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ON EXTREMAL LOG ENRIQUES SURFACES, II

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Abstract. We shall show that there is only one (resp. two) rational log Enriques surface(s) of Dynkin type $\boldsymbol{D}$-eighteen (resp. $\boldsymbol{A}$-eighteen).

Introduction. This is a sequel to our paper [OZ1], where we characterized the unique $K3$ surface of Picard number 20 and discriminant 3 or 4, and also showed that there is only one rational log Enriques surface of type D_{19} and one of type A_{19}; this uniqueness result is an affirmative answer to a question raised by Reid and Naruki (see [R, Example 6]). In the present paper, we shall show that there is exactly one (resp. two) rational log Enriques surface of type D_{18} (resp. A_{18}).

We begin with some definitions. Let Z be a normal projective surface defined over the complex number field $\boldsymbol{C}$ and with at worst quotient singularities. Z is *a log Enriques surface* if, by definition, the irregularity $\dim H^1(Z, \mathcal{O}_Z)=0$ and a positive multiple IK_Z of the canonical Weil divisor K_Z is linearly equivalent to zero [Z1, Definition 1.1].

Let Z be a log Enriques surface and let $I(Z):=\min\{n\in \boldsymbol{Z}_{>0} \mid \mathcal{O}_Z(nK_Z)\cong\mathcal{O}_Z\}$ be the index. The canonical cover of Z is defined as

$$\pi\colon S_{\text{can}}:=\mathcal{S}pec_{\mathcal{O}_Z}(\textstyle\bigoplus_{i=0}^{I-1}\mathcal{O}_Z(-iK_Z))\to Z\,.$$

REMARK 1. (1) A log Enriques surface Z of index I is nothing but the quotient space of a surface S_{can} which is either an abelian surface or a $K3$ surface with at worst Du Val singular points, modulo the group $\boldsymbol{Z}/I\boldsymbol{Z}$ each of whose non-trivial element neither acts trivially on a non-zero holomorphic 2-form of S_{can} nor point-wise fixes a curve.

(2) A log Enriques surface Z is irrational if and only if Z is a $K3$ or Enriques surface with at worst Du Val singular points [Z1, Proposition 1.3].

A log Enriques surface Z is *of type* D_{18} (resp. of type A_{18}) if, by definition, its canonical cover S_{can} has a singular point of Dynkin type D_{18} (resp. A_{18}).

Log Enriques surfaces, which have been intensively studied by Alexeev, Blache, Reid and the authors, are closely related to the study of fibered Calabi-Yau threefolds [O1, 2, 3, 4; Vo, W].

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Our main results are as follows.

THEOREM 1. *There is only one rational log Enriques surface of type D_{18} up to isomorphism.*

THEOREM 2. *There are exactly two rational log Enriques surfaces of type A_{18} up to isomorphism.*

The procedure to prove the theorems above is as follows. Let Z be a rational log Enriques surface of type D_{18} or A_{18}, $\pi: S_{\mathrm{can}} \to Z$ the canonical cover of Z and $\nu: S \to S_{\mathrm{can}}$ the minimal resolution of S_{can}. Let $\langle g \rangle$ be the automorphism group of S induced from the Galois group of π, and Δ the exceptional locus of ν.

First, we shall prove that $(S, \langle g \rangle)$ is isomorphic to Shioda-Inose's pair $(S_3, \langle g_3 \rangle)$ (cf. Example 1.1 below and [OZ1, Example 1]). So we can and will identify $(S, \langle g \rangle)$ with Shioda-Inose's pair. Next we will reduce ourselves to type D_{19} case.

More precisely, we shall prove:

THEOREM 3. *Let Δ be a reduced divisor of Dynkin type D_{18} on S_3. Then there is a smooth rational curve C_1 on S_3 such that $C_1 + \Delta$ has Dynkin type D_{19}. Moreover, $(S_3, \langle g_3 \rangle, C_1 + \Delta)$ is isomorphic to Shioda-Inose's triple $(S_3, \langle g_3 \rangle, \Delta_3)$ in [OZ1, Example 1].*

THEOREM 4. *Let Δ be a reduced divisor of Dynkin type A_{18} on S_3. Then there is a smooth rational curve F on S_3 such that $\Delta + F$ has Dynkin type D_{19}. Moreover, $(S_3, \langle g_3 \rangle, \Delta + F)$ is isomorphic to Shioda-Inose's triple $(S_3, \langle g_3 \rangle, \Delta_3)$.*

REMARK 2. There is no divisors of Dynkin type A_{19} on S_3. See Lemma 1.4 in §1.

To show Theorems 3 and 4, we will first find a curve on $S_3/\langle g_3 \rangle$ so that its strict transform E' on S_3, together with Δ, either forms a graph of Dynkin type D_{19} or contains a singular elliptic fiber. In the latter case, we will find a smooth rational curve F in another singular elliptic fiber so that $F + \Delta$ has Dynkin type D_{19}.

Note that there are two symmetric ways to get a graph of Dynkin type A_{18} by deleting a vertex in a graph of Dynkin type D_{19}. This explains intuitively why we have two isomorphism classes Z_{α_1}, Z_{α_2} of rational log Enriques surfaces of type A_{18} (see Example 1.3). One hard part of the paper is to prove that Z_{α_1} and Z_{α_2} are not isomorphic to each other, though constructed extremely symmetrically (see Theorem 1.6).

From the proofs of Theorems 1 and 2 in §4, we obtain:

COROLLARY 1. *Let Z be a rational log Enriques surface of type D_{18} or A_{18}. Then the minimal resolution S of the (global) canonical cover S_{can} of Z is isomorphic to the unique K3 surface of Picard number 20 and discriminant 3.*

REMARK 3. If Z is a rational log Enriques surface of type D_{19} (resp. A_{19}) then the minimal resolution S of the canonical cover S_{can} of Z is isomorphic to the unique K3 surface of Picard number 20 and discriminant 3 (resp. 4) (cf. [OZ1]). Normally, more $K3$ surfaces like S above, should appear when we decrease the "weight" 19 of D_{19} or A_{19}. So Corollary 1 is a surprise. However, we shall see in our forthcoming paper that the case A_{17} will produce a $K3$ surface of Picard number 18 and discriminant 5.

From some different aspect, Kato and Naruki [KN] constructed a quartic surface in $\boldsymbol{P}^3$ with Du Val singular point of Dynkin type D_{18} or A_{18}. We believe that the canonical covers of our log Enriques surfaces of tyep D_{18} and A_{18} are not isomorphic to theirs.

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1. Rational log Enriques surfaces of type D_{18} or A_{18}. In this section we shall construct one rational log Enriques surface of type D_{18} and two of type A_{18}. It will turn out that these three are all of rational log Enriques surfaces of type D_{18} or A_{18} by Theorems 1, 2 and 1.6.

EXAMPLE 1.1 (a log Enriques surface of type D_{19}, compare [Z1, Example 6.11] and [R, Example 6]). In [OZ1, Example 1], we constructed the triple $(S_3, \langle g_3\rangle, \Delta_3)$, where S_3 is the unique $K3$ surface of Picard number 20 and discriminant 3, g_3 is an order 3 automorphism on S_3 satisfying $g_3^*\omega_{S_3}=\zeta\omega_{S_3}$ for a non-zero holomorphic 2-form ω on S_3 and the primitive cubic root $\zeta=\exp(2\pi\sqrt{-1}/3)$ of unity, and Δ_3 is a rational tree of Dynkin type D_{19} on S_3.

