \frac{1}{2}) \\ \vdots & \vdots \\ \widetilde{m}_n(\gamma) & \widetilde{m}_n(\gamma + \frac{1}{2}) \end{pmatrix}. \quad (1.3)$$

Formulated for trigonometric masks, the MEP reads as follows:

Proposition 1.1 *Assume that $\varphi, \widetilde{\varphi} \in L^2(\mathbb{R})$ satisfy the conditions in the general setup, with associated masks m_0, \widetilde{m}_0 . For each $\ell = 1, \dots, n$, let $m_\ell, \widetilde{m}_\ell$ be trigonometric polynomials and define $\psi_\ell, \widetilde{\psi}_\ell \in L^2(\mathbb{R})$ by (1.2). Assume that $\{D^j T_k \psi_i\}_{i=1, \dots, n; j, k \in \mathbb{Z}}$ and $\{D^j T_k \widetilde{\psi}_i\}_{i=1, \dots, n; j, k \in \mathbb{Z}}$ are Bessel sequences. If the corresponding matrix-valued functions M and \widetilde{M} satisfy*

$$\widetilde{M}(\gamma)^* M(\gamma) = I, \quad \gamma \in \mathbb{T}, \quad (1.4)$$

then $\{D^j T_k \psi_i\}_{i=1, \dots, n; j, k \in \mathbb{Z}}$ and $\{D^j T_k \widetilde{\psi}_i\}_{i=1, \dots, n; j, k \in \mathbb{Z}}$ form dual frames for $L^2(\mathbb{R})$.

The MEP was later extended to the *mixed oblique extension principle* in [9, 12]; these papers also contain several explicit examples. Other papers about the MEP include [1, 13, 15].

Most of the concrete wavelet frame constructions in the literature are obtained via the related unitary extension principle [18] and its variants, which lead to tight frame constructions (see, e.g., the papers [19, 17, 14, 3, 6], just to mention a few out of many). But it is already noted in, e.g., [12, 9] that the extra flexibility in the MEP frequently leads to more attractive constructions, a claim that is also supported by some of the results in the current paper. For example, we consider a case where the extension of a wavelet system to a tight frame introduces a wavelet generator without compact support, while the extension to a dual pair of frames is possible with compactly supported generators.

Note that the analysis in the current paper is complementary to the one in [5]. In [5] we formulated the general question whether any pair of Bessel sequences $\{D^j T_k \psi_1\}_{j,k \in \mathbb{Z}}$ and $\{D^j T_k \tilde{\psi}_1\}_{j,k \in \mathbb{Z}}$ can be extended to a pair of dual frames by adding a wavelet system to each of the given Bessel sequences. A sufficient condition for a positive answer turned out to be that ψ_1 is compactly supported on $[-1, 1]$. In contrast, the extension principle applied in the current paper involves functions that are compactly supported in time.

Finally, for the sake of the non-specialist, we note that it is known that the dual frames of a wavelet frame not necessarily have wavelet structure: there are cases where no dual wavelet frame exist at all (see, e.g., the books by Chui [7] and Daubechies [10]), and there are cases where some duals have wavelet structure and some do not (see the paper by Bownik and Weber [2]). This issue is one of the key motivations behind the various extension principles, which construct, simultaneously, a frame and a dual with wavelet structure.

2 Preliminaries on Bessel sequences and the MEP

In the entire paper we assume that we have given trigonometric polynomials $m_0, m_1, \tilde{m}_0, \tilde{m}_1$ as described in the general setup. We will search for trigonometric polynomials m_2, \dots, m_n and $\tilde{m}_2, \dots, \tilde{m}_n$ for the cases $n = 2$ and $n = 3$ satisfying the condition (1.4), i.e.,

$$\widetilde{M}(\gamma)^* M(\gamma) = I, \quad \gamma \in \mathbb{T}. \quad (2.1)$$

Note that (2.1) is equivalent to the two conditions

$$\sum_{\ell=0}^n \overline{m_\ell(\gamma)} \tilde{m}_\ell(\gamma) = 1, \quad \gamma \in \mathbb{T}, \quad (2.2)$$

$$\sum_{\ell=0}^n \overline{m_\ell(\gamma)} \tilde{m}_\ell(\gamma + 1/2) = 0, \quad \gamma \in \mathbb{T}. \quad (2.3)$$

In Proposition 1.1 we need that $\{D^j T_k \psi_i\}_{i=1, \dots, n; j, k \in \mathbb{Z}}$ and $\{D^j T_k \tilde{\psi}_i\}_{i=1, \dots, n; j, k \in \mathbb{Z}}$ are Bessel sequences. Since these systems are finite union of wavelet systems, it is sufficient that each of these form a Bessel sequence. The following lemma provides necessary and sufficient conditions for this:

Lemma 2.1 Let $\varphi \in L^2(\mathbb{R})$ be a scaling function, with a refinement mask m_0 satisfying the conditions in the general setup. Let m be a trigonometric polynomial and define ψ by

$$\widehat{\psi}(2\gamma) = m(\gamma)\widehat{\varphi}(\gamma), \quad \ell = 1, \dots, n. \quad (2.4)$$

Then the following are equivalent:

- (a) $\{D^j T_k \psi\}_{j,k \in \mathbb{Z}}$ is a Bessel sequence;
- (b) $m(0) = 0$.

Proof. (a) \Rightarrow (b) : Assume that $\{D^j T_k \psi\}_{j,k \in \mathbb{Z}}$ is a Bessel sequence with bound B . Using [8, Theorem 1] (or see Proposition 11.2.2 in [4]), we have $\sum_{j \in \mathbb{Z}} \left| \widehat{\psi}(2^j \gamma) \right|^2 \leq B$. By the conditions in the general setup, $\widehat{\psi}$ is continuous at the origin, so it follows that $\widehat{\psi}(0) = 0$. Hence, via (2.4) and the assumption $\widehat{\varphi}(0) = 1$ we conclude that $m(0) = 0$.
(b) \Rightarrow (a) : See [11, Lemma 2.1]. \square

Example 2.2 Let B_N , $N \in \mathbb{N}$, denote the N th order B-spline, defined recursively by

$$B_1 := \chi_{[0,1]}, \quad B_{N+1} := B_1 * B_N.$$

It is well known that $\varphi := B_N$ satisfies the conditions in the general setup with $m_0(\gamma) = (e^{-\pi i \gamma} \cos(\pi \gamma))^N$. Thus Lemma 2.1 shows that $\{D^j T_k B_N\}_{j,k \in \mathbb{Z}}$ is not a Bessel sequence in $L^2(\mathbb{R})$. On the other hand, a finite linear combination

$$\psi_1(x) = \sum_{k=-K}^K c_k B_N(2x - k)$$

generates a Bessel sequences $\{D^j T_k \psi_1\}_{j,k \in \mathbb{Z}}$ if and only if $\sum_{k=-K}^K c_k = 0$. \square

