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ON EXTRAORDINARY SEMISIMPLE MATRIX N(v) FOR ANISOTROPIC ELASTIC MATERIALS

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Abstract. The 6×6 real matrix $\mathbf{N}(v)$ for anisotropic elastic materials under a two-dimensional steady-state motion with speed v is *extraordinary semisimple* when $\mathbf{N}(v)$ has three identical complex eigenvalues p and three independent associated eigenvectors. We show that such an $\mathbf{N}(v)$ exists when $v \neq 0$. The eigenvalues are purely imaginary. The material can sustain a steady-state motion such as a moving line dislocation. Explicit expressions of the Barnett-Lothe tensors for $v \neq 0$ are presented. However, $\mathbf{N}(v)$ cannot be extraordinary semisimple for surface waves. When $v = 0$, $\mathbf{N}(0)$ can be extraordinary semisimple if the strain energy of the material is allowed to be *positive semidefinite*. Explicit expressions of the Barnett-Lothe tensors and Green's functions for the infinite space and half-space are presented. An unusual phenomenon for the material with positive semidefinite strain energy considered here is that it can support an edge dislocation with zero stresses everywhere. In the special case when $p = i$ is a triple eigenvalue, this material is an *un-pressurable material* in the sense that it can change its (two-dimensional) volume with zero pressure. It is a counterpart of an incompressible material (whose strain energy is also positive semidefinite) that can support pressure with zero volume change.

1. Introduction. A key role in two-dimensional deformations of anisotropic elastic materials under a steady-state motion with speed v is the eigenrelation [1,2]

$$\mathbf{N}(v)\xi = p\xi,$$

$$\mathbf{N}(v) = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 + \rho v^2 \mathbf{I} & \mathbf{N}_1^{\mathrm{T}} \end{bmatrix}, \qquad \xi = \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{bmatrix}, \tag{1}$$

$$\mathbf{N}_1 = -\mathbf{T}^{-1}\mathbf{R}^{\mathrm{T}}, \qquad \mathbf{N}_2 = \mathbf{T}^{-1}, \qquad \mathbf{N}_3 = \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^{\mathrm{T}} - \mathbf{Q}.$$

In the above ρ is the mass density, $\boldsymbol{a}$ and $\boldsymbol{b}$ are 3-vectors, $\mathbf{I}$ is the unit matrix, the superscript T stands for the transpose, and $\mathbf{Q}, \mathbf{R}, \mathbf{T}$ are 3×3 matrices whose elements

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are

$$Q_{ik} = C_{i1k1}, \qquad R_{ik} = C_{i1k2}, \qquad T_{ik} = C_{i2k2},$$

in which C_{ijks} are the elastic stiffnesses. When the eigenvalues p are complex they consist of three pairs of complex conjugates [3, 4]. If p_1, p_2, p_3 are assumed to have a positive imaginary part, the remaining three eigenvalues are complex conjugates of p_1, p_2, p_3. The 6×6 matrix $\mathbf{N}(v)$ is *simple* when p_1, p_2, p_3 are distinct. If $p_1 = p_2 \neq p_3$ and there exist three independent eigenvectors $\xi_1, \xi_2, \xi_3, \mathbf{N}(v)$ is *semisimple.* Isotropic material is an example for which $\mathbf{N}(v)$ is semisimple when $v \neq 0$. For $v = 0$, we will present an example of semisimple $\mathbf{N}(0)$ at the end of Sec. 4. If $p_1 = p_2 = p_3$ and there exist three independent eigenvectors $\xi_1, \xi_2, \xi_3, \mathbf{N}(v)$ is *extraordinary semisimple.* No anisotropic materials have been found to possess an extraordinary semisimple $\mathbf{N}(v)$. We will show that an extraordinary semisimple $\mathbf{N}(v)$ exists if $v \neq 0$. When $v = 0$ it exists if the strain energy of the material is allowed to be positive semidefinite.

It should be noted that there are numerous anisotropic materials for which $p_1 = p_2 = p_3$ are the eigenvalues of $\mathbf{N}(0)$ [5]. However, they are all found to possess only two, not three, independent eigenvectors. Certain properties related to $\mathbf{N}(0)$ are discussed in [6].

2. Extraordinary semisimple $\mathbf{N}(v)$. Equation (1) is equivalent to

$$\mathbf{N}_1\boldsymbol{a} + \mathbf{N}_2\boldsymbol{b} = p\boldsymbol{a}, \qquad (\mathbf{N}_3 + \rho v^2\mathbf{I})\boldsymbol{a} + \mathbf{N}_1^{\mathrm{T}}\boldsymbol{b} = p\boldsymbol{b}. \tag{2}$$

Elimination of $\boldsymbol{b}$ leads to

$$\{\mathbf{Q} - \rho v^2\mathbf{I} + p(\mathbf{R} + \mathbf{R}^{\mathrm{T}}) + p^2\mathbf{T}\}\boldsymbol{a} = \mathbf{0}, \tag{3}$$

which recovers the result first obtained in [3, 4]. This provides an equation for the 3-vector $\boldsymbol{a}$. The 3-vector $\boldsymbol{b}$ is obtained from $(2)_1$ as

$$\boldsymbol{b} = \{\mathbf{R}^{\mathrm{T}} + p\mathbf{T}\}\boldsymbol{a}. \tag{4}$$

When $\mathbf{N}(v)$ is extraordinary semisimple there exist three independent eigenvectors ξ. Since $\boldsymbol{b}$ is determined from $\boldsymbol{a}$ through (4), this implies that there exist three independent vectors $\boldsymbol{a}$. If (3) has three independent solutions $\boldsymbol{a}$ for the same p we must have

$$\mathbf{Q} - \rho v^2\mathbf{I} + p(\mathbf{R} + \mathbf{R}^{\mathrm{T}}) + p^2\mathbf{T} = \mathbf{0}. \tag{5}$$

Setting

$$p = \alpha + i\beta, \qquad \lambda = (\alpha^2 + \beta^2)^{1/2},$$

where α, β are real and $\beta > 0$, the real and imaginary parts of (5) give

$$\mathbf{Q} - \lambda^2\mathbf{T} = \rho v^2\mathbf{I}, \qquad \mathbf{R} + \mathbf{R}^{\mathrm{T}} = -2\alpha\mathbf{T}. \tag{6}$$

Conversely, when (6) holds (3) reduces to

$$(p^2 - 2\alpha p + \lambda^2)\mathbf{T}\boldsymbol{a} = \mathbf{0}.$$

The vanishing of the determinant of the 3×3 coefficient matrix of $\boldsymbol{a}$ yields

$$(p^2 - 2\alpha p + \lambda^2)^3 |\mathbf{T}| = 0.$$

Since $\mathbf{T}$ is nonsingular we have

$$[p - (\alpha + i\beta)]^3 [p - (\alpha - i\beta)]^3 = 0.$$

Thus $p = \alpha \pm i\beta$ is a triple root, and there are three independent vectors $\boldsymbol{a}$.