As described in [OZ1, Example 1], the fixed locus $(S_3)^{g_3}$ is contained in Δ_3, except one point P_{32}. Let $v_3: S_3\to S_{3,\mathrm{can}}$ be the contraction of Δ_3 to a point Q_3. Then g_3 acts on $S_{3,\mathrm{can}}$ so that $(S_{3,\mathrm{can}})^{g_3}=\{Q_3, v_3(P_{32})\}$. Now the quotient surface $Z_3:=S_{3,\mathrm{can}}/\langle g_3\rangle$ is a rational log Enriques surface of type D_{19} and of index 3. Note that Z_3 has exactly two singular points: one is of type D_9' and the other is of type $(1/3)(1, 1)$ under the two g_3-fixed points Q_3 and $v_3(P_{32})$, respectively (see [R, Example 6] for the notation).

EXAMPLE 1.2 (a rational log Enriques surface of type D_{18}). We use the notation in Example 1.1 above and [OZ1, Example 1]. We rename the components of Δ_3 in the following way:

$$
\begin{array}{l}
C_{18}\\
\ \ |\\
C_{17}-C_{16}-C_{15}-C_{14}-C_{13}-C_{12}-C_{11}-C_{10}-C_{9}-C_{8}-C_{7}-C_{6}-C_{5}-C_{4}-C_{3}-C_{2}-C_{1}\,.\\
\ \ |\\
C_{19}
\end{array}
$$

So $C_1=E_{13}'$, $C_2=F_1, \ldots, C_{17}=G_1$, $C_{18}=E_{11}$, $C_{19}=E_{21}$. Let $\delta: S_3\to S_\delta$ be the contraction of the rational tree Δ_3-C_1 of Dynkin type D_{18} to a point Q_δ. Then g_3 acts on S_δ so that $(S_\delta)^{g_3}=\{Q_\delta, Q_\delta', \delta(P_{32})\}$, where Q_δ' is the g_3-fixed point on $\delta(C_1)-\delta(C_2)$. Now the quotient surface $Z_\delta:=S_\delta/\langle g_3\rangle$ is a rational log Enriques surface of type D_{18}

and of index 3. Note that $\mathrm{Sing}(Z_\delta)$ consists of exactly one singular point of type D_8' and two of type $(1/3)(1, 1)$ under the three g_3-fixed points Q_δ, Q_δ' and $\delta(P_{32})$, respectively.

EXAMPLE 1.3 (two rational log Enriques surfaces of type A_{18}). We use the notation in Examples 1.1 and 1.2. For $i=1$ (resp. $i=2$), let α_i: $S_3 \to S_{\alpha_i}$ be the contraction of the rational tree $\Delta_3 - C_{18}$ (resp. $\Delta_3 - C_{19}$) of Dynkin type A_{18} to a point Q_{α_i}. Then g_3 acts on S_{α_i} so that $(S_{\alpha_i})^{g_3} = \{Q_{\alpha_i}, Q_{\alpha_i}', \sigma(P_{32})\}$ where Q_{α_1}' (resp. Q_{α_2}') is the g_3-fixed point on $\alpha_1(C_{18}) - \alpha_1(C_{17})$ (resp. $\alpha_2(C_{19}) - \alpha_2(C_{17})$). Now the quotient surfaces $Z_{\alpha_i} := S_{\alpha_i}/\langle g_3 \rangle$ are rational log Enriques surfaces of type A_{18} and of index 3. Note that $\mathrm{Sing}(Z_{\alpha_i})$ consists of exactly one singular point of type A_8' and two of type $(1/3)(1, 1)$ under the three g_3-fixed points Q_{α_i}, Q_{α_i}', $\alpha_i(P_{32})$, respectively.

We shall prove that Z_{α_1} is not isomorphic to Z_{α_2}. First, we need the following Proposition 1.5. We also prove Lemma 1.4 below which implies Remark 2 in the Introduction.

LEMMA 1.4. (1) *Let Δ be a reduced divisor of Dynkin type D_{19} (resp. D_{18} or A_{18}) on S_3. Let v: $S_3 \to S_{\mathrm{can}}$ be the contraction of Δ to a point q. Then g_3 acts on S_{can} with $(S_{\mathrm{can}})^{g_3} = \{q, q_0\}$ (resp. $(S_{\mathrm{can}})^{g_3} = \{q, q_0, q_{19}\}$) where q_i is a point. Hence the quotient surface $S_{\mathrm{can}}/\langle g_3 \rangle$ is a rational log Enriques surface of index* 3 *and type D_{19} (resp. D_{18} or A_{18}).*

(2) *There is no divisors of Dynkin type A_{19} on S_3.*

PROOF. (1) We consider the case where Δ is of Dynkin type A_{18}, while the other two cases are similar. Write $\Delta = C_1 + C_2 + \cdots + C_{18}$ so that $C_i.C_{i+1} = 1$ $(1 \le i \le 17)$. By [OZ1, Lemmas 2.2 and 2.3 and Remark 3 in §1], $(S_3)^{g_3}$ is equal to

$$\mathrm{Supp}(C_2 + C_5 + C_8 + C_{11} + C_{14} + C_{17}) \coprod \{q_0, q_1, q_{3,4}, q_{6,7}, q_{9,10}, q_{12,13}, q_{15,16}, q_{18}, q_{19}\},$$

where $q_{i,i+1} = C_i \cap C_{i+1}$, q_k is a point on C_k $(k=1, 18)$ and q_0, q_{19} are points not on Δ. Now (1) follows after we identify q_i with $v(q_i)$ $(i=0, 19)$.

(2) Suppose to the contrary that $\Delta = C_1 + C_2 + \cdots + C_{19}$ is a reduced divisor of Dynkin type A_{19} on S_3, where $C_i.C_{i+1} = 1$ $(1 \le i \le 18)$. By [ibid.], either $C_1 + C_4 + C_7 + C_{10} + C_{13} + C_{16} + C_{19}$ or $C_2 + C_5 + C_8 + C_{11} + C_{14} + C_{17}$ is contained in $(S_3)^{g_3}$, after relabelling Δ if necessary. The first case is impossible because $(S_3)^{g_3}$ consists of exactly six irreducible curves and nine isolated points [OZ1, Lemma 2.3]. In the second case, there must be a g_3-fixed curve C_{20} such that $C_{20}.C_{19} = 1$ by [OZ1, Lemma 2.2]. This leads to the conclusion that $(S_3)^{g_3}$ contains at least seven fixed curves C_i $(i = 2, 5, 8, 11, 14, 17, 20)$, again a contradiction. So (2) is true. q.e.d.