The following example demonstrates that the matrix condition in Theorem 1.1 is not sufficient for duality, i.e., the assumption of the sequences $\{D^j T_k \psi_i\}_{i=1, \dots, n; j, k \in \mathbb{Z}}$ and $\{D^j T_k \widetilde{\psi}_i\}_{i=1, \dots, n; j, k \in \mathbb{Z}}$ being Bessel sequences is essential:

Example 2.3 Consider the scaling functions $\varphi(x) = \widetilde{\varphi}(x) = B_2(x)$ and

$$\begin{aligned} \psi_1(x) &= \widetilde{\psi}_1(x) = -\frac{1}{2}B_2(2x) + B_2(2x - 1) - \frac{1}{2}B_2(2x - 2), \\ \psi_2(x) &= -\frac{1}{4}B_2(2x + 2) + \frac{1}{2}B_2(2x) - \frac{1}{4}B_2(2x - 2), \quad \widetilde{\psi}_2(x) = 2B_2(2x), \end{aligned}$$

with the associated masks

$$\begin{aligned} m_0(\gamma) &= \widetilde{m}_0(\gamma) = e^{-2\pi i \gamma} \cos^2(\pi \gamma), \\ m_1(\gamma) &= \widetilde{m}_1(\gamma) = e^{-2\pi i \gamma} \sin^2(\pi \gamma), \quad m_2(\gamma) = 2 \cos^2(\pi \gamma) \sin^2(\pi \gamma), \quad \widetilde{m}_2(\gamma) = 1. \end{aligned}$$

Then the MEP-condition (1.4) with $n = 2$ in Proposition 1.1 is satisfied. But by Lemma 2.1 we know that $\{D^j T_k \psi_2\}_{j,k \in \mathbb{Z}}$ is not a Bessel sequence. Hence $\{D^j T_k \psi_i\}_{i=1,2; j,k \in \mathbb{Z}}$ and $\{D^j T_k \widetilde{\psi}_i\}_{i=1,2; j,k \in \mathbb{Z}}$ does not form dual frames for $L^2(\mathbb{R})$. \square

In Lemma 2.5 we will state three necessary conditions for the existence of MEP-type wavelet systems $\{D^j T_k \psi_i\}_{i=1, \dots, n; j, k \in \mathbb{Z}}$ and $\{D^j T_k \tilde{\psi}_i\}_{i=1, \dots, n; j, k \in \mathbb{Z}}$. We first need the following factorizations:

Lemma 2.4

- (i) *Let f be a 1-periodic trigonometric polynomial with $f(0) = 0$. Then $f(\gamma) = e^{-\pi i \gamma} \sin(\pi \gamma) \Lambda_1(\gamma)$ for a 1-periodic trigonometric polynomial Λ_1 ;*
- (ii) *Let g be a 1-periodic trigonometric polynomial with $g(1/2) = 0$. Then $g(\gamma) = e^{-\pi i \gamma} \cos(\pi \gamma) \Lambda_2(\gamma)$ for a 1-periodic trigonometric polynomial Λ_2 .*

Proof. For the proof of (i), write f as $f(\gamma) = \sum_{k \in \mathbb{Z}} c_k e^{-2\pi i k \gamma}$. Since $f(0) = \sum_{k \in \mathbb{Z}} c_k = 0$, we have $f(\gamma) = \sum_{k \neq 0} c_k (e^{-2\pi i k \gamma} - 1)$. Define f_+ and f_- by

$$f_+(\gamma) := \sum_{k \in \mathbb{N}} c_k (e^{-2\pi i k \gamma} - 1), \quad f_-(\gamma) := \sum_{k \in \mathbb{N}} c_{-k} (e^{2\pi i k \gamma} - 1).$$

Then we see that

$$\begin{aligned} f_+(\gamma) &= \sum_{k \in \mathbb{N}} c_k (e^{-2\pi i k \gamma} - 1) \sum_{\ell=0}^{k-1} e^{-2\pi i \ell \gamma} = e^{-\pi i \gamma} \sin(\pi \gamma) \left(-2i \sum_{k \in \mathbb{N}} c_k \sum_{\ell=0}^{k-1} e^{-2\pi i \ell \gamma} \right) \\ &=: e^{-\pi i \gamma} \sin(\pi \gamma) \Lambda_+(\gamma). \end{aligned}$$

Similarly,

$$f_-(\gamma) = e^{-\pi i \gamma} \sin(\pi \gamma) \left(2i \sum_{k \in \mathbb{N}} c_{-k} \sum_{\ell=1}^k e^{2\pi i \ell \gamma} \right) =: e^{-\pi i \gamma} \sin(\pi \gamma) \Lambda_-(\gamma).$$

Then we have $f(\gamma) = f_+(\gamma) + f_-(\gamma) = e^{-\pi i \gamma} \sin(\pi \gamma) \Lambda_1(\gamma)$, where $\Lambda_1(\gamma) := \Lambda_+(\gamma) + \Lambda_-(\gamma)$ is a 1-periodic trigonometric polynomial. This proves (i).

For the proof of (ii), let $\tilde{g}(\gamma) := g(\gamma + 1/2)$. Since $\tilde{g}(0) = 0$, there exists a 1-periodic trigonometric polynomial Λ such that $\tilde{g}(\gamma) = e^{-\pi i \gamma} \sin(\pi \gamma) \Lambda(\gamma)$. Then we have $g(\gamma) = \tilde{g}(\gamma - 1/2) = e^{-\pi i \gamma} \cos(\pi \gamma) \Lambda_2(\gamma)$, where $\Lambda_2(\gamma) := -i \Lambda(\gamma - 1/2)$. This proves (ii). \square

Lemma 2.5 *Under the hypothesis of Proposition 1.1, the following hold:*

- (a) $m_\ell(0) = \tilde{m}_\ell(0) = 0$, $\ell = 1, 2, \dots, n$;
- (b) $m_0(1/2) = \tilde{m}_0(1/2) = 0$;
- (c) $1 - \overline{m_0(\gamma)} \tilde{m}_0(\gamma) = \sin^2(\pi \gamma) \Lambda(\gamma)$ for some 1-periodic trigonometric polynomial Λ .

Proof. (a) : This follows by Lemma 2.1.

(b) : Note that $m_0(0) = \tilde{m}_0(0) = 1$ by the assumptions in the general setup. This together with (a) and (2.3) implies $m_0(1/2) = \tilde{m}_0(1/2) = 0$.