Employing the contracted notation for C_{ijks}, (6) is written in full as

$$\begin{gathered} C_{16} = -\alpha C_{66}, \qquad C_{26} = -\alpha C_{22}, \qquad C_{45} = -\alpha C_{44}, \\ C_{16} = \lambda^2 C_{26}, \qquad C_{15} = \lambda^2 C_{46}, \qquad C_{56} = \lambda^2 C_{24}, \\ C_{12} + C_{66} = -2\alpha C_{26}, \qquad C_{14} + C_{56} = -2\alpha C_{46}, \qquad C_{25} + C_{46} = -2\alpha C_{24}, \\ C_{11} - \lambda^2 C_{66} = C_{55} - \lambda^2 C_{44} = C_{66} - \lambda^2 C_{22} = \rho v^2. \end{gathered} \tag{7}$$

Let $\mathbf{C}$ be the 6×6 matrix whose elements are C_{ks}. Since C_{3s} $(s = 1, 2, \ldots, 6)$ do not appear in (7) we will consider the 5×5 matrix $\mathbf{C}^0$ obtained by deleting the third rows and the third column of the 6×6 matrix $\mathbf{C}$. The structure of $\mathbf{C}^0$ that satisfies (7) depends on whether $C_{66} = \lambda^2 C_{22}$ or not. This is due to the three equations in $(7)_{1,2,4}$. Elimination of C_{16} and C_{26} among these three equations yields

$$\alpha(C_{66} - \lambda^2 C_{22}) = 0.$$

Let $C_{66} = \lambda^2 C_{22}$. Then α can be arbitrary, including $\alpha = 0$. If $C_{66} \neq \lambda^2 C_{22}$, α must vanish and $\lambda = \beta$.

Consider the case $C_{66} = \lambda^2 C_{22}$ first. The last equation in (7) tells us that

$$\rho v^2 = 0.$$

Equations (7) are satisfied by

$$\mathbf{C}^0 = \begin{bmatrix} \lambda^4 C_{22} & C_{12} & C_{14} & \lambda^2 C_{46} & -\lambda^2 \alpha C_{22} \\ C_{12} & C_{22} & C_{24} & C_{25} & -\alpha C_{22} \\ C_{14} & C_{24} & C_{44} & -\alpha C_{44} & C_{46} \\ \lambda^2 C_{46} & C_{25} & -\alpha C_{44} & \lambda^2 C_{44} & \lambda^2 C_{24} \\ -\lambda^2 \alpha C_{22} & -\alpha C_{22} & C_{46} & \lambda^2 C_{24} & \lambda^2 C_{22} \end{bmatrix}, \tag{8a}$$

in which

$$C_{12} = (2\alpha^2 - \lambda^2) C_{22}, \qquad C_{14} = -\lambda^2 C_{24} - 2\alpha C_{46}, \qquad C_{25} = -C_{46} - 2\alpha C_{24}. \tag{8b}$$

Hence (8a,b) apply to the statics case $v = 0$ regardless of whether α is zero or nonzero. In view of (8b) C_{22}, C_{24}, C_{44}, and C_{46} are the only independent elastic constants in addition to the parameters α and $\lambda > 0$.

For the case $C_{66} \neq \lambda^2 C_{22}$, $\alpha = 0$ and $\lambda = \beta$. This means that $p = i\beta$. Equations (7) are satisfied by

$$\mathbf{C}^0 = \begin{bmatrix} \beta^2 C_{66} + \eta & -C_{66} & -\beta^2 C_{24} & \beta^2 C_{46} & 0 \\ -C_{66} & C_{22} & C_{24} & -C_{46} & 0 \\ -\beta^2 C_{24} & C_{24} & C_{44} & 0 & C_{46} \\ \beta^2 C_{46} & -C_{46} & 0 & \beta^2 C_{44} + \eta & \beta^2 C_{24} \\ 0 & 0 & C_{46} & \beta^2 C_{24} & C_{66} \end{bmatrix}, \tag{9a}$$

in which, from the the last equation of (7),

$$C_{66} = \beta^2 C_{22} + \eta, \qquad \eta = \rho v^2 > 0. \tag{9b}$$

Thus (9a,b) apply to a steady-state motion with a nonzero v. In view of (9b) C_{22}, C_{24}, C_{44}, and C_{46} are the only independent elastic constants in addition to the parameters $\beta > 0$ and $\eta > 0$. When $v \to 0$ (i.e., $\eta \to 0$), the $\mathbf{C}^0$ in (9) is identical to the $\mathbf{C}^0$ in (8) with $\alpha = 0$ (and hence $\lambda = \beta$).

The strain energy is positive if the 6×6 matrix $\mathbf{C}$ is positive definite. When the 5×5 matrix $\mathbf{C}^0$ is positive definite, it is always possible to choose C_{3s} $(s = 1, 2, \ldots, 6)$ such that the 6×6 matrix $\mathbf{C}$ is positive definite [5]. It suffices therefore to study if the $\mathbf{C}^0$ in (8) and (9) are positive definite. If $\mathbf{C}^0$ is positive definite, an extraordinary semisimple $\mathbf{N}(v)$ exists.

We will consider the case $v = 0$ first. The case $v \neq 0$ will be studied in Sec. 7.

3. Extraordinary semisimple $\mathbf{N}(0)$ for elastostatics. A set of necessary and sufficient conditions for a symmetric matrix to be positive definite is that the determinants of the 1×1, $2 \times 2, \ldots$ submatrices obtained from the upper left corner of the matrix be positive [7]. Instead of taking the submatrices from the upper left corner, we could take the submatrices from the lower right corner of the matrix. It can be shown that the determinants of the 1×1, $2 \times 2, \ldots$ submatrices taken from the lower right corner of the 5×5 matrix $\mathbf{C}^0$ in (8a) (with the use of (8b)) are, respectively,

$$\lambda^2 C_{22}, \quad \lambda^4 \delta, \quad \lambda^2 C_{44} \Delta, \quad \Delta^2, \quad 0, \tag{10}$$

in which

$$\begin{gathered} \delta = C_{22} C_{44} - C_{24}^2, \\ \Delta = \beta^2 (C_{22} C_{44} - C_{24}^2) - (\alpha C_{24} + C_{46})^2. \end{gathered} \tag{11}$$

That the determinant of $\mathbf{C}^0$ vanishes can be proved easily by observing that addition of the first row, the second row multiplied by λ^2, and the fifth row multiplied by 2α vanishes. Consequently the matrix $\mathbf{C}^0$ is singular, and hence it is not positive definite.