PROPOSITION 1.5. *Suppose that the two rational log Enriques surfaces Z_{α_i} $(i=1, 2)$ in Example* 1.3 *are isomorphic to each other. Then there is a common integer solution to the following system of four quadratic equations*:

(1) $$38x^2+2y^2+19xy+4x+y=0$$

(2) $$38z^2+2w^2+19zw+36z+9w+7=0$$

(3) $$76xz+19xw+19yz+4yw+27x+6y+4z+w+2=0$$

$(4)_\pm$ $$-19xw+19yz-19x+2y-w-1=\pm 1\,.$$

PROOF. Claim (1). (1) $C_1, C_2, \ldots, C_{19}, C_{20}:=E'_{21}-E_{13}$ form a $\boldsymbol{Z}$-basis of $\mathrm{Pic}(S_3)$.
(2) There exists an isometry ψ of the lattice $\mathrm{Pic}(S_3)$ such that

$$\psi(C_1)=C_{19},\ \psi(C_i)=C_{19-i}\ (2\leq i\leq 17),\ \psi(C_{18})=E'_{11},\ \psi(C_{19})=C_1,\ \psi(C_{20})=-C_{20}\,.$$

The assertion (1) can be verified by computing that the determinant of the intersection matrix of the twenty curves in (1) equals -3, which is also the determinant of that of $\mathrm{Pic}(S_3)$.

By (1), there exists a group-automorphism ψ of $\mathrm{Pic}(S_3)$ satisfying the equalities in (2). A direct checking shows that $\psi(C_i).\psi(C_j)=C_i.C_j$ $(i,j=1,2,\ldots,20)$. So ψ is an isometry of the lattics $\mathrm{Pic}(S_3)$. Claim (1) is proved.

Suppose that Z_{α_1} is isomorphic to Z_{α_2}. Then there exists an automorphism φ such that $g_3\circ\varphi=\varphi\circ g_3$ and $\varphi(\Delta_3-C_{19})=\Delta_3-C_{18}$. So either

$(*)$ $$\varphi(C_i)=C_i\ (1\leq i\leq 17)\,,\quad \varphi(C_{18})=C_{19}\,,\quad \text{or}$$

$(**)$ $$\varphi(C_1)=C_{19}\,,\quad \varphi(C_i)=C_{19-i}\ (2\leq i\leq 18)\,.$$

Replacing φ by $\psi\circ\varphi$ if necessary, we may assume that there exists an isometry φ of the lattice $\mathrm{Pic}(S_3)$ satisfying the hypothesis $(*)$.

Set $M:=\varphi(C_{19})$, $N:=\varphi(C_{20})$. Since φ is a lattice isometry, there are integers a_i, b, α_i, β such that $M=\sum_{i=1}^{19}a_iC_i+bC_{20}$, $N=\sum_{i=1}^{19}\alpha_iC_i+\beta C_{20}$.

Note that $M.C_i=C_{19}.C_i$ is equal to 1 if $i=17$, and 0 if $1\leq i\leq 16$, and that $M.C_{19}=C_{19}.C_{18}=0$. On the other hand, $M.C_i$ can be written as a linear combination of a_i, b. So we get eighteen linear equations in a_i, b. Solving them, we obtain:

$$a_i=ia_1+(i-1)b\ (1\leq i\leq 8)\,,\quad a_j=ja_1+7b\ (j=9,10,11)\,,$$

$$a_k=ka_1+(k-4)b\ (12\leq k\leq 17)\,,$$

$$a_{18}=(19a_1+14b+2)/2\,,\quad a_{19}=(17a_1+14b)/2\,.$$

Substituting these into the calculation $2=-C_{19}^2=-M^2=-(\sum_{i=1}^{19}a_iC_i+bC_{20})^2$, we get:

$$19a_1^2+4b^2+19a_1b+4a_1+2b=0\,.$$

From the expression of a_{18} in terms of a_1, b, we see that a_1 is an even integer. Write $a_1=2a$. Then $(x,y)=(a,b)$ satisfies the equation (1) of Proposition 1.5.

Note that $N.C_i=C_{20}.C_i$ is equal to -1 if $i=1, 11$, equal to 1 if $i=8$, and equal to 0 if $i\neq 1, 8, 11$ and $1\leq i\leq 17$, that $N.C_{19}=C_{20}.C_{18}=0$ and that $N^2=C_{20}^2=-4$. As in the case for M, we obtain the following equalities, where we set $\beta_1 := \beta - 1$:

$$\alpha_i = i\alpha_1 + (i-1)\beta_1 \ (1 \le i \le 8), \quad \alpha_j = j\alpha_1 + 7\beta_1 \ (j = 9, 10, 11),$$
$$\alpha_k = k\alpha_1 + (k-4)\beta_1 \ (12 \le k \le 17),$$
$$\alpha_{18} = (19\alpha_1 + 14\beta_1 - 1)/2, \quad \alpha_{19} = (17\alpha_1 + 14\beta_1 + 1)/2,$$
$$19\alpha_1^2 + 4\beta_1{}^2 + 19\alpha_1\beta_1 - 2\alpha_1 - \beta_1 - 3 = 0.$$

The expression of α_{18} implies that α_1 is an odd integer. Write $\alpha_1 = 2\alpha + 1$. The last equation shows that $(z, w) = (\alpha, \beta_1)$ satisfies the equation (2) of Proposition 1.5.

Now each α_i is a function in α_1, β_1. Substituting these into the calculation $1 = C_{19}.C_{20} = M.N = (\sum_{i=1}^{19} a_i C_i + bC_{20})(\sum_{i=1}^{19} \alpha_i C_i + \beta C_{20})$, we see that $(x, y, z, w) = (a, b, \alpha, \beta_1)$ satisfies the equation (3) of Proposition 1.5.

To finish the proof, we still need to show that the quadruple (x, y, z, w) satisfies the equation $(4)_\pm$. Note that φ, regarded as an automorphism of the lattice $\mathrm{Pic}(S_3)$, has the following transition matrix, with repect to the basis $C_1, C_2, \dots, C_{20}$ in Claim (1)

$$A_\varphi = \begin{pmatrix} I_{17} & 0 & 0 & 0 \\ 0 \cdots 0 & 0 & 1 & 0 \\ a_1 \cdots a_{17} & a_{18} & a_{19} & b \\ \alpha_1 \cdots \alpha_{17} & \alpha_{18} & \alpha_{19} & \beta \end{pmatrix}.$$

Now the equation $(4)_\pm$ follows from the observation $\pm 1 = \det A_\varphi = b\alpha_{18} - \beta a_{18}$ and the substitutions of α_{18}, a_{18} in x, y, z, w. This proves Proposition 1.5. q.e.d.

THEOREM 1.6. *The two rational log Enriques surfaces Z_{α_1} and Z_{α_2} in Example* 1.3 *are not isomorphic to each other.*

PROOF. In view of Proposition 1.5, we have only to show that there are no common integer solutions to the system there.

First we consider the system $(+)$ consisting of four eqautions (1), (2), (3), $(4)_+$, where we choose "$+1$" on the right of the equation $(4)_\pm$ in Proposition 1.5. One can verify that $(-1/4, 1/2, 0, -1)$, $(-5/2, 7, 2, -7)$ are common rational solutions of the system $(+)$. One can also check that $(-5, -9, 0, -1)$, $(7, 7, 2, -7)$ are the only solutions of the system $(+)$ modulo 19.

We apply Cramer's rule to the equations (3) and $(4)_+$ and write x, y in terms of z, w:

$$x = (6z - w + 1)/(171z + 57w + 49), \quad y = (-38z - 4w - 8)/(171z + 57w + 49).$$

Here we note that the denominator function in z, w, in the above expression has no integer zeros because 19 divides 171 and 57 but not 49.