(c) : By (a) and Lemma 2.4, the functions $m_\ell, \tilde{m}_\ell, \ell = 1, 2, \dots, n$, can be factorized as

$$m_\ell(\gamma) = e^{-\pi i \gamma} \sin(\pi \gamma) \Lambda_\ell(\gamma), \quad \tilde{m}_\ell(\gamma) = e^{-\pi i \gamma} \sin(\pi \gamma) \tilde{\Lambda}_\ell(\gamma),$$

for some 1-periodic trigonometric polynomials $\Lambda_\ell, \tilde{\Lambda}_\ell$. Combining this with (2.2) leads to $1 - \overline{m_0(\gamma)} \tilde{m}_0(\gamma) = \sin^2(\pi \gamma) \Lambda(\gamma)$, where $\Lambda(\gamma) = \sum_{\ell=1}^n \overline{\Lambda_\ell(\gamma)} \tilde{\Lambda}_\ell(\gamma)$. \square

In Section 4 we will show that, on the other hand, the assumptions (a), (b) and (c) in Lemma 2.5 implies that the wavelet systems $\{D^j T_k \psi_1\}_{j,k \in \mathbb{Z}}, \{D^j T_k \tilde{\psi}_1\}_{j,k \in \mathbb{Z}}$ can be extended to pairs of dual wavelet frames by adding two wavelet systems.

3 Extension with one pair of generators

In the rest of the paper we will consider scaling functions $\varphi, \tilde{\varphi} \in L^2(\mathbb{R})$ as in the general setup, with associated trigonometric polynomial masks m_0, \tilde{m}_0 . Assuming that we have given trigonometric polynomials m_1, \tilde{m}_1 and defined the associated functions $\psi_1, \tilde{\psi}_1$ by (1.2), our goal is to extend the Bessel sequences $\{D^j T_k \psi_1\}_{j,k \in \mathbb{Z}}, \{D^j T_k \tilde{\psi}_1\}_{j,k \in \mathbb{Z}}$ to pairs of dual wavelet frames. It turns out to be convenient to consider the functions \tilde{M}_α and \tilde{M}_β , defined by

$$\tilde{M}_\alpha(\gamma) := 1 - \overline{m_0(\gamma)} \tilde{m}_0(\gamma) - \overline{m_1(\gamma)} \tilde{m}_1(\gamma); \quad (3.1)$$

$$\tilde{M}_\beta(\gamma) := -\overline{m_0(\gamma)} \tilde{m}_0(\gamma + 1/2) - \overline{m_1(\gamma)} \tilde{m}_1(\gamma + 1/2). \quad (3.2)$$

Lemma 3.1 *If the conditions (a), (b) and (c) in Lemma 2.5 are satisfied, then \tilde{M}_α and \tilde{M}_β can be factorized as*

$$\tilde{M}_\alpha(\gamma) = \sin^2(\pi \gamma) \Lambda_\alpha(\gamma), \quad \tilde{M}_\beta(\gamma) = -i \sin(\pi \gamma) \cos(\pi \gamma) \Lambda_\beta(\gamma) \quad (3.3)$$

for some 1-periodic trigonometric polynomials Λ_α and Λ_β .

Proof. By Lemma 2.4, and (a) and (b) of Lemma 2.5, $m_\ell, \tilde{m}_\ell, \ell = 0, 1$ can be factorized as

$$\begin{aligned} m_0(\gamma) &= e^{-\pi i \gamma} \cos(\pi \gamma) \Lambda_0(\gamma), & \tilde{m}_0(\gamma) &= e^{-\pi i \gamma} \cos(\pi \gamma) \tilde{\Lambda}_0(\gamma), \\ m_1(\gamma) &= e^{-\pi i \gamma} \sin(\pi \gamma) \Lambda_1(\gamma), & \tilde{m}_1(\gamma) &= e^{-\pi i \gamma} \sin(\pi \gamma) \tilde{\Lambda}_1(\gamma), \end{aligned}$$

for some 1-periodic trigonometric polynomials $\Lambda_\ell, \tilde{\Lambda}_\ell$ for $\ell = 0, 1$. Together with (c) of Lemma 2.5 this implies

$$\tilde{M}_\alpha(\gamma) = \sin^2(\pi \gamma) \left(\Lambda(\gamma) - \overline{\Lambda_1(\gamma)} \tilde{\Lambda}_1(\gamma) \right) =: \sin^2(\pi \gamma) \Lambda_\alpha(\gamma),$$

and

$$\begin{aligned}\widetilde{M}_\beta(\gamma) &= -i \sin(\pi\gamma) \cos(\pi\gamma) \left(\overline{\Lambda_0(\gamma)} \widetilde{\Lambda}_0(\gamma + \pi/2) - \overline{\Lambda_1(\gamma)} \widetilde{\Lambda}_1(\gamma + \pi/2) \right) \\ &=: -i \sin(\pi\gamma) \cos(\pi\gamma) \Lambda_\beta(\gamma),\end{aligned}$$

as desired. \square

We are now ready to state a condition for extension of MRA-type wavelet systems $\{D^j T_k \psi_1\}_{j,k \in \mathbb{Z}}$, $\{D^j T_k \widetilde{\psi}_1\}_{j,k \in \mathbb{Z}}$ to dual frames $\{D^j T_k \psi_\ell\}_{\ell=1,2; j,k \in \mathbb{Z}}$, $\{D^j T_k \widetilde{\psi}_\ell\}_{\ell=1,2; j,k \in \mathbb{Z}}$. Note that in Theorem 3.2 below, the condition (i) in (I) means that $\{D^j T_k \psi_2\}_{j,k \in \mathbb{Z}}$ and $\{D^j T_k \widetilde{\psi}_2\}_{j,k \in \mathbb{Z}}$ are Bessel sequences, while (ii) simply is the MEP-condition. We also note that the proof shows how to choose the corresponding masks m_2, \widetilde{m}_2 . With this information we can find the functions $\psi_2, \widetilde{\psi}_2$ explicitly: if, e.g., $m_2(\gamma) = \sum d_k e^{2\pi i k \gamma}$, then $\psi_2 = 2 \sum d_k \varphi(2x + k)$.