The matrix $\mathbf{C}^0$ is *positive semidefinite* if the first four quantities in (10) are positive. It is easily seen that

$$C_{44} > 0 \quad \text{and} \quad \Delta > 0$$

are the necessary and sufficient conditions for $\mathbf{C}^0$ to be positive semidefinite. By (11) the conditions $\delta > 0$ and $C_{22} > 0$ are redundant. One can then choose C_{3s} $(s = 1, 2, \ldots, 6)$ such that $\mathbf{C}$ is positive semidefinite [5].

Even though $\mathbf{C}^0$ is not positive definite, it can be shown that the matrices $\mathbf{Q}$ and $\mathbf{T}$ deduced from (8) are. The three matrices $\mathbf{N}_2, \mathbf{N}_1, \mathbf{N}_3$, computed in that order, have the expression

$$\mathbf{N}_2 = \frac{1}{C_{22}\Delta}\begin{bmatrix} \delta & C_{24}C_{46} + \alpha C_{22}C_{44} & -C_{22}(\alpha C_{24} + C_{46}) \\ C_{24}C_{46} + \alpha C_{22}C_{44} & \lambda^2 C_{22}C_{44} - C_{46}^2 & -C_{22}(\alpha C_{46} + \lambda^2 C_{24}) \\ -C_{22}(\alpha C_{24} + C_{46}) & -C_{22}(\alpha C_{46} + \lambda^2 C_{24}) & \beta^2 C_{22}^2 \end{bmatrix},$$

$$\mathbf{N}_1 = \frac{1}{C_{22}}\begin{bmatrix} 2\alpha C_{22} & -C_{22} & -C_{24} \\ \lambda^2 C_{22} & 0 & C_{46} \\ 0 & 0 & \alpha C_{22} \end{bmatrix}, \qquad \mathbf{N}_3 = \frac{-\Delta}{C_{22}}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{12}$$

4. The Barnett-Lothe tensors. When p_k $(k = 1, 2, 3)$ are distinct, let $\boldsymbol{a}_k$ $(k = 1, 2, 3)$ be the solution of (3) associated with p_k. The vectors $\boldsymbol{b}_k$ are determined from (4). They automatically satisfy the orthogonality relations [1, 2]

$$\boldsymbol{a}_k \cdot \boldsymbol{b}_s + \boldsymbol{a}_s \cdot \boldsymbol{b}_k = 0 \quad (k \neq s). \tag{13}$$

The vectors $\boldsymbol{a}_k, \boldsymbol{b}_k$ are not unique. They can be normalized by

$$2\boldsymbol{a}_k \cdot \boldsymbol{b}_k = 1 \quad (k = 1, 2, 3). \tag{14}$$

Let the 3×3 matrices $\mathbf{A}$ and $\mathbf{B}$ be defined by

$$\mathbf{A} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3], \qquad \mathbf{B} = [\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3].$$

When $\boldsymbol{a}_k$ and $\boldsymbol{b}_k$ are normalized according to (14), the three Barnett-Lothe tensors $\mathbf{S}, \mathbf{H}, \mathbf{L}$ given by

$$\mathbf{S} = i(2\mathbf{A}\mathbf{B}^{\mathrm{T}} - \mathbf{I}), \qquad \mathbf{H} = 2i\mathbf{A}\mathbf{A}^{\mathrm{T}}, \qquad \mathbf{L} = -2i\mathbf{B}\mathbf{B}^{\mathrm{T}} \tag{15}$$

are real [1, 2]. They are related by

$$\mathbf{HL} - \mathbf{SS} = \mathbf{I}. \tag{16}$$

If the strain energy is positive definite, $\mathbf{H}$ and $\mathbf{L}$ are symmetric and positive definite, and

$$\mathbf{LS}, \qquad \mathbf{SH}, \qquad \mathbf{SL}^{-1}, \qquad \mathbf{H}^{-1}\mathbf{S}$$

are skew-symmetric. Making use of the identities

$$\mathbf{B}\mathbf{A}^{-1} = (\mathbf{A}\mathbf{B}^{\mathrm{T}})^{\mathrm{T}}(\mathbf{A}\mathbf{A}^{\mathrm{T}})^{-1}, \qquad \mathbf{A}\mathbf{B}^{-1} = (\mathbf{A}\mathbf{B}^{\mathrm{T}})(\mathbf{B}\mathbf{B}^{\mathrm{T}})^{-1},$$

which can be verified easily, the impedance tensor $\mathbf{M}$ and its inverse $\mathbf{M}^{-1}$ have the expressions

$$\begin{aligned} \mathbf{M} &= -i\mathbf{B}\mathbf{A}^{-1} = \mathbf{H}^{-1} + i\mathbf{H}^{-1}\mathbf{S}, \\ \mathbf{M}^{-1} &= i\mathbf{A}\mathbf{B}^{-1} = \mathbf{L}^{-1} - i\mathbf{S}\mathbf{L}^{-1}. \end{aligned} \tag{17}$$

They are positive definite Hermitian. It should be mentioned that, while the normalization (14) is required for the columns of $\mathbf{A}$ and $\mathbf{B}$ in (15), it is redundant for the columns of $\mathbf{A}$ and $\mathbf{B}$ in (17). This is so because the normalization factors cancel out after the matrix multiplication [8]. Thus (17) and (16) can be employed to compute $\mathbf{S}, \mathbf{H}, \mathbf{L}$ without normalizing the vectors $\boldsymbol{a}_k$ and $\boldsymbol{b}_k$.