Substituting the above solutions of x, y into the equation (1), we obtain, by getting rid of the denominator, the following:

$$(1') \qquad 2470z^2 + 310w^2 + 1748zw + 1332z + 492w + 182 = 0.$$

Using (1′) and (2), one can write z in terms of w:

$$z=(180w^2-93w-273)/(-513w+1008)\,.$$

Now substituting this into the equation (2) multiplied by the denominator and divided by 18, we get

$$f(w)=171w^4+3192w^3+16090w^2+15176w+2107=0\,.$$

One can verify that

$$f(w)=(w+1)(w+7)(171w^2+1824w+301)\,.$$

Thus, only $w=-1,\ -7$ are integer zeros of $f(w)$. Substituting them into the functions z, x, y, we see that $(x, y, z, w)=(-1/4, 1/2, 0, -1), (-5/2, 7, 2, -7)$ are the only solutions of the system $(+)$ with integer w. Thus there is no integer solutions to the system $(+)$. This proves Theorem 1.6 in the present case.

Nex we consider the system $(-)$ consisting of four equations (1), (2), (3), $(4)_-$, where we choose "-1" on the right of the equation $(4)_\pm$ in Propositon 1.5. One can check that $(x, y, z, w)=(0, -1/2, 0, -1), (-1/2, 5/4, 2, -7)$ are common solutions to the system $(-)$, and that $(0, 9, 0, -1), (9, 6, 2, -7)$ are the only solutions of the system $(-)$ modulo 19.

As in the previous case, one can solve the system $(-)$ in the following procedure:

$$x=-(13z+5w+5)/(171z+57w+49)\,,\quad y=(38z+15w+19)/(171z+57w+49)\,,$$

$$(1')\qquad 2470z^2+310w^2+1748zw+1332z+492w+182=0\,,$$

$$z=(180w^2-93w-273)/(-513w+1008)\,,$$

$$f(w)=(w+1)(w+7)(171w^2+1824w+301)=0\,.$$

As in the case of system $(+)$, we see that $(x, y, z, w)=(0, -1/2, 0, -1), (-1/2, 5/4, 2, -7)$ are the only solutions of the system $(-)$ with integer w. Thus there is no integer solutions to the system $(-)$. This completes the proof of Theorem 1.6. q.e.d.

We prove the following lemma to be used in §4.

LEMMA 1.7. *Let S be a $K3$ surface with at worst Du Val singular points. Suppose that σ is an order I $(I\geq 2)$ automorphism of S such that no curve on S is point-wise fixed by any non-trivial element of $\langle\sigma\rangle$ and that $\sigma^*\omega_S=\zeta_I\omega_S$ for a primitive I-th root ζ_I of unity and a nowhere vanishing holomorphic 2-form ω_S on S. Suppose further that S contains a singular point p_0 of Dynkin type A_r or D_r for some $r\geq 10$. Then the quotient surface $S/\langle\sigma\rangle$ is a rational log Enriques surface of index I with S as its canonical cover.*

PROOF. Clearly, $S/\langle\sigma\rangle$ is a log Enriques surface of index I with the quotient morphism $\pi: S\to S/\langle\sigma\rangle$ as its canonical cover. We only need to show the rationality of $S/\langle\sigma\rangle$. Let $T\to S$ be a minimal resolution of S. Then T is a $K3$ surface. We first prove the following:

Claim (1). The singular point $p_0 \in S$ is σ-fixed.

If Claim (1) is false, then p_0 and $\sigma(p_0)$ are two distinct Du Val singular points of Dynkin type A_r or D_r for some $r \geq 10$. This leads to the conclusion that the minimal resolution T of S has Picard number $\geq 2r+1 \geq 21$, a contradiction. So Claim (1) is true.

Now suppose to the contrary that $S/\langle\sigma\rangle$ is not rational. Then, by the classification of surfaces, $S/\langle\sigma\rangle$ is an Enriques surface with at worst Du Val singular points and $I=2$. By Claim (1), the inverse image Δ on T of the σ-fixed point p_0 is stable under the induced σ-action on T. It is easy to see that Δ contains a point fixed by the involution σ.

On the other hand, $\sigma^*\omega_T = -\omega_T$ implies that σ has no isolated σ-fixed points, and that the fixed locus T^σ is a disjoint union of smooth rational curves by the hypothesis on the σ-action on S. Thus $T/\langle\sigma\rangle$ is smooth and rational by the ramification formula. But then the Enriques surface $S/\langle\sigma\rangle$ with Du Val singularities, is birational to the rational surface $T/\langle\sigma\rangle$, a contradiction. This proves Lemma 1.7. q.e.d.

2. Extend D_{18} to D_{19} on S_3. In this section, we shall prove the following, where S_3 is given in Example 1.1.

PROPOSITION 2.1. *Let Δ be a reduced divisor of Dynkin type D_{18} on S_3. Then there exists a smooth rational curve C_1 on S_3 such that $C_1+\Delta$ has Dynkin type D_{19}.*

The proof of Proposition 2.1 consists of the following Lemmas 2.4, 2.6–2.10.

We write $\Delta = \sum_{i=2}^{19} C_i$ whose dual graph is the same as the one given at the beginning of §4. By [OZ1, Lemmas 2.2 and 2.3] the fixed locus $(S_3)^{g_3}$ consists of exactly six curves C_2, C_5, C_8, C_{11}, C_{14}, C_{17} and nine isolated points. To be precise, $(S_3)^{g_3}$ is equal to

$$\mathrm{Supp}(C_2+C_5+C_8+C_{11}+C_{14}+C_{17}) \coprod \{p_{3,4}, p_{6,7}, p_{9,10}, p_{12,13}, p_{15,16}, p_{18}, p_{19}, l_1, l_2\},$$

where $p_{i,i+1}$ is the intersection point $C_i \cap C_{i+1}$, p_j ($j=18$, 19) is a point on C_j and l_1, l_2 are points not on Δ.

Let $\nu: S_3 \to S_{\mathrm{can}}$ be the contraction of Δ to a point q_3. Then $\langle g_3\rangle$ acts on S_{can} with $(S_{\mathrm{can}})^{g_3} = \{q_3, \nu(l_1), \nu(l_2)\}$. Put $Z = S_{\mathrm{can}}/\langle g_3\rangle$ and let $\pi: S_{\mathrm{can}} \to Z$ be the quotient morphism. Then Z is a rational log Enriques surface of type D_{18} and index 3. This Z has one singular point $\pi(q_3)$ of type D'_8, two singular points $\pi\nu(l_i)$ ($i=1, 2$) of type $(1/3)(1, 1)$ and no other singular points.

Let $\mu: X \to Z$ be the minimal resolution of Z and denote the exceptional locus of μ by $\Gamma = \Gamma_2+\Gamma_5+\Gamma_8+\Gamma_{11}+\Gamma_{14}+\Gamma_{17}+\Pi_{18}+\Pi_{19}+\Lambda_1+\Lambda_2$:

$$\begin{array}{l} \Pi_{18} \\ \quad | \\ \Gamma_{17}-\Gamma_{14}-\Gamma_{11}-\Gamma_8-\Gamma_5-\Gamma_2\,, \quad \Lambda_1\,, \quad \Lambda_2\,. \\ \quad | \\ \Pi_{19} \end{array}$$

Here $\Gamma_2^2=-4$, $\Gamma_i^2=-2$ $(i=5, 8, 11, 14, 17)$, $\Pi_j^2=-2$ $(j=18, 19)$, $\Lambda_k^2=-3$ $(k=1, 2)$, and $\Gamma_2+\Gamma_5+\Gamma_8+\Gamma_{11}+\Gamma_{14}+\Gamma_{17}+\Gamma_{18}+\Gamma_{19}=\mu^{-1}(\pi(q_3))$, $\Lambda_i=\mu^{-1}(\pi\nu(l_i))$ $(i=1, 2)$.