Theorem 3.2 *Let $\varphi, \widetilde{\varphi} \in L^2(\mathbb{R})$ be as in the general setup, with trigonometric polynomial masks m_0, \widetilde{m}_0 , respectively. Let m_1, \widetilde{m}_1 be trigonometric polynomials, and define $\psi_1, \widetilde{\psi}_1 \in L^2(\mathbb{R})$ by (1.2). Assume that the conditions (a), (b) and (c) in Lemma 2.5 for $m_0, \widetilde{m}_0, m_1, \widetilde{m}_1$ are satisfied. Then the following are equivalent:*

(I) *There exist 1-periodic trigonometric polynomials m_2, \widetilde{m}_2 such that*

$$(i) \quad m_2(0) = \widetilde{m}_2(0) = 0;$$

(ii) *the matrix-valued functions M, \widetilde{M} in (1.3) with $n = 2$ satisfy*

$$\widetilde{M}(\gamma)^* M(\gamma) = I, \quad \gamma \in \mathbb{T},$$

$$(II) \quad \widetilde{M}_\alpha(\gamma) \widetilde{M}_\alpha(\gamma + 1/2) = \widetilde{M}_\beta(\gamma) \widetilde{M}_\beta(\gamma + 1/2), \quad \gamma \in \mathbb{T}.$$

In the affirmative case, the multi-wavelet systems $\{D^j T_k \psi_\ell\}_{\ell=1,2; j,k \in \mathbb{Z}}$ and $\{D^j T_k \widetilde{\psi}_\ell\}_{\ell=1,2; j,k \in \mathbb{Z}}$, with $\psi_2, \widetilde{\psi}_2$ defined by (1.2), form dual frames for $L^2(\mathbb{R})$.

Proof. (I) \Rightarrow (II): This follows from (2.2) and (2.3):

$$\begin{aligned}\widetilde{M}_\alpha(\gamma) \widetilde{M}_\alpha(\gamma + 1/2) &= \overline{m_2(\gamma)} \widetilde{m}_2(\gamma) \overline{m_2(\gamma + 1/2)} \widetilde{m}_2(\gamma + 1/2) \\ &= \widetilde{M}_\beta(\gamma) \widetilde{M}_\beta(\gamma + 1/2).\end{aligned}$$

(II) \Rightarrow (I): Let $\widetilde{M}_\alpha, \widetilde{M}_\beta$ be factorized as (3.3) for 1-periodic trigonometric polynomials $\Lambda_\alpha, \Lambda_\beta$. Then Lemma 2.5 (b) implies

$$\Lambda_\alpha(\gamma) \Lambda_\alpha(\gamma + 1/2) = \Lambda_\beta(\gamma) \Lambda_\beta(\gamma + 1/2). \quad (3.4)$$

Let Γ be the common factor of Λ_α and Λ_β , that is,

$$\Lambda_\alpha(\gamma) = \Gamma(\gamma) \Gamma_\alpha(\gamma), \quad \Lambda_\beta(\gamma) = \Gamma(\gamma) \Gamma_\beta(\gamma), \quad (3.5)$$

for some 1-periodic trigonometric polynomials $\Gamma_\alpha, \Gamma_\beta$ with no common factors. This together with (3.4) implies

$$\Gamma_\alpha(\gamma)\Gamma_\alpha(\gamma + 1/2) = \Gamma_\beta(\gamma)\Gamma_\beta(\gamma + 1/2).$$

Since Γ_α and Γ_β have no common factor, we have

$$\Gamma_\beta(\gamma) = \Gamma_\alpha(\gamma + 1/2) \quad (3.6)$$

up to constant. Define m_2 and \tilde{m}_2 by

$$m_2(\gamma) := e^{-\pi i \gamma} \sin(\pi \gamma) \overline{\Gamma(\gamma)}, \quad \tilde{m}_2(\gamma) := e^{-\pi i \gamma} \sin(\pi \gamma) \Gamma_\alpha(\gamma).$$

Then (i) is trivial. By (3.5) and (3.6), we have

$$\overline{m_2(\gamma)} \tilde{m}_2(\gamma) = \sin^2(\pi \gamma) \Gamma(\gamma) \Gamma_\alpha(\gamma) = \widetilde{M}_\alpha(\gamma),$$

and

$$\overline{m_2(\gamma)} \tilde{m}_2(\gamma + 1/2) = -i \sin(\pi \gamma) \cos(\pi \gamma) \Gamma(\gamma) \Gamma_\alpha(\gamma + 1/2) = \widetilde{M}_\beta(\gamma).$$

These lead to (ii) by (2.2) and (2.3) with $n = 2$. \square

For the case where $\varphi = \tilde{\varphi} = B_2$, we can characterize the possible trigonometric polynomials m_1, \tilde{m}_1 with at most three terms that satisfy the conditions in Theorem 3.2. Our main reason for stating this is that we can use the result to identify concrete candidates for pairs of wavelet systems that can not be extended to a pair of dual wavelet frames by adding a single pair of wavelet systems, see Example 3.5.

Corollary 3.3 *Let $d_0, d_1, \tilde{d}_0, \tilde{d}_1 \in \mathbb{C}$. Define ψ_1 and $\tilde{\psi}_1$ by*

$$\psi_1(x) := d_0 B_2(2x) + (d_1 - d_0) B_2(2x - 1) - d_1 B_2(2x - 2); \quad (3.7)$$

$$\tilde{\psi}_1(x) := \tilde{d}_0 B_2(2x) + (\tilde{d}_1 - \tilde{d}_0) B_2(2x - 1) - \tilde{d}_1 B_2(2x - 2). \quad (3.8)$$

Then the following are equivalent:

- (a) *There exist 1-periodic trigonometric polynomials m_2 and \tilde{m}_2 such that (I) in Theorem 3.2 holds;*
- (b) $3\bar{d}_0\tilde{d}_0 + 3\bar{d}_1\tilde{d}_1 - \bar{d}_1\tilde{d}_0 - \bar{d}_0\tilde{d}_1 = 2.$

Proof. Let $\varphi := \tilde{\varphi} := B_2$ with the associated masks $m_0(\gamma) := \tilde{m}_0(\gamma) := (e^{-\pi i \gamma} \cos(\pi \gamma))^2$. From (3.7) and (3.8), m_1 and \tilde{m}_1 are defined by

$$m_1(\gamma) = \frac{d_0}{2} + \frac{d_1 - d_0}{2} e^{-2\pi i \gamma} - \frac{d_1}{2} e^{-4\pi i \gamma}, \quad \tilde{m}_1(\gamma) = \frac{\tilde{d}_0}{2} + \frac{\tilde{d}_1 - \tilde{d}_0}{2} e^{-2\pi i \gamma} - \frac{\tilde{d}_1}{2} e^{-4\pi i \gamma}. \quad (3.9)$$