When the real part of an eigenvalue p is nonzero (i.e., $\alpha \neq 0$), one can rotate the coordinate system about the x_3-axis by an angle ψ given by

$$\tan 2\psi = \frac{-(p+\bar{p})}{p\bar{p}-1} = \frac{-2\alpha}{\lambda^2 - 1}$$

where the overbar denotes the complex conjugate. The eigenvalue p referred to the rotated coordinate system reduces to purely imaginary [9]. Since the Barnett-Lothe tensors are tensors of rank two when the transformation is a rotation about the x_3-axis [9], it suffices to compute $\mathbf{S}, \mathbf{H}, \mathbf{L}$ referred to the rotated coordinate system for which p is purely imaginary. Therefore we set $\alpha = 0$ in (8) and consider

$$\mathbf{C}^0 = \begin{bmatrix} \beta^4 C_{22} & -\beta^2 C_{22} & -\beta^2 C_{24} & \beta^2 C_{46} & 0 \\ -\beta^2 C_{22} & C_{22} & C_{24} & -C_{46} & 0 \\ -\beta^2 C_{24} & C_{24} & C_{44} & 0 & C_{46} \\ \beta^2 C_{46} & -C_{46} & 0 & \beta^2 C_{44} & \beta^2 C_{24} \\ 0 & 0 & C_{46} & \beta^2 C_{24} & \beta^2 C_{22} \end{bmatrix}. \tag{18}$$

The Δ in (11) is now given by

$$\Delta = \beta^2 (C_{22} C_{44} - C_{24}^2) - C_{46}^2. \tag{19}$$

Equation (3) with $v = 0$ has the expression

$$\begin{bmatrix} \beta^2(\beta^2+p^2)C_{22} & 0 & (\beta^2+p^2)C_{46} \\ 0 & (\beta^2+p^2)C_{22} & (\beta^2+p^2)C_{24} \\ (\beta^2+p^2)C_{46} & (\beta^2+p^2)C_{24} & (\beta^2+p^2)C_{44} \end{bmatrix} \boldsymbol{a} = \mathbf{0}. \tag{20}$$

The vanishing of the determinant of the 3×3 matrix yields

$$(\beta^2 + p^2)^3 \Delta = 0$$

and hence $p = i\beta$ is a triple root.

The matrix $\mathbf{C}^0$ in (18) tells us that addition of the first column and the second column multiplied by β^2 vanishes. This means that all stresses vanish if the nonzero strains are ε_{11} and ε_{22} with $\varepsilon_{22} = \beta^2 \varepsilon_{11}$. When $\beta = 1$, it implies that a change in the volume caused by $\varepsilon_{11} = \varepsilon_{22}$ requires no pressure. Therefore the material is an *un-pressurable material.* It is a counterpart of an incompressible material (whose strain energy is also positive semidefinite) that can support a pressure with no volume change. This peculiar phenomenon is due to the positive semidefiniteness of the strain energy.

When $\mathbf{N}(0)$ is extraordinary semisimple, $\boldsymbol{a}_k$ $(k = 1, 2, 3)$ obtained from (20) and $\boldsymbol{b}_k$ from (4) may not satisfy the orthogonality relations (13). We outline below how one can compute a set of $\boldsymbol{a}_k, \boldsymbol{b}_k$ $(k = 1, 2, 3)$ that satisfy (13).

The vector $\boldsymbol{a}_1$ can be chosen arbitrarily. Let

$$\boldsymbol{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

The vector $\boldsymbol{b}_1$ is then determined from (4). This provides an unnormalized ξ. Let the vector $\boldsymbol{a}_2$ be given by

$$\boldsymbol{a}_2 = \begin{bmatrix} d \\ e \\ 0 \end{bmatrix},$$

where d and e are arbitrary. With $\boldsymbol{b}_2$ obtained from (4), d and e are determined, within a constant multiplier, by (13) for $(k,s) = (1,2)$. Finally, the vectors $\boldsymbol{a}_3$ and $\boldsymbol{b}_3$ obtained from (4) must satisfy (13) for $(k,s) = (1,3)$ and $(2,3)$. Again, $\boldsymbol{a}_3$ and $\boldsymbol{b}_3$ are determined within a constant multiplier. The constant multipliers can be ignored when $\boldsymbol{a}_k, \boldsymbol{b}_k$ $(k = 1,2,3)$ are not normalized.

With the procedure outlined above the unnormalized $\boldsymbol{a}_k, \boldsymbol{b}_k$ $(k = 1,2,3)$ can be computed. The result is

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -C_{46} \\ 0 & 1 & -\beta^2 C_{24} \\ 0 & 0 & \beta^2 C_{22} \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} i\beta^3 C_{22} & \beta^2 C_{22} & 0 \\ -\beta^2 C_{22} & i\beta C_{22} & 0 \\ i\beta(C_{46} + i\beta C_{24}) & C_{46} + i\beta C_{24} & i\beta\Delta \end{bmatrix}.$$

Hence

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 & C_{46}/(\beta^2 C_{22}) \\ 0 & 1 & C_{24}/C_{22} \\ 0 & 0 & 1/(\beta^2 C_{22}) \end{bmatrix}.$$

Notice that the unnormalized matrix $\mathbf{A}$ is real, which is unusual in anisotropic elasticity. The matrix $\mathbf{A}$ is not singular. The matrix $\boldsymbol{b}$ is singular because the first column is equal to $i\beta$ times the second column. After performing the product $-i\mathbf{B}\mathbf{A}^{-1}$ and using $(17)_2$ we have

$$\mathbf{H}^{-1} = \beta \begin{bmatrix} \beta^2 C_{22} & 0 & C_{46} \\ 0 & C_{22} & C_{24} \\ C_{46} & C_{24} & C_{44} \end{bmatrix},$$

$$\mathbf{H}^{-1}\mathbf{S} = \begin{bmatrix} 0 & -\beta^2 C_{22} & -\beta^2 C_{24} \\ \beta^2 C_{22} & 0 & C_{46} \\ \beta^2 C_{24} & -C_{46} & 0 \end{bmatrix}.$$

The Barnett-Lothe tensors are

$$\mathbf{H} = \frac{1}{C_{22}\beta\Delta} \begin{bmatrix} \delta & C_{24}C_{46} & -C_{22}C_{46} \\ C_{24}C_{46} & \Delta + \beta^2 C_{24}^2 & -\beta^2 C_{22}C_{24} \\ -C_{22}C_{46} & -\beta^2 C_{22}C_{24} & \beta^2 C_{22}^2 \end{bmatrix}, \tag{21}$$

$$\mathbf{S} = \frac{1}{\beta C_{22}} \begin{bmatrix} 0 & -C_{22} & -C_{24} \\ \beta^2 C_{22} & 0 & C_{46} \\ 0 & 0 & 0 \end{bmatrix} \tag{22}$$

and, from (16),

$$\mathbf{L} = \frac{\Delta}{\beta C_{22}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{23}$$

Since the strain energy is positive semidefinite, it affects the tensor $\mathbf{L}$ in that $\mathbf{L}$ is positive semidefinite. However, $\mathbf{H}$ is positive definite.