The following result follows from the construction of Z (see [Z1, Table 1, p. 449]).

LEMMA 2.2. (1) *$3(K_X+\Gamma^*)=\mu^*(3K_Z)\sim 0$, where $\Gamma^*=2(\Gamma_2+\Gamma_5+\Gamma_8+\Gamma_{11}+\Gamma_{14}+\Gamma_{17})/3+(\Pi_{18}+\Pi_{19}+\Lambda_1+\Lambda_2)/3$.*

(2) *Let $\nu_1: \tilde{S}_3\to S_3$ be the blowing up of four points p_{18}, p_{19}, l_1, l_2 on S_3 to four (-1)-curves P_{18}, P_{19}, L_1, L_2. Then there exists a degree three morphism $\tilde{\pi}: \tilde{S}_3\to X$ such that $\pi\circ\nu\circ\nu_1=\mu\circ\tilde{\pi}$ and*

$$\tilde{\pi}_*(C_i)=3\Gamma_i\ (i=2, 5, 8, 11, 14, 17),\quad \tilde{\pi}_*(P_j)=3\Pi_j\ (j=18, 19),\quad \tilde{\pi}_*(L_k)=3\Lambda_k\ (k=1, 2).$$

In the following lemma, by a $(-n)$-curve on X we mean a smooth rational curve of self-intersection number $-n$.

LEMMA 2.3. (1) rank Pic$(Z)=2$, rank Pic$(X)=12$ *and* $K_X^2=-2$.

(2) *For any (-1)-curve E on X we have $E.\Gamma^*=1$. If H is an irreducible curve on X with $H^2<0$, then H is either a component of Γ or a (-1)-curve.*

PROOF. By Lemma 2.2, $K_X^2=(\Gamma^*)^2=-2$. Thus (1) follows. Now $3(K_X+\Gamma^*)\sim 0$ in Lemma 2.2 and the genus formula imply the first half of (2) and that H with $H^2<0$ either satisfies (2), or is a (-2)-curve disjoint from Γ. The latter case is impossible because $g_3^*|_{\mathrm{Pic}(S_3)}=\mathrm{id}$ (cf. [OZ1, Lemma 2.3]). q.e.d.

LEMMA 2.4. *There exists one (-1)-curve E or two disjoint (-1)-curves E_1, E_2 on X such that one of the following cases occurs (after exchanging the roles of Π_{18} with Π_{19} and Λ_1 with Λ_2 if necessary):*

Case $(\delta 1)$ *$E.\Lambda_1=E.\Gamma_i=1$ for either one of $i=2, 5, 8, 11, 14$ or 17,*

Case $(\delta 2)$ *$E.\Lambda_1=E.\Pi_{18}=E.\Pi_{19}=1$,*

Case $(\delta 3)$ *$E.\Lambda_1=E.\Lambda_2=E.\Pi_{19}=1$,*

Case $(\delta 4)$ *$E_i.(\Lambda_i+\Pi_{18}+\Pi_{19})=3$ and $E_i.\Lambda_i\in\{1, 2\}$ for both $i=1, 2$, and*

Case $(\delta 5)$ *$E_i.\Lambda_1\in\{1, 2\}$ and $E_1.(\Lambda_1+\Pi_{19})=E_2.(\Lambda_1+\Lambda_2)=3$ for both $i=1, 2$.*

PROOF. Let $f: X\to\Sigma_n$ be a smooth contraction of smooth rational curves to points on some Hirzebruch surface Σ_n of degree n. Since $K_{\Sigma_n}+f_*\Gamma^*\equiv 0$ (Lemma 2.2 (1)), $f_*\Gamma$ contains at least one horizontal component and is connected.

Claim (1). Supp $f(\Gamma)=$ Supp $f_*\Gamma$, that is, no connected component of Γ is contracted to a point not lying on $f_*\Gamma$.

Suppose to the contrary that a maximal union Γ' of connected components of Γ is contracted to a point p not lying on $f_*\Gamma$ so that $f(\Gamma')\cap f(\Gamma-\Gamma')=\varnothing$. Decompose $f=f_3\circ f_2\circ f_1$ so that $f_1(\Gamma')$ is a (-1)-curve and f_2 is the blowing down of $f_1(\Gamma')$. Then we have $0=f_1(\Gamma').f_{1*}(K_X+\Gamma^*)=-1-\alpha<0$, where α is the coefficient in Γ^* of the proper transform $f_1'(f_1(\Gamma'))$. This is a contradiction. Claim (1) is proved.

Claim (1) and its preceding argument imply that $f(\Gamma)$ is connected. So $f^{-1}f(\Gamma)$ is connected. We can write $f^{-1}f(\Gamma)=\Gamma+E_{-1}+C_{-2}$ where E_{-1} is a union of (-1)-curves, and C_{-2} is a union of (-2)-curves disjoint from Γ (Lemma 2.3 (2)). Since $E_{-1}+C_{-2}$ is f-exceptional and hence has negative definite intersection matrix, each connected component of C_{-2} is a twig of $f^{-1}f(\Gamma)$ sprouting from a (-1)-curve in E_{-1}. So $\Gamma+E_{-1}$ is connected. Now Lemma 2.4 follows from Lemma 2.3 (2) and the fact that E_{-1} consists of disjoint (-1)-curves. q.e.d.

We need the following lemma which is a consequence of Kodaira's classification of singular elliptic fibers, "Three Go" Lemma [OZ1, Lemma 2.2] and the fact that $g_3^*|_{\mathrm{Pic}(S_3)}=\mathrm{id}$ in [OZ1, Lemma 2.3]. The condition $n\le 18$ (resp. $n\le 17$) in the type (2) (resp. the type (3)) comes from the fact that rank $\mathrm{Pic}(S_3)<21$.

LEMMA 2.5. *Let ξ be a singular fiber of an elliptic fibration Φ: $S_3\to \boldsymbol{P}^1$. Suppose that all curves of $(S_3)^{g_3}$ are contained in fibers of Φ and ξ contains at least one curve of $(S_3)^{g_3}$. Then ξ has one of the following types*:

(1) *$\xi=H_1+H_2+H_3$, where H_i's share one and the same point. After relabelling the components of ξ if necessary, H_1 is the only common component of ξ with $(S_3)^{g_3}$.*

(2) *$\xi=H_1+H_2+\cdots+H_n$ is a loop with $H_i.H_{i+1}=H_n.H_1=1$ $(1\le i\le n-1)$. n is one of* 3, 6, 9, 12, 15, 18. *After relabelling the components of ξ if necessary, H_1, H_4, $H_7,\dots,H_{n-2}$ are the only common components of ξ with $(S_3)^{g_3}$.*

(3) *$\xi=H_1+H_2+2(H_3+H_4+\cdots+H_{n-2})+H_{n-1}+H_n$, where $H_1.H_3=H_i.H_{i+1}=H_{n-2}.H_n=1$ $(2\le i\le n-2)$. n is one of* 5, 8, 11, 14, 17. *$H_3, H_6, H_9,\dots,H_{n-2}$ are the only common components of ξ with $(S_3)^{g_3}$.*