Trivially, (a), (b) and (c) in Lemma 2.5 for $m_0, \tilde{m}_0, m_1, \tilde{m}_1$ are satisfied. We now check the condition (II) in Theorem 3.2. Note that

$$\widetilde{M}_\alpha(\gamma) = 1 - \cos^4(\pi\gamma) - \overline{m_1(\gamma)}\tilde{m}_1(\gamma), \quad \widetilde{M}_\beta(\gamma) = \cos^2(\pi\gamma)\sin^2(\pi\gamma) - \overline{m_1(\gamma)}\tilde{m}_1(\gamma + 1/2).$$

Then we have

$$\begin{aligned} \widetilde{M}_\alpha(\gamma)\widetilde{M}_\alpha(\gamma + 1/2) &= (1 - \cos^4(\pi\gamma))(1 - \sin^4(\pi\gamma)) - (1 - \cos^4(\pi\gamma))\overline{m_1(\gamma + 1/2)}\tilde{m}_1(\gamma + 1/2) \\ &\quad - (1 - \sin^4(\pi\gamma))\overline{m_1(\gamma)}\tilde{m}_1(\gamma) + \overline{m_1(\gamma)m_1(\gamma + 1/2)}\tilde{m}_1(\gamma)\tilde{m}_1(\gamma + 1/2) \end{aligned}$$

and

$$\begin{aligned} &\widetilde{M}_\beta(\gamma)\widetilde{M}_\beta(\gamma + 1/2) \\ &= \cos^4(\pi\gamma)\sin^4(\pi\gamma) - \sin^2(\pi\gamma)\cos^2(\pi\gamma)\overline{m_1(\gamma + 1/2)}\tilde{m}_1(\gamma) \\ &\quad - \cos^2(\pi\gamma)\sin^2(\pi\gamma)\overline{m_1(\gamma)}\tilde{m}_1(\gamma + 1/2) + \overline{m_1(\gamma)m_1(\gamma + 1/2)}\tilde{m}_1(\gamma)\tilde{m}_1(\gamma + 1/2). \end{aligned}$$

Using the identities

$$\begin{aligned} 1 - \cos^4(\pi\gamma) &= \sin^2(\pi\gamma)(1 + \cos^2(\pi\gamma)), \quad 1 - \sin^4(\pi\gamma) = \cos^2(\pi\gamma)(1 + \sin^2(\pi\gamma)), \\ \cos^2(\pi\gamma) - \cos^4(\pi\gamma) &= \sin^2(\pi\gamma) - \sin^4(\pi\gamma) = \cos^2(\pi\gamma)\sin^2(\pi\gamma), \end{aligned}$$

the condition (II) in Theorem 3.2 is equivalent to

$$\begin{aligned} &\sin^2(\pi\gamma)\cos^2(\pi\gamma)\left(2 - \left(\overline{m_1(\gamma)} - \overline{m_1(\gamma + 1/2)}\right)(\tilde{m}_1(\gamma) - \tilde{m}_1(\gamma + 1/2))\right) \\ &= \sin^2(\pi\gamma)\overline{m_1(\gamma + 1/2)}\tilde{m}_1(\gamma + 1/2) + \cos^2(\pi\gamma)\overline{m_1(\gamma)}\tilde{m}_1(\gamma). \end{aligned} \quad (3.10)$$

From (3.9), m_1 and \tilde{m}_1 can be factorized as

$$m_1(\gamma) = e^{-\pi i \gamma} \sin(\pi\gamma)\Lambda_1(\gamma), \quad \tilde{m}_1(\gamma) = e^{-\pi i \gamma} \sin(\pi\gamma)\tilde{\Lambda}_1(\gamma),$$

where

$$\Lambda_1(\gamma) := i(d_0 + d_1 e^{-2\pi i \gamma}), \quad \tilde{\Lambda}_1(\gamma) := i(\tilde{d}_0 + \tilde{d}_1 e^{-2\pi i \gamma}). \quad (3.11)$$

Since $\overline{m_1(\gamma)}\tilde{m}_1(\gamma) = \sin^2(\pi\gamma)\overline{\Lambda_1(\gamma)}\tilde{\Lambda}_1(\gamma)$, (3.10) is equivalent with

$$2 - \left(\overline{m_1(\gamma)} - \overline{m_1(\gamma + 1/2)}\right)(\tilde{m}_1(\gamma) - \tilde{m}_1(\gamma + 1/2)) = \overline{\Lambda_1(\gamma + 1/2)}\tilde{\Lambda}_1(\gamma + 1/2) + \overline{\Lambda_1(\gamma)}\tilde{\Lambda}_1(\gamma),$$

This together with (3.9) and (3.11) leads to $2 - (\bar{d}_1 - \bar{d}_0)(\tilde{d}_1 - \tilde{d}_0) = 2(\bar{d}_0\tilde{d}_0 + \bar{d}_1\tilde{d}_1)$, that is, $3\bar{d}_0\tilde{d}_0 + 3\bar{d}_1\tilde{d}_1 - \bar{d}_1\tilde{d}_0 - \bar{d}_0\tilde{d}_1 = 2$, as desired. \square

Let us consider a concrete case and find the functions $\psi_2, \tilde{\psi}_2$ explicitly:

Example 3.4 Let $d_0 = 1, d_1 = 0, \tilde{d}_0 = \frac{1}{2}, \tilde{d}_1 = -\frac{1}{2}$ and consider

$$\psi_1(x) = B_2(2x) - B_2(2x - 1), \tilde{\psi}_1(x) = \frac{1}{2}B_2(2x) - B_2(2x - 1) + \frac{1}{2}B_2(2x - 1),$$

with the associated masks

$$m_1(\gamma) = ie^{-\pi i \gamma} \sin(\pi \gamma), \quad \tilde{m}_1(\gamma) = -e^{-2\pi i \gamma} \sin^2(\pi \gamma).$$

Then (b) in Corollary 3.3 holds. We now use (II) \Rightarrow (I) in Theorem 3.2 to construct ψ_2 and $\tilde{\psi}_2$ such that the multi-wavelet systems $\{D^j T_k \psi_\ell\}_{\ell=1,2;j,k \in \mathbb{Z}}$ and $\{D^j T_k \tilde{\psi}_\ell\}_{\ell=1,2;j,k \in \mathbb{Z}}$ form dual frames for $L^2(\mathbb{R})$. A direct calculation shows that

$$\widetilde{M}_\alpha(\gamma) = \sin^2(\pi \gamma) \Lambda_\alpha(\gamma), \quad \widetilde{M}_\beta(\gamma) = -i \sin(\pi \gamma) \cos(\pi \gamma) \Lambda_\beta(\gamma),$$

where

$$\Lambda_\alpha(\gamma) = (1 + e^{-2\pi i \gamma}) (e^{2\pi i \gamma} + 3), \quad \Lambda_\beta(\gamma) = (1 + e^{-2\pi i \gamma}) (e^{2\pi i \gamma} - 3)$$