While the matrices $\mathbf{A}$ and $\mathbf{B}$ are not unique depending on the choices of $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, the Barnett-Lothe tensors are unique. We will prove this in the Appendix where we consider the general case $\alpha \neq 0$, and present explicit expressions of $\mathbf{S}, \mathbf{H}, \mathbf{L}$ directly in terms of $\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3$ given in (12).

In closing this section we point out that the matrix $\mathbf{C}^0$ in (18) can be made positive definite if we replace the C_{11} element by

$$C_{11} = (\beta^2 + \varepsilon)C_{22}, \quad \varepsilon > 0.$$

The eigenvalues p_1 and p_2 remain equal to $i\beta$ while

$$p_3 = i\sqrt{\beta^2 + \varepsilon(\delta/\Delta)}.$$

The 6×6 matrix $\mathbf{N}(0)$ is now *semisimple.* Another example of semisimple $\mathbf{N}(0)$ can be found in [10]. The tensor $\mathbf{L}$ is positive definite and its inverse $\mathbf{L}^{-1}$ exists.

5. The Green's function for the infinite space. In a fixed rectangular coordinate system x_i $(i = 1, 2, 3)$ let the infinite anisotropic elastic material be subjected to a line force $\mathbf{f}$ and a line dislocation with Burgers' vector $\mathbf{b}$ (not to be confused with the italic $\boldsymbol{b}$ employed earlier) at $x_1 = x_2 = 0$. The solution for the displacement $\mathbf{u}$ and the stress function ϕ for the extraordinary semisimple case with the triple eigenvalues $p = i\beta$ can be deduced from the solution for distinct eigenvalue obtained in [1] as

$$\mathbf{u} = \frac{1}{\pi}\operatorname{Im}\{(\ln z)\mathbf{A}\mathbf{q}^\infty\}, \qquad \phi = \frac{1}{\pi}\operatorname{Im}\{(\ln z)\mathbf{B}\mathbf{q}^\infty\}. \tag{24}$$

In the above, Im stands for the imaginary part, $\mathbf{q}^\infty$ is a constant vector to be determined, and

$$z = x_1 + px_2.$$

The stresses σ_{ij} are determined from [11]:

$$\sigma_{i2} = \partial\phi_i/\partial x_1, \qquad \sigma_{i1} = -\partial\phi_i/\partial x_2. \tag{25}$$

When the columns of $\mathbf{A}$ and $\mathbf{B}$ are normalized, it is shown in [12] that

$$\mathbf{q}^\infty = \mathbf{A}^{\mathrm{T}}\mathbf{f} + \mathbf{B}^{\mathrm{T}}\mathbf{b}. \tag{26}$$

With the use of (15) we obtain

$$\mathbf{A}\mathbf{q}^\infty = \tfrac{1}{2}\{\mathbf{b} - i(\mathbf{S}\mathbf{b} + \mathbf{H}\mathbf{f})\}, \qquad \mathbf{B}\mathbf{q}^\infty = \tfrac{1}{2}\{\mathbf{f} - i(\mathbf{S}^{\mathrm{T}}\mathbf{f} - \mathbf{L}\mathbf{b})\}. \tag{27}$$

Thus the $\mathbf{Aq}^\infty$ and $\mathbf{Bq}^\infty$ in (24) are replaced by the right-hand sides of (27). The question of normalization of the columns of $\mathbf{A}$, $\mathbf{B}$, and the nonuniqueness of $\mathbf{A}, \mathbf{B}$ becomes immaterial.

With the expressions of $\mathbf{S}$ and $\mathbf{L}$ given in (22) and (23), $\mathbf{Bq}^\infty$ in $(27)_2$ can be written explicitly as

$$\mathbf{Bq}^\infty = \frac{1}{2}\begin{bmatrix} -i\beta g \\ g \\ h \end{bmatrix} \tag{28}$$

where

$$\begin{aligned} g &= f_2 + i\beta^{-1} f_1, \\ h &= f_3 + i(\beta C_{22})^{-1}(C_{24} f_1 - C_{46} f_2 + \Delta b_3). \end{aligned} \tag{29}$$

Notice that the b_1, b_2 components of the Burgers' vector $\mathbf{b}$ are absent in (29). This implies that $\mathbf{Bq}^\infty$ vanishes if the infinite space is subjected to an edge dislocation only. By (24) ϕ vanishes, so do stresses. We therefore have the result that the anisotropic material considered here can support an edge dislocation in the infinite space with zero stresses everywhere. To find the displacement field due to an edge dislocation, we set $\mathbf{f} = \mathbf{0}$ and $b_3 = 0$ in $(27)_1$. We have

$$\mathbf{Aq}^\infty = \frac{1}{2}(b_1 + i\beta^{-1} b_2)\begin{bmatrix} 1 \\ -i\beta \\ 0 \end{bmatrix}, \tag{30}$$

and $(24)_1$ provides the displacement. A direct differentiation of $(24)_1$ shows that the only nonzero strains are ε_{11} and ε_{22} with $\varepsilon_{22} = \beta^2 \varepsilon_{11}$. This confirms the statement made earlier that the stresses vanish under these strains for the material considered here.

6. The Green's functions for the half-space. In this section we consider the half-space subjected to a line force $\mathbf{f}$ and a line dislocation with Burgers' vector $\mathbf{b}$ located inside the half-space. Since we have chosen the coordinate system (x_1, x_2) such that the eigenvalue p is purely imaginary, the boundary of the half-space may not be at $x_2 = 0$. If the boundary makes an angle θ with the x_1-axis, let

$$\mathbf{n} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}, \qquad \mathbf{m} = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix}$$

be two mutually orthogonal unit vectors. The material occupies the region

$$\mathbf{x} \cdot \mathbf{m} \geq 0.$$

Let the line force $\mathbf{f}$ and the line dislocation $\mathbf{b}$ be located at a distance d from the boundary, i.e., at

$$\mathbf{x} \cdot \mathbf{n} = 0, \qquad \mathbf{x} \cdot \mathbf{m} = d. \tag{31}$$