(4) *$\xi=3H_1+2H_2+H_3+2H_4+H_5+2H_6+H_7$, where $H_1.H_i=H_i.H_{i+1}=1$ $(i=2, 4, 6)$. H_1 is the only common component of ξ with $(S_3)^{g_3}$.*

(5) *$\xi=4H_1+2H_2+3H_3+2H_4+H_5+3H_6+2H_7+H_8$, where $H_1.H_i=H_{j-1}.H_j=H_j.H_{j+1}=1$ $(i=2, 3, 6; j=4, 7)$. H_1, H_5, H_8 are the only common components of ξ with $(S_3)^{g_3}$.*

(6) *$\xi=6H_1+3H_2+4H_3+2H_4+5H_5+4H_6+3H_7+2H_8+H_9$, where $H_1.H_i=H_3.H_4=H_j.H_{j+1}=1$ $(i=2, 3, 5;\ 5\le j\le 8)$. H_1, H_7 are the only common components of ξ with $(S_3)^{g_3}$.*

We now treat the cases in Lemma 2.4 separately to conclude Proposition 2.1.

LEMMA 2.6. *If Case $(\delta 1)$ of Lemma* 2.4 *occurs then Proposition* 2.1 *is true.*

PROOF. Let E be as in Case $(\delta 1)$. By Lemma 2.2 (2), we see that the strict transform E' on S_3 of E is a smooth rational curve such that $E'.\Delta=E'.C_i=1$ for $i=2$, 5, 8, 11, 14 or 17. If $i=2$, we let $C_1=E'$ and Proposition 2.1 is proved.

So we may assume that $i=5$, 8, 11, 14 or 17. Let $\xi_0:=E'+C_{i-1}+2\sum_{k=i}^{17}C_k+C_{18}+C_{19}$. Applying the Riemann-Roch theorem to this nef divisor ξ_0 we see that there exists an elliptic fibration Φ: $S_3\to\boldsymbol{P}^1$ with ξ_0 as its singular fiber. Let ξ_1 be the singular fiber of Φ containing $\sum_{k=2}^{i-3}C_k$. Then ξ_1 fits one of the six types in Lemma 2.5. If ξ_1

has either of the type (1), (2), (3), (4) or (6) then, after relabelling, we can take H_2 or H_8 (only for the type (6)) as C_1, which satisfies the condition of Proposition 2.1.

We may assume now that ξ_1 is of the type (5). So $i=11$ and $\xi_1=4C_5+2H_2+3C_4+2C_3+C_2+3C_6+2C_7+C_8$ where $H_2.C_5=1$. Consider a new elliptic fibration $\Psi: S_3 \to \boldsymbol{P}^1$ with $\eta_0=H_2+C_4+2\sum_{k=5}^{17} C_k+C_{18}+C_{19}$ as a singular fiber. Let η_1 be the singular fiber of Ψ containing C_2. Then η_1 has one of the six types in Lemma 2.5. Since the Euler number $\chi(\eta_0)=18$, one has $\chi(\eta_1)\le\chi(S_3)-18=6$. Hence η_1 is not of the type (5). (Actually η_1 has the type (2) with $n=3$.) Now we can find from η_1, as in the previous paragraph, a smooth rational curve C_1 which satisfies the condition of Proposition 2.1. This proves Lemma 2.6. q.e.d.

LEMMA 2.7. *If Case* ($\delta 2$) *of Lemma* 2.4 *occurs then Proposition* 2.1 *is true.*

PROOF. Let E be as in Case ($\delta 2$). Then the strict transform E' on S_3 of E is a smooth elliptic curve such that $E'.\Delta=2$ and $E'.C_i=1$ for both $i=18$, 19 (cf. Lemma 2.2 (2)).

Consider the elliptic fibration $\Phi: S_3 \to \boldsymbol{P}^1$ with E' as a fiber. Let ξ_1 be the singular fiber of Φ containing $\sum_{k=2}^{17} C_k$. Then ξ_1 fits the type (2) of Lemma 2.5 with $n=18$. Now let C_1 ($\neq C_3$) be the curve in ξ_1 meeting C_2. This C_1 satisfies the condition of Proposition 2.1. Lemma 2.7 is proved. q.e.d.

LEMMA 2.8. *If Case* ($\delta 3$) *of Lemma* 2.4 *occurs then Proposition* 2.1 *is true.*

PROOF. Let E be as in Case ($\delta 3$). Then the strict transform E' on S_3 of E is a smooth elliptic curve such that $E'.\Delta=E'.C_{19}=1$.

Claim (1). There is a smooth rational curve H_2 on S_3 such that $H_2.\Delta=H_2.C_5=1$.

By Lemma 2.5, there exists a smooth rational curve G_1 such that $G_1.C_2=G_1.C_{18}=1$ and $G_1+\sum_{i=2}^{18} C_i$ is a singular fiber of type (2) of the elliptic fibration $\Phi_{|E'|}: S_3 \to \boldsymbol{P}^1$. By the same lemma, we see that there is a smooth rational curve H_2 satisfying the conditions in Claim (1) such that $6C_5+3H_2+4C_4+2C_3+5C_6+4C_7+3C_8+2C_9+C_{10}$ and $6C_{17}+3C_{19}+4C_{18}+2G_1+5C_{16}+4C_{15}+3C_{14}+2C_{13}+C_{12}$ are two distinct fibers of an elliptic fibration on S_3. This proves Claim (1).

Now letting $\xi_0:=H_2+C_4+2(C_5+\cdots+C_{17})+C_{18}+C_{19}$ and arguing as in Lemma 2.6, we can see that Proposition 2.1 is true. This proves Lemma 2.8. q.e.d.

LEMMA 2.9. *Case* ($\delta 4$) *of Lemma* 2.4 *does not occur.*

PROOF. Consider Case ($\delta 4$). Denote by E_i' the strict transform on S_3 of E_i. Then E_i' is a nodal elliptic or type-(2.5)-cuspidal rational curve of self intersection number 2. Set $G_{i-1}:=C_i$ ($2\le i\le 19$), $G_{18+i}:=E_i'$ ($i=1, 2$). Since the discriminant of S_3 is 3, $\det(G_i.G_j)=-3n^2$ for a non-negative integer n. Here n is the index of the sublattice $\sum_{i=1}^{20} \boldsymbol{Z}G_i$ in $\mathrm{Pic}(S_3)$ if G_i's are linearly independent, and zero otherwise. After exchanging the roles of Π_{18} with Π_{19} or E_1, Δ_1 with E_2, Δ_2 if necessary, one of the following subcases occurs. Here we use also the fact that $E'_1.E'_2>0$ for both E'_1, E'_2 are nef and big divisors.

Case (δ4.1) $E_i.\Pi_{19}=2$ and $E_i.\Lambda_i=1$ for both $i=1,2$. Then $E'_i.C_{19}=2$ $(i=1,2)$ and $E'_1.E'_2=4$. Now $-3n^2=\det(G_i.G_j)=-336$, which is impossible.

Case (δ4.2) $E_1.\Pi_{19}=2$, $E_2.\Pi_{19}=1$, $E_1.\Lambda_1=1$, $E_2.\Lambda_2=2$. Then $E'_1.C_{19}=2$, $E'_2.C_{19}=1$, $E'_1.E'_2=2$. Now $-3n^2=\det(G_i.G_j)=36$, which is impossible.