Letting $\Gamma(\gamma) = -i(1 + e^{-2\pi i \gamma})$ we have $\Lambda_\alpha(\gamma) = \Gamma(\gamma)\Gamma_\alpha(\gamma), \Lambda_\beta(\gamma) = \Gamma(\gamma)\Gamma_\beta(\gamma)$, where $\Gamma_\alpha(\gamma) = i(e^{2\pi i \gamma} + 3), \Gamma_\beta(\gamma) = i(e^{2\pi i \gamma} - 3)$. Let

$$m_2(\gamma) = e^{-\pi i \gamma} \sin(\pi \gamma) \overline{\Gamma(\gamma)} = e^{-\pi i \gamma} \sin(\pi \gamma) i (e^{2\pi i \gamma} + 1), \\ \tilde{m}_2(\gamma) = e^{-\pi i \gamma} \sin(\pi \gamma) \Gamma_\alpha(\gamma) = e^{-\pi i \gamma} \sin(\pi \gamma) i (e^{2\pi i \gamma} + 3).$$

and define $\psi_2, \tilde{\psi}_2$ by (1.2), i.e.,

$$\psi_2(x) = B_2(2x + 1) - B_2(2x - 1), \quad \tilde{\psi}_2(x) = B_2(2x + 1) + 2B_2(2x) - 3B_2(2x - 1).$$

By Theorem 3.2, $\{D^j T_k \psi_\ell\}_{\ell=1,2;j,k \in \mathbb{Z}}$ and $\{D^j T_k \tilde{\psi}_\ell\}_{\ell=1,2;j,k \in \mathbb{Z}}$ form dual frames. \square

Remember that our main motivation is the question whether any pair of Bessel wavelet systems can be extended to a pair of dual wavelet frames by adding a single pair of wavelet systems. Based on Corollary 3.3 we can state a concrete candidate for a counter example:

Example 3.5 Let $d_0 = \tilde{d}_0 = 1, d_1 = \tilde{d}_1 = 0$ and consider

$$\psi_1(x) = \tilde{\psi}_1(x) = B_2(2x) - B_2(2x - 1).$$

Then we have

$$3\bar{d}_0\tilde{d}_0 + 3\bar{d}_1\tilde{d}_1 - \bar{d}_1\tilde{d}_0 - \bar{d}_0\tilde{d}_1 = 3 \neq 2.$$

By Corollary 3.3, there does not exist a pair of $\psi_2, \tilde{\psi}_2$ such that $\widetilde{M}(\gamma)^* M(\gamma) = I$ with $n = 2$. Thus the wavelet systems $\{D^j T_k \psi_1\}_{j,k \in \mathbb{Z}}$ and $\{D^j T_k \tilde{\psi}_1\}_{j,k \in \mathbb{Z}}$ can not be extended to dual pairs of MRA-based wavelet frames $\{D^j T_k \psi_\ell\}_{\ell=1,2;j,k \in \mathbb{Z}}, \{D^j T_k \tilde{\psi}_\ell\}_{\ell=1,2;j,k \in \mathbb{Z}}$. On the other hand it is an open question whether the systems $\{D^j T_k \psi_1\}_{j,k \in \mathbb{Z}}, \{D^j T_k \tilde{\psi}_1\}_{j,k \in \mathbb{Z}}$ can be extended to dual wavelet pairs $\{D^j T_k \psi_\ell\}_{\ell=1,2;j,k \in \mathbb{Z}}, \{D^j T_k \tilde{\psi}_\ell\}_{\ell=1,2;j,k \in \mathbb{Z}}$ for which the functions $\psi_2, \tilde{\psi}_2$ do not have the MRA-structure. \square

The following example illustrates the flexibility of the MEP-approach compared with the UEP. In fact, we will consider masks m_0 and m_1 as in the general setup, with the property that the wavelet system $\{D^j T_k \psi_1\}_{j,k \in \mathbb{Z}}$ can not be extended to a tight wavelet frame $\{D^j T_k \psi_\ell\}_{\ell=1,2;j,k \in \mathbb{Z}}$ using the UEP. On the other hand, by considering a second pair of masks $\widetilde{m}_0, \widetilde{m}_1$, the wavelet systems $\{D^j T_k \psi_1\}_{j,k \in \mathbb{Z}}$ and $\{D^j T_k \widetilde{\psi}_1\}_{j,k \in \mathbb{Z}}$ can be extended to dual pairs $\{D^j T_k \psi_\ell\}_{\ell=1,2;j,k \in \mathbb{Z}}, \{D^j T_k \widetilde{\psi}_\ell\}_{\ell=1,2;j,k \in \mathbb{Z}}$ via the MEP:

Example 3.6 Consider

$$\varphi(x) := B_1(x), \quad \psi_1(x) := \frac{1}{4}B_1(2x+2) + \frac{1}{4}B_1(2x+1) - \frac{1}{4}B_1(2x) - \frac{1}{4}B_1(2x+2)$$

with the associated masks

$$m_0(\gamma) = e^{-\pi i \gamma} \cos(\pi \gamma), \quad m_1(\gamma) = i e^{\pi i \gamma} \sin(\pi \gamma) \cos^2(\pi \gamma).$$

Let

$$M_\alpha(\gamma) = 1 - |m_0(\gamma)|^2 - |m_1(\gamma)|^2, \quad M_\beta(\gamma) = -m_0(\gamma)\overline{m}_0(\gamma + 1/2) - m_1(\gamma)\overline{m}_1(\gamma + 1/2).$$

A direct calculation shows that $M_\alpha(\gamma)M_\alpha(\gamma + 1/2) \not\equiv M_\beta(\gamma)M_\beta(\gamma + 1/2)$; by [6, Theorem 2.2], this implies that there does not exist $m_2 \in L^\infty(\mathbb{R})$ such that the multi-wavelet system $\{D^j T_k \psi_\ell\}_{\ell=1,2;j,k \in \mathbb{Z}}$, with ψ_2 defined by (1.2), forms a Parseval frame for $L^2(\mathbb{R})$.