We now consider the solution

$$\begin{aligned} \mathbf{u} &= \frac{1}{\pi}\,\mathrm{Im}\{\ln(\hat{z} - p(\theta)d)\mathbf{Aq}^\infty\} + \frac{1}{\pi}\,\mathrm{Im}\{\ln(\hat{z} - \bar{p}(\theta)d)\mathbf{Aq}\}, \\ \phi &= \frac{1}{\pi}\,\mathrm{Im}\{\ln(\hat{z} - p(\theta)d)\mathbf{Bq}^\infty\} + \frac{1}{\pi}\,\mathrm{Im}\{\ln(\hat{z} - \bar{p}(\theta)d)\mathbf{Bq}\}, \end{aligned} \tag{32}$$

in which

$$\hat{z} = \mathbf{x} \cdot \mathbf{n} + p(\theta)\mathbf{x} \cdot \mathbf{m},$$
$$p(\theta) = \frac{p\cos\theta - \sin\theta}{p\sin\theta + \cos\theta}. \tag{33}$$

$\mathbf{A}\mathbf{q}^\infty$ and $\mathbf{B}\mathbf{q}^\infty$ are given in (27) while $\mathbf{q}$ is a constant vector to be determined by the condition at the boundary $\mathbf{x} \cdot \mathbf{m} = 0$. In view of the nonuniqueness of $\mathbf{A}$ and $\mathbf{B}$, we will determine $\mathbf{A}\mathbf{q}$ and $\mathbf{B}\mathbf{q}$ instead of $\mathbf{q}$.

The use of $\hat{z}$ instead of z is very common in the study of surface waves [1, 2, 13] but it has rarely been employed for an oblique boundary. The $p(\theta)$ in (33) is the eigenvalue p of the material referred to the rotated coordinate system in which the boundary of the half-space is the new x_1-axis [9]. The first terms on the right-hand sides of (32) represent the Green's function for the infinite space with the line force $\mathbf{f}$ and the line dislocation $\mathbf{b}$ at the location given in (31). The second terms represent an image singularity [12]. The location of the image singularity is at

$$\mathbf{x} \cdot \mathbf{n} + p(\theta)\mathbf{x} \cdot \mathbf{m} = \bar{p}(\theta)d$$

or

$$\mathbf{x} \cdot \mathbf{m} = -d, \qquad \mathbf{x} \cdot \mathbf{n} = 2\,\mathrm{Re}\{p(\theta)\}\,d \tag{34}$$

where Re denotes the real part. It is outside the half-space.

(I) *Rigidly clamped boundary.* When the boundary of the half-space is a rigidly clamped surface, $\mathbf{u} = \mathbf{0}$ at $\mathbf{x} \cdot \mathbf{m} = 0$. By $(32)_1$,

$$\mathbf{0} = \frac{1}{\pi}\,\mathrm{Im}\{\ln(\mathbf{x} \cdot \mathbf{n} - p(\theta)d)\mathbf{A}\mathbf{q}^\infty\} + \frac{1}{\pi}\,\mathrm{Im}\{\ln(\mathbf{x} \cdot \mathbf{n} - \bar{p}(\theta)d)\mathbf{A}\mathbf{q}\}.$$

The first term is identical to the negative of its complex conjugate. Hence we obtain

$$\mathbf{A}\mathbf{q} = \bar{\mathbf{A}}\bar{\mathbf{q}}^\infty. \tag{35}$$

Writing $\mathbf{B}\mathbf{q}$ as

$$\mathbf{B}\mathbf{q} = (\mathbf{B}\mathbf{A}^{-1})\mathbf{A}\mathbf{q} = (\mathbf{B}\mathbf{A}^{-1})\bar{\mathbf{A}}\bar{\mathbf{q}}^\infty$$

and using $(17)_2$, we have

$$\mathbf{B}\mathbf{q} = (i\mathbf{H}^{-1} - \mathbf{H}^{-1}\mathbf{S})\bar{\mathbf{A}}\bar{\mathbf{q}}^\infty. \tag{36}$$

Equations (35) and (36) provide the $\mathbf{A}\mathbf{q}$ and $\mathbf{B}\mathbf{q}$ in (32) in terms of $\bar{\mathbf{A}}\bar{\mathbf{q}}^\infty$, which can be deduced from $(27)_1$.

(II) *Traction-free boundary.* If the boundary of the half-space is a traction-free surface, $\phi = \mathbf{0}$ at $\mathbf{x} \cdot \mathbf{m} = 0$. A similar analysis leads to

$$\mathbf{B}\mathbf{q} = \bar{\mathbf{B}}\bar{\mathbf{q}}^\infty. \tag{37}$$

We cannot write

$$\mathbf{A}\mathbf{q} = (\mathbf{A}\mathbf{B}^{-1})\mathbf{B}\mathbf{q} = (\mathbf{A}\mathbf{B}^{-1})\bar{\mathbf{B}}\bar{\mathbf{q}}^\infty$$

because $\mathbf{B}$ is singular, and $\mathbf{B}^{-1}$ does not exist. Despite $\mathbf{B}$ being singular, (37) has a solution for $\mathbf{q}$ if $\bar{\mathbf{B}}\bar{\mathbf{q}}^{\infty}$ is orthogonal to the left null vector of $\mathbf{B}$, which is

$$\begin{bmatrix} 1 \\ i\beta \\ 0 \end{bmatrix}.$$

With $\bar{\mathbf{B}}\bar{\mathbf{q}}^{\infty}$ deduced from (28), this condition leads to $\bar{g} = 0$ or

$$f_1 = f_2 = 0. \tag{38}$$

Therefore, a solution for $\mathbf{q}$ exists if the applied force is an antiplane force. The Burgers' vector $\mathbf{b}$ for the dislocation can be arbitrary.