Case (δ4.3) $E_i.\Pi_{19}=1$ and $E_i.\Lambda_i=2$ for both $i=1,2$. Then $E'_i.C_{19}=1$ $(i=1,2)$ and $E'_1.E'_2=1$. Now $-3n^2=\det(G_i.G_j)=48$, which is impossible. q.e.d.

LEMMA 2.10. *If Case* (δ5) *of Lemma* 2.4 *occurs then Proposition* 2.1 *is true.*

PROOF. Let E_1, E_2 be as in Case (δ5). Then the strict transform G_{18+i} on S_3 of E_i is a curve of self intersection number 2. Set $G_{i-1}:=C_i$ $(2\le i\le 19)$. Then $\det(G_i.G_j)=-3n^2$ for a non-negative integer n. This implies, as in Lemma 2.9, that $E_1.\Pi_{19}=E_2.\Lambda_2=1$, and $E_i.\Lambda_1=2$ for both $i=1,2$. Moreover, $\det(G_i.G_j)=-12$.

Let $\eta_0:=2(E_1+\Pi_{19}+\Gamma_{17})+\Pi_{18}+\Gamma_{14}$ and $\Psi: X\to \boldsymbol{P}^1$ the $\boldsymbol{P}^1$-fibration with η_0 as a fiber. Let η_1 be the fiber containing $E_2+\Lambda_2$. By Lemma 2.3, there are (-1)-curves E_3, E_4 such that either $E_3.\Gamma_{11}=E_j.\Lambda_2=1$ $(j=3,4)$, $E_4.\Lambda_1=2$ and $\eta_1=\Lambda_2+\sum_{j=2}^4 E_j$, or $E_3.\Gamma_2=E_3.\Lambda_2=E_4.\Gamma_5=E_4.\Lambda_1=1$ and $\eta_1=2(E_3+E_4+\Gamma_5)+E_2+\Lambda_2+\Gamma_2+\Gamma_8$. In both cases, we are reduced to Case (δ1) with Λ_2 (resp. E) replaced by Λ_1 (resp. E_3). So Proposition 2.1 is true by Lemma 2.6. q.e.d.

3. **Extend A_{18} to D_{19} on S_3.**

In this section, we shall prove the following, where S_3 is given in Example 1.1.

PROPOSITION 3.1. *Let Δ be a reduced divisor of Dynkin type A_{18} on S_3. Then there exists a smooth rational curve F on S_3 such that $\Delta+F$ has Dynkin type D_{19}.*

The proof of Proposition 3.1 consists of the following Lemmas 3.5–3.9.

Write $\Delta=\sum_{i=1}^{18} C_i$ where $C_i.C_{i+1}=1$. By [OZ1, Lemmas 2.2 and 2.3], $(S_3)^{g_3}$ equals

$$\mathrm{Supp}(C_2+C_5+C_8+C_{11}+C_{14}+C_{17})\coprod\{p_1, p_{3,4}, p_{6,7}, p_{9,10}, p_{12,13}, p_{15,16}, p_{18}, l_1, l_2\},$$

where $p_{i,i+1}$ is the intersection point $C_i\cap C_{i+1}$, p_j $(j=1, 18)$ is a point on C_j, and l_1, l_2 are points not on Δ.

Let $\nu: S_3\to S_{\mathrm{can}}$ be the contraction of Δ to a point q_3. Then $\langle g_3\rangle$ acts on S_{can} with $(S_{\mathrm{can}})^{g_3}=\{q_3, \nu(l_1), \nu(l_2)\}$. Put $Z=S_{\mathrm{can}}/\langle g_3\rangle$ and let $\pi: S_{\mathrm{can}}\to Z$ be the quotient morphism. Then Z is a rational log Enriques surface of type A_{18} and index 3. Z has one singular point $\pi(q_3)$ of type A'_8, two singular points $\pi\nu(l_i)$ $(i=1,2)$ of type (1/3)(1,1) and no other singular points.

Let $\mu: X\to Z$ be the minimal resolution of Z and denote the exceptional locus of μ by $\Gamma=\Pi_1+\Gamma_2+\Gamma_5+\Gamma_8+\Gamma_{11}+\Gamma_{14}+\Gamma_{17}+\Pi_{18}+\Lambda_1+\Lambda_2$:

$$\Pi_{18}-\Gamma_{17}-\Gamma_{14}-\Gamma_{11}-\Gamma_8-\Gamma_5-\Gamma_2-\Pi_1\,,\quad \Lambda_1\,,\quad \Lambda_2\,.$$

Here $\Pi_i^2=-2$ $(i=1, 18)$, $\Gamma_j^2=-3$ $(j=2, 17)$, $\Gamma_k^2=-2$ $(i=5, 8, 11, 14)$, $\Lambda_r^2=-3$ $(r=1, 2)$, and $\Pi_1+\Gamma_2+\Gamma_5+\Gamma_8+\Gamma_{11}+\Gamma_{14}+\Gamma_{17}+\Pi_{18}=\mu^{-1}(\pi(q_3))$, $\Lambda_i=\mu^{-1}(\pi v(l_i))$ $(i=1, 2)$.

The following Lemmas 3.2, 3.3 and 3.4 can be proved similarly as in Lemmas 2.2, 2.3 and 2.4.

LEMMA 3.2. (1) $3(K_X+\Gamma^*)=\mu^*(3K_Z)\sim 0$, *where* $\Gamma^*=2(\Gamma_2+\Gamma_5+\Gamma_8+\Gamma_{11}+\Gamma_{14}+\Gamma_{17})/3+(\Pi_1+\Pi_{18}+\Lambda_1+\Lambda_2)/3$.

(2) *Let* $v_1: \tilde{S}_3\to S_3$ *be the blowing up of four points* p_1, p_{18}, l_1, l_2 *on* S_3 *to four* (-1)-*curves* P_1, P_{18}, L_1, L_2. *Then there exists a degree three morphism* $\tilde{\pi}: \tilde{S}_3\to X$ *such that* $\pi\circ v\circ v_1=\mu\circ\tilde{\pi}$ *and*

$$\tilde{\pi}_*(C_i)=3\Gamma_i\ (i=2, 5, 8, 11, 14, 17),\quad \tilde{\pi}_*(P_j)=3\Pi_j\ (j=1, 18),\quad \tilde{\pi}_*(L_k)=3\Lambda_k\ (k=1, 2).$$

LEMMA 3.3. (1) rank $\mathrm{Pic}(Z)=2$, rank $\mathrm{Pic}(X)=12$ *and* $K_X^2=-2$.