On the other hand, consider now

$$\widetilde{\varphi}(x) := B_3(x+1), \quad \widetilde{\psi}_1(x) := B_3(2x+2) - B_3(2x+1)$$

with the associated masks $\widetilde{m}_0(\gamma) = e^{-\pi i \gamma} \cos^3(\pi \gamma), \widetilde{m}_1(\gamma) = i e^{\pi i \gamma} \sin(\pi \gamma)$. A direct calculation shows that

$$\widetilde{M}_\alpha(\gamma) = \sin^2(\pi \gamma), \quad \widetilde{M}_\beta(\gamma) = -i \sin(\pi \gamma) \cos(\pi \gamma).$$

Then we have

$$\widetilde{M}_\alpha(\gamma)\widetilde{M}_\alpha(\gamma + 1/2) = \widetilde{M}_\beta(\gamma)\widetilde{M}_\beta(\gamma + 1/2) = \sin^2(\pi \gamma) \cos^2(\pi \gamma).$$

It is easy to check that the conditions (a), (b) and (c) in Lemma 2.5 are satisfied, so by Theorem 3.2 there exists functions $\psi_2, \widetilde{\psi}_2$ such that $\{D^j T_k \psi_\ell\}_{\ell=1,2;j,k \in \mathbb{Z}}$ and $\{D^j T_k \widetilde{\psi}_\ell\}_{\ell=1,2;j,k \in \mathbb{Z}}$ form dual frames. By the proof of Theorem 3.2 we can define ψ_2 and $\widetilde{\psi}_2$ by

$$\psi_2(x) = B_1(2x) - B_1(2x-1), \quad \widetilde{\psi}_2(x) = B_3(2x+1) - B_3(2x)$$

with the associated masks $m_2(\gamma) = \widetilde{m}_2(\gamma) = i e^{-\pi i \gamma} \sin(\pi \gamma)$. □

4 Extension with two pairs of generators

For the sake of completeness we now prove that if the necessary conditions in Lemma 2.5 are satisfied, then we can always extend $\{D^j T_k \psi_1\}_{j,k \in \mathbb{Z}}$, $\{D^j T_k \widetilde{\psi}_1\}_{j,k \in \mathbb{Z}}$ to dual wavelet frame $\{D^j T_k \psi_\ell\}_{\ell=1,2,3;j,k \in \mathbb{Z}}$, $\{D^j T_k \widetilde{\psi}_\ell\}_{\ell=1,2,3;j,k \in \mathbb{Z}}$, by adding two pairs of wavelet systems with MRA-structure. Note that if it is not required that the functions $\psi_2, \psi_3, \widetilde{\psi}_2, \widetilde{\psi}_3$ have MRA-structure, it is trivial that such an extension exists: we can just take $\psi_2 = \psi_1, \widetilde{\psi}_2 = -\widetilde{\psi}_1$ to “cancel” the action of the given wavelet system, and then take ψ_3 and $\widetilde{\psi}_3$ to be any pair of functions generating dual wavelet frames. But this argument does not work under the assumptions in the current paper. For example, if we consider the MRA-setup where the functions $\psi_\ell, \widetilde{\psi}_\ell$ are generated by a B-spline, $\varphi = B_N, N > 1$, then there does not exist dual wavelet frames $\{D^j T_k \psi_3\}_{j,k \in \mathbb{Z}}, \{D^j T_k \widetilde{\psi}_3\}_{j,k \in \mathbb{Z}}$ where

$$\psi_3 = \sum c_k B_N(2x - k), \quad \widetilde{\psi}_3 = \sum \widetilde{c}_k B_N(2x - k),$$

with finite coefficient sequences, i.e., the masks m_3, \widetilde{m}_3 can not be trigonometric polynomials; see [9] (the result is repeated in Theorem 14.5.1 in [4]). Thus, we have to provide a different argument:

Theorem 4.1 *Let $\varphi, \widetilde{\varphi} \in L^2(\mathbb{R})$ be as in the general setup, with trigonometric polynomial masks m_0, \widetilde{m}_0 . Let m_1, \widetilde{m}_1 be trigonometric polynomials, and define $\psi_1, \widetilde{\psi}_1$ by (1.2). Assume that the conditions (a), (b) and (c) in Lemma 2.5 for $m_0, \widetilde{m}_0, m_1, \widetilde{m}_1$ are satisfied. Then there exist trigonometric polynomials $m_2, m_3, \widetilde{m}_2, \widetilde{m}_3$ such that $\{D^j T_k \psi_\ell\}_{\ell=1,2,3;j,k \in \mathbb{Z}}$ and $\{D^j T_k \widetilde{\psi}_\ell\}_{\ell=1,2,3;j,k \in \mathbb{Z}}$, with $\psi_2, \widetilde{\psi}_2, \psi_3, \widetilde{\psi}_3$ defined by (1.2), form dual frames for $L^2(\mathbb{R})$.*

Proof. We construct $m_2, \widetilde{m}_2, m_3, \widetilde{m}_3$ such that

- (1) $m_2(0) = \widetilde{m}_2(0) = m_3(0) = \widetilde{m}_3(0) = 0$;
- (2) the matrix-valued functions M, \widetilde{M} in (1.3) with $n = 3$ satisfy

$$\widetilde{M}(\gamma)^* M(\gamma) = I, \quad \gamma \in \mathbb{T}.$$

Choose M_α, M_β as in (3.1)–(3.2). We first define m_2 and \widetilde{m}_2 by

$$m_2(\gamma) := \overline{\widetilde{M}_\alpha(\gamma)} + \overline{\widetilde{M}_\beta(\gamma)} = \sin(\pi\gamma) \left(\sin(\pi\gamma) \overline{\Lambda_\alpha(\gamma)} + i \cos(\pi\gamma) \overline{\Lambda_\beta(\gamma)} \right) \quad (4.1)$$

$$\widetilde{m}_2(\gamma) := \sin^2(\pi\gamma), \quad (4.2)$$

where Λ_α and Λ_β are defined as in (3.3). Then

$$\begin{aligned} 1 - \sum_{\ell=0}^2 \overline{m_\ell(\gamma)} \widetilde{m}_\ell(\gamma) &= \widetilde{M}_\alpha(\gamma) - \overline{m_2(\gamma)} \widetilde{m}_2(\gamma) = \widetilde{M}_\alpha(\gamma) - \left(\overline{\widetilde{M}_\alpha(\gamma)} + \overline{\widetilde{M}_\beta(\gamma)} \right) \sin^2(\pi\gamma) \\ &= \sin^2(\pi\gamma) \cos(\pi\gamma) (\cos(\pi\gamma) \Lambda_\alpha(\gamma) + i \sin(\pi\gamma) \Lambda_\beta(\gamma)) \end{aligned}$$

and

$$\begin{aligned} \sum_{\ell=0}^2 \overline{m_\ell(\gamma)} \tilde{m}_\ell(\gamma + 1/2) &= -\widetilde{M}_\beta(\gamma) + \overline{m_2(\gamma)} \tilde{m}_2(\gamma + 1/2) \\ &= \sin^2(\pi\gamma) \cos(\pi\gamma) (\cos(\pi\gamma) \Lambda_\alpha(\gamma) + i \sin(\pi\gamma) \Lambda_\beta(\gamma)) \end{aligned}$$