When (38) holds (37) reduces to

$$\mathbf{Bq} = \gamma \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \gamma = \frac{1}{2}\{f_3 + i\Delta b_3(\beta C_{22})^{-1}\}. \tag{39}$$

It can be shown that the solution for $\mathbf{q}$ is

$$\mathbf{q} = \frac{-i\gamma}{\beta\Delta} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + k\mathbf{q}_0 \tag{40}$$

where k is an arbitrary constant and

$$\mathbf{q}_0 = \begin{bmatrix} 1 \\ -i\beta \\ 0 \end{bmatrix}$$

is the right null vector of $\mathbf{B}$. The product $\mathbf{Aq}_0$ is

$$\mathbf{Aq}_0 = \begin{bmatrix} 1 \\ -i\beta \\ 0 \end{bmatrix}.$$

According to (30) this term represents the solution due to an edge dislocation at the location (34) for the image singularity. This term produces no stresses. Hence the traction-free boundary condition is not violated. We obtain from (40)

$$\mathbf{Aq} = \frac{-i\gamma}{\beta\Delta} \begin{bmatrix} -C_{46} \\ -\beta^2 C_{24} \\ \beta^2 C_{22} \end{bmatrix} + k \begin{bmatrix} 1 \\ -i\beta \\ 0 \end{bmatrix}. \tag{41}$$

Equation (41) and (39) provide the $\mathbf{Aq}$ and $\mathbf{Bq}$ in (32). The solution for the displacement is not unique. (In contrast, the solution for the stress is not unique if the material is incompressible.) In fact, we could also add any edge dislocation at any point outside the half-space without violating the traction-free condition at the boundary.

7. Extraordinary semisimple $\mathbf{N}(v)$ for steady-state motion. The matrix $\mathbf{C}^0$ in (9) applies when $v \neq 0$. Let I_k $(k = 1, 2, \ldots, 5)$ be the determinants of the 1×1, $2 \times 2, \ldots$ submatrices taken from the lower right corner of the matrix $\mathbf{C}^0$. It can be shown that

$$\begin{aligned}
I_1 &= \beta^2 C_{22} + \eta, \\
I_2 &= (\beta^2 C_{22} + \eta)(\beta^2 C_{44} + \eta) - \beta^4 C_{24}^2, \\
I_3 &= C_{44}(I_2 - \beta^2 C_{46}^2) - \eta C_{46}^2, \\
I_4 &= \beta^{-2}\Delta(I_2 - \beta^2 C_{46}^2) + \eta^2 \beta^{-2} C_{46}^2, \\
I_5 &= \eta(1 - \beta^2) I_4 - \eta^2 I_3,
\end{aligned}$$

where Δ is defined in (19). I_5 can be written explicitly as

$$\eta(I_2 - \beta^2 C_{46}^2)[\beta^{-2}(1 - \beta^2)\Delta - \eta C_{44}] + \eta^3 \beta^{-2} C_{46}^2.$$

Keeping in mind that η and β are positive and nonzero, it is not difficult to see that $I_1, I_2, \ldots, I_5$ are positive and nonzero if

$$\beta < 1, \quad C_{44} > 0, \quad I_3 > 0, \quad \beta^{-2}(1 - \beta^2)\Delta - \eta C_{44} > 0. \tag{42}$$

The last two inequalities in (42) can be satisfied by choosing a sufficiently large C_{22}. Hence $\mathbf{C}^0$ can be positive definite, and an extraordinary semisimple $\mathbf{N}(v)$ exists when $v \neq 0$. Explicit expressions of the Barnett-Lothe tensors $\mathbf{H}, \mathbf{L}, \mathbf{S}$ for $v \neq 0$ are presented in the Appendix. Also shown in the Appendix is that $\mathbf{N}(v)$ can be extraordinary semisimple only for $v > v_R$ where v_R is the Rayleigh wave speed for the anisotropic material.

As an example, consider the matrix

$$\mathbf{C}^0 = \begin{bmatrix} 3 & -4 & 0 & 0 & 0 \\ -4 & 6 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

which is clearly positive definite. Equation (3) becomes

$$\begin{bmatrix} 3 - \rho v^2 + 4p^2 & 0 & 0 \\ 0 & 4 - \rho v^2 + 6p^2 & 0 \\ 0 & 0 & 2 - \rho v^2 + 2p^2 \end{bmatrix} \boldsymbol{a} = \mathbf{0}. \tag{43}$$

The eigenvalues are

$$p_1 = i\frac{\sqrt{3 - \rho v^2}}{2}, \qquad p_2 = i\frac{\sqrt{4 - \rho v^2}}{\sqrt{6}}, \qquad p_3 = i\frac{\sqrt{2 - \rho v^2}}{\sqrt{2}}.$$

When $\rho v^2 = 1$ we have

$$p_1 = p_2 = p_3 = i\frac{1}{\sqrt{2}},$$

and the 3×3 matrix in (43) vanishes identically.

Subjected to the conditions (42) the wave speed v is arbitrary for a steady-state motion such as a moving line dislocation. For surface waves in a half-space $x_2 > 0$, v has to be chosen such that the surface traction at $x_2 = 0$ vanishes. We will show that $\mathbf{N}(v)$ cannot be extraordinary semisimple for subsonic surface waves.

Suppose that $\mathbf{N}(v)$ is extraordinary semisimple and that a subsonic surface wave exists. This means that $p_1 = p_2 = p_3 = i\beta$, and the solution can be written as [2, 4, 13]

$$\mathbf{u} = \mathbf{A}\mathbf{q}e^{ikz}, \qquad \phi = \mathbf{B}\mathbf{q}e^{ikz},$$
$$z = x_1 + i\beta x_2 - vt,$$

where k is a positive parameter and $\mathbf{q}$ is a complex vector to be determined. Equation (4) for $p_1 = p_2 = p_3 = i\beta$ can be combined into one equation as

$$\mathbf{B} = (\mathbf{R}^{\mathrm{T}} + i\beta\mathbf{T})\mathbf{A}.$$

Hence

$$\mathbf{B}\mathbf{q} = (\mathbf{R}^{\mathrm{T}} + i\beta\mathbf{T})\mathbf{A}\mathbf{q}.$$

That the traction at $x_2 = 0$ vanishes demands that $\mathbf{B}\mathbf{q} = \mathbf{0}$. The matrix $\mathbf{A}$ is not singular because the columns of $\mathbf{A}$ are $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$, which are independent vectors. This means that the matrix

$$\mathbf{R}^{\mathrm{T}} + i\beta\mathbf{T} = \begin{bmatrix} i\beta(\beta^2 C_{22} + \eta) & (\beta^2 C_{22} + \eta) & \beta(\beta C_{24} + iC_{46}) \\ -(\beta^2 C_{22} + \eta) & i\beta C_{22} & i(\beta C_{24} + iC_{46}) \\ -\beta(\beta C_{24} - iC_{46}) & i(\beta C_{24} - iC_{46}) & i\beta C_{44} \end{bmatrix}$$

must be singular. The determinant of the 3×3 matrix can be shown to be

$$i\beta\eta(\Delta + \eta C_{44}),$$

which is nonzero because β, η, C_{44} are all positive and nonzero, and so is Δ as can be deduced from $(42)_4$. This leads to a contradiction. Therefore, $\mathbf{N}(v)$ cannot be extraordinary semisimple for surface waves. An alternate and simpler proof is given at the end of the Appendix.