We now define m_3, \tilde{m}_3 by

$$m_3(\gamma) := \sin(\pi\gamma) \left(\cos(\pi\gamma) \overline{\Lambda_\alpha(\gamma)} - i \sin(\pi\gamma) \overline{\Lambda_\beta(\gamma)} \right) \quad (4.3)$$

$$\tilde{m}_3(\gamma) := \sin(\pi\gamma) \cos(\pi\gamma). \quad (4.4)$$

Then it is easy to see that

$$\begin{aligned} 1 - \sum_{\ell=0}^2 \overline{m_\ell(\gamma)} \tilde{m}_\ell(\gamma) &= \overline{m_3(\gamma)} \tilde{m}_3(\gamma), \\ \sum_{\ell=0}^2 \overline{m_\ell(\gamma)} \tilde{m}_\ell(\gamma + 1/2) &= -\overline{m_3(\gamma)} \tilde{m}_3(\gamma + 1/2), \end{aligned}$$

which is equivalent to the condition (2). By construction, (1) holds. \square

Let us illustrate Theorem 4.1 by returning to a construction presented in Example 2.5 in [6]. In that example, masks m_0, m_1 as in the general setup were considered, and it was shown how to extend the Bessel sequence $\{D^j T_k \psi_1\}_{j,k \in \mathbb{Z}}$ to a tight frame $\{D^j T_k \psi_\ell\}_{\ell=1,2,3; j,k \in \mathbb{Z}}$ using the UEP. However, the masks defining the functions ψ_2, ψ_3 are not trigonometric polynomials, so the functions ψ_2, ψ_3 are not compactly supported. Using the MEP we can now show that it is possible to extend $\{D^j T_k \psi_1\}_{j,k \in \mathbb{Z}}$ to a pair of dual wavelet frames with compactly supported generators. The example deals again with the B-splines, for convenience shifted to be centered around $x = 0$:

Example 4.2 Let $\ell \geq 2$. Consider

$$\varphi(x) := B_{2\ell}(x + \ell), \quad \psi_1(x) := \sum_{k=-\ell}^{\ell} \frac{1}{2^{2\ell-1}} \binom{2\ell}{\ell+k} B_{2\ell}(2x - k + \ell)$$

with the associated masks

$$m_0(\gamma) = \cos^{2\ell}(\pi\gamma), \quad m_1(\gamma) = \sin^{2\ell}(\pi\gamma).$$

In [6, Example 2.5] it was shown that by letting

$$\lambda_1(\gamma) := 1 - (\cos^{2\ell}(\pi\gamma) + \sin^{2\ell}(\pi\gamma))^2, \quad \lambda_2(\gamma) := 1 - (\cos^{2\ell}(\pi\gamma) - \sin^{2\ell}(\pi\gamma))^2,$$

the masks $m_2, m_3 \in L^\infty(\mathbb{T})$ defined by

$$m_2(\gamma) := \begin{cases} \left(\sqrt{\lambda_1(\gamma)} + \sqrt{\lambda_2(\gamma)} \right) / 2, & \gamma \in [-1/2, 0] + \mathbb{Z} \\ \left(\sqrt{\lambda_1(\gamma - 1/2)} - \sqrt{\lambda_2(\gamma - 1/2)} \right) / 2, & \gamma \in [0, 1/2] + \mathbb{Z} \end{cases}$$

and

$$m_3(\gamma) := \begin{cases} \left(\sqrt{\lambda_1(\gamma)} - \sqrt{\lambda_2(\gamma)} \right) / 2, & \gamma \in [-1/2, 0] + \mathbb{Z} \\ \left(\sqrt{\lambda_1(\gamma - 1/2)} + \sqrt{\lambda_2(\gamma - 1/2)} \right) / 2, & \gamma \in [0, 1/2] + \mathbb{Z} \end{cases},$$

satisfy the UEP conditions. We note that these masks are not trigonometric polynomials.

Using the MEP we can now provide an alternative construction using trigonometric polynomials. Consider now $\tilde{\varphi} := \varphi$, $\tilde{\psi}_1 := \psi_1$ with the associated masks

$$\tilde{m}_0(\gamma) := m_0(\gamma) = \cos^{2\ell}(\pi\gamma), \quad \tilde{m}_1(\gamma) := m_1(\gamma) = \sin^{2\ell}(\pi\gamma).$$

A direct calculation shows that

$$\widetilde{M}_\alpha(\gamma) = \sin^2(\pi\gamma)\Lambda_\alpha(\gamma), \quad \widetilde{M}_\beta(\gamma) = -i \sin(\pi\gamma) \cos(\pi\gamma)\Lambda_\beta(\gamma),$$

where

$$\Lambda_\alpha(\gamma) = \left(\sum_{k=0}^{2\ell-1} \cos^{2k}(\pi\gamma) \right) - \sin^{4\ell-2}(\pi\gamma), \quad \Lambda_\beta(\gamma) = -2i \sin^{2\ell-1}(\pi\gamma) \cos^{2\ell-1}(\pi\gamma).$$

Define trigonometric polynomials $m_2, \tilde{m}_2, m_3, \tilde{m}_3$ as in (4.1)-(4.4):

$$\begin{aligned} m_2(\gamma) &:= \sin(\pi\gamma) \left(\sin(\pi\gamma) \overline{\Lambda_\alpha(\gamma)} + i \cos(\pi\gamma) \overline{\Lambda_\beta(\gamma)} \right), \quad \tilde{m}_2(\gamma) := \sin^2(\pi\gamma); \\ m_3(\gamma) &:= \sin(\pi\gamma) \left(\cos(\pi\gamma) \overline{\Lambda_\alpha(\gamma)} - i \sin(\pi\gamma) \overline{\Lambda_\beta(\gamma)} \right), \quad \tilde{m}_3(\gamma) := \sin(\pi\gamma) \cos(\pi\gamma). \end{aligned}$$

Then we see that the masks satisfy the conditions in Theorem 4.1. Hence we conclude that $\{D^j T_k \psi_\ell\}_{\ell=1,2,3; j,k \in \mathbb{Z}}$ and $\{D^j T_k \tilde{\psi}_\ell\}_{\ell=1,2,3; j,k \in \mathbb{Z}}$, with $\psi_2, \tilde{\psi}_2, \psi_3, \tilde{\psi}_3$ defined by (1.2), form dual frames for $L^2(\mathbb{R})$. By construction all the generators are compactly supported. \square

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