8. Remarks. A related problem is the question of whether there exists an $\mathbf{N}(v)$ that has three identical eigenvalues but has only one independent eigenvector. The answer is that it exists for $v = 0$, and for a steady-state motion with $v \neq 0$ such as a moving line dislocation [14]. It also exists for surface waves [17].

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Appendix. We will derive explicit expressions of the Barnett-Lothe tensors and show that they are unique even though the matrices **A** and **B** are not.

Equation (4) for $p_1 = p_2 = p_3$ can be combined into one equation as

$$\mathbf{B} = (\mathbf{R}^{\mathrm{T}} + p\mathbf{T})\mathbf{A}. \tag{A1}$$

When (13) and (14) are written as

$$\boldsymbol{a}_k \cdot \boldsymbol{b}_s + \boldsymbol{a}_s \cdot \boldsymbol{b}_k = \delta_{ks}$$

where δ_{ks} is the Kronecker delta, we have

$$\mathbf{A}^{\mathrm{T}}\mathbf{B} + \mathbf{B}^{\mathrm{T}}\mathbf{A} = \mathbf{I}$$

or, using (A1),

$$\mathbf{A}^{\mathrm{T}}\{(\mathbf{R} + \mathbf{R}^{\mathrm{T}}) + 2p\mathbf{T}\}\mathbf{A} = \mathbf{I}.$$

By virtue of $(6)_2$ this is simplified to

$$(2i\beta)\mathbf{A}^{\mathrm{T}}\mathbf{T}\mathbf{A} = \mathbf{I}. \tag{A2}$$

Since **T** is symmetric and positive definite, there exists a unique square root $\mathbf{T}^{1/2}$ such that it is symmetric and positive definite. Equation (A2) is satisfied by

$$\mathbf{A} = (2i\beta)^{-1/2}\mathbf{T}^{-1/2}\Omega, \qquad \Omega^{\mathrm{T}}\Omega = \mathbf{I} = \Omega\Omega^{\mathrm{T}}, \tag{A3}$$

where Ω is any orthogonal tensor. Hence **A** is not unique.

For $v = 0$, substitution of $(\mathrm{A3})_1$ into $(15)_2$ yields

$$\mathbf{H} = \frac{1}{\beta}\mathbf{T}^{-1} = \frac{1}{\beta}\mathbf{N}_2. \tag{A4}$$

Inserting (A1) into $(15)_3$ and making use of $(15)_2$, (A4), and (6), it is readily shown that

$$\mathbf{L} = \frac{-1}{\beta}\mathbf{N}_3. \tag{A5}$$

Finally, substituting (A1) into $(15)_1$ and following the derivation of **L** we have

$$\mathbf{S} = \frac{1}{\beta}(\mathbf{N}_1 - \alpha\mathbf{I}). \tag{A6}$$

With $\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3$ presented in (12) we have obtained explicitly $\mathbf{S}, \mathbf{H}, \mathbf{L}$ for the general case $\alpha \neq 0$. They do not depend on the arbitrary orthogonal tensor Ω, and hence are unique. For the special case $\alpha = 0$ they recover $\mathbf{S}, \mathbf{H}, \mathbf{L}$ given in (21)–(23).

For $v \neq 0$, the results obtained above apply if we replace $\mathbf{N}_3$ by $\mathbf{N}_3 + \rho v^2\mathbf{I}$ and set $\alpha = 0$. We have

$$\mathbf{H} = \frac{1}{\beta}\mathbf{N}_2, \qquad \mathbf{L} = \frac{-1}{\beta}(\mathbf{N}_3 + \rho v^2\mathbf{I}), \qquad \mathbf{S} = \frac{1}{\beta}\mathbf{N}_1. \tag{A7}$$

The matrices $\mathbf{N}_2, \mathbf{N}_1, \mathbf{N}_3$ for $v \neq 0$ obtained from (9) can be shown to be

$$\mathbf{N}_2 = \frac{1}{\zeta}\begin{bmatrix} \delta & C_{24}C_{46} & -C_{22}C_{46} \\ C_{24}C_{46} & C_{44}C_{66} - C_{46}^2 & -C_{24}C_{66} \\ -C_{22}C_{46} & -C_{24}C_{66} & C_{22}C_{66} \end{bmatrix},$$

$$\mathbf{N}_1 = \frac{1}{\zeta}\begin{bmatrix} \eta C_{24}C_{46} & -\zeta & -C_{24}\Delta \\ C_{66}(\Delta + \eta C_{44}) & 0 & C_{46}(\Delta + \eta C_{44}) \\ -\eta C_{24}C_{66} & 0 & -\eta C_{24}C_{46} \end{bmatrix}, \qquad \mathbf{N}_3 = \frac{1}{\zeta}\begin{bmatrix} X & 0 & Z \\ 0 & 0 & 0 \\ Z & 0 & Y \end{bmatrix}, \tag{A8}$$

where δ and Δ are defined in $(11)_1$ and (19), and

$$\eta = \rho v^2, \qquad C_{66} = \beta^2 C_{22} + \eta, \qquad \zeta = C_{22}\Delta + \delta\eta,$$
$$X = -\eta[(1-\beta^2)\zeta - \eta(C_{44}C_{66} - C_{46}^2)],$$
$$Y = -[\Delta^2 + \eta\Delta(C_{22} + C_{44}) + \delta\eta^2], \qquad Z = \eta C_{46}(\Delta + \eta C_{44}).$$

With (A8), the Barnett-Lothe tensors for $v \neq 0$ are obtained from (A7).

The 3×3 matrix $\mathbf{N}_3$ has zero elements in the second row and the second column (see [11, 15] and (A8) above). With this property, $(A7)_2$ tells us that $-\rho v^2/\beta < 0$ is an eigenvalue of $\mathbf{L}$. This is not possible for $v \leq v_R$ where v_R is the Rayleigh wave speed for the anisotropic material. It is shown in [16] that $\mathbf{L}$ is positive definite for $v < v_R$ and that, at $v = v_R$, two of the eigenvalues of $\mathbf{L}$ vanish while the third one is positive. Therefore, $\mathbf{N}(v)$ cannot be extraordinary semisimple for $v < v_R$ and $v = v_R$; the latter applies to surface waves.

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