(2) *For any* (-1)-*curve* E *on* X *we have* $E.\Gamma^*=1$. *If* H *is an irreducible curve on* X *with* $H^2<0$, *then* H *is either a component of* Γ *or a* (-1)-*curve.*

LEMMA 3.4. *There exists a* (-1)-*curve* E *or two disjoint* (-1)-*curves* E_1, E_2 *on* X *such that one of the following cases occurs* (*after exchanging the roles of* Λ_1 *with* Λ_2 *and relabelling* $\mu^{-1}(\pi(q_3))$ *if necessary*):

Case (α1) $E.\Lambda_1=E.\Gamma_i=1$ *for either* $i=11$, 14, *or* 17,

Case (α2) $E.\Lambda_1=E.\Pi_1=E.\Pi_{18}=1$,

Case (α3) $E.\Lambda_1=E.\Lambda_2=E.\Pi_{18}=1$,

Case (α4) $E_i.(\Lambda_i+\Pi_1+\Pi_{18})=3$ *and* $E_i.\Lambda_i\in\{1, 2\}$ *for both* $i=1, 2$, *and*

Case (α5) $E_i.\Lambda_1\in\{1, 2\}$ *and* $E_1.(\Lambda_1+\Pi_{18})=E_2.(\Lambda_1+\Lambda_2)=3$ *for both* $i=1, 2$.

We now treat the cases in Lemma 3.4 separately to conclude Proposition 3.1.

LEMMA 3.5. *If Case* (α1) *of Lemma* 3.4 *occurs then Proposition* 3.1 *is true.*

PROOF. Let E be as in Case (α1). By Lemma 3.2 (2), we see that the strict transform E' on S_3 of E is a smooth rational curve such that $E'.\Delta=E'.C_i=1$ for $i=11$, 14 or 17. If $i=17$, we let $F=E'$ and Proposition 3.1 is proved.

So we may assume that $E'.C_i=1$ for $i=11$ or 14.

Claim (1). Assume that $E'.C_{14}=1$. Then either Proposition 3.1 is true or there is a (-2)-curve E_1' such that $E_1'.(\Delta+E')=E_1'.(C_2+C_{18})$, $E_1'.C_2=E_1'.C_{18}=1$.

Let $\xi_0:=4C_{14}+2E'+3C_{13}+2C_{12}+C_{11}+3C_{15}+2C_{16}+C_{17}$. Applying the Riemann-Roch theorem to this nef divisor ξ_0 we see that there is an elliptic fibration $\Phi: S_3\to \boldsymbol{P}^1$ with ξ_0 as its singular fiber. Let ξ_1 be the singular fiber of Φ containing $\sum_{i=1}^{9} C_i$. Then ξ_1 must have the type (3) with $n=11$ in Lemma 2.5. So there are two smooth rational curves E_1', E_2' such that $\xi_1=E_1'+C_1+2\sum_{i=2}^{8} C_i+C_9+E_2'$ where $E_1'.C_2=E_2'.C_8=1$. Note that the cross-section C_{18} meets either E_2' or E_1'. Thus, Claim (1) is true. Indeed, if C_{18} meets E_2' then $C_{18}.E_1'=0$ and $\Delta+E_1'$ has Dynkin type D_{19} and hence Proposition 3.1 is true, otherwise the second case in Claim (1) occurs.

Claim (2). If the second case in Claim (1) occurs then Proposition 3.1 is true.

Let E_1' be as in Claim (1). Let $\eta_0 := E_1' + C_1 + 2\sum_{i=2}^{14} C_i + C_{15} + E'$ and let $\Psi: S_3 \to \boldsymbol{P}^1$ be the elliptic fibration with η_0 as its singular fiber. Let η_1 be the singular fiber of Ψ containing C_{17}. Then η_1 fits one of the six types in Lemma 2.5. (Actually η_1 is of the type (1) or (2) there.) Taking as F a component in η_1 adjacent to C_{17}, we see that $\Delta + F$ is of Dynkin type D_{19}.

To finish the proof of Lemma 3.5, we have only to show the following Claim (3). In fact, if Claim (3) is true then by relabelling Δ and replacing E' by E_1' in Claim (3), we are reduced to the case where $E'.C_{14} = 1$.

Claim (3). Assume that $E'.C_{11} = 1$. Then either Proposition 3.1 is true or we can find a smooth rational curve E_1' such that $E_1'.\Delta = E_1'.C_5 = 1$.

Let $\theta_0 = 4C_{11} + 2E' + 3C_{10} + 2C_9 + C_8 + 3C_{12} + 2C_{13} + C_{14}$ and let $\Theta: S_3 \to \boldsymbol{P}^1$ be the elliptic fibration with θ_0 as its singular fiber. Let θ_1 be the singular fiber of Θ containing $\sum_{i=1}^{6} C_i$. Then θ_1 must have the type (3) in Lemma 2.5. More precisely, if $\sum_{i=16}^{18} C_i$ is not contained in θ_1 then $\theta_1 = E_1' + C_6 + 2\sum_{i=2}^{5} C_i + C_1 + E_2'$ where E_1', E_2' are smooth rational curves with $E_1'.C_5 = E_2'.C_1 = 1$; if $\sum_{i=1}^{6} C_i$ is contained in θ_1 then $\theta_1 = E_1' + C_6 + 2(\sum_{i=1}^{5} C_i + E_2' + C_{17}) + C_{16} + C_{18}$ where E_1', E_2' are smooth rational curves with $E_1'.C_5 = E_2'.C_1 = E_2'.C_{17} = 1$. (Actually the first case here does not occur by counting the number of g_3-fixed points in the fiber of Θ containing $\sum_{i=16}^{18} C_i$.) If the cross-section C_{15} intersects E_1' then the first case here occurs and Proposition 3.1 is true because now $C_{15}.E_2' = 0$ and $\Delta + E_2'$ has Dynkin type D_{19}. If C_{15} does not intersect E_1' then the second case in Claim (3) occurs. This proves Claim (3) and also Lemma 3.5. □

LEMMA 3.6. *If Case* (α2) *of Lemma* 3.4 *occurs then Proposition* 3.1 *is true.*

PROOF. Let E be as in Case (α2). Then the strict transform E' on S_3 of E is a smooth elliptic curve such that $E'.\Delta = 2$ and $E'.C_i = 1$ for both $i = 1$, 18 (cf. Lemma 3.2 (2)).

Consider the elliptic fibration $\Phi: S_3 \to \boldsymbol{P}^1$ with E' as a fiber. Let ξ_1 be the singular fiber of Φ containing $\sum_{i=2}^{17} C_i$. Then ξ_1 fits the type (2) of Lemma 2.5 with $n = 18$. Now let $F (\neq C_{16})$ be the curve in ξ_1 meeting C_{17}. Then $\Delta + F$ has Dynkin type D_{19}. Lemma 3.6 is proved. q.e.d.

LEMMA 3.7. *If Case* (α3) *of Lemma* 3.4 *occurs then Proposition* 3.1 *is true.*

PROOF. Let E be as in Case (α3). Then the strict transform E' on S_3 of E is a smooth elliptic curve such that $E'.\Delta = E'.C_{18} = 1$ (cf. Lemma 3.2 (2)).

Consider the elliptic fibration $\Phi: S_3 \to \boldsymbol{P}^1$ with E' as a fiber. Let ξ_1 be the singular fiber of Φ containing $\sum_{i=1}^{17} C_i$. Then ξ_1 has the type (2) in Lemma 2.5 with $n = 18$. To be precise, $\xi_1 = E_1' + \sum_{i=1}^{17} C_i$ where E_1' is a smooth rational curve with $E_1'.C_1 =$

$E_1'.C_{17}=1$. In order to finish the proof of Lemma 3.7, it suffices to show the following Claim (1). Indeed, replacing E' by E_3' in Claim (1), we are reduced to the case of Lemma 3.5.

Claim (1). There is a smooth rational curve E_3' such that $E_3'.\Delta=E_3'.C_{11}=1$.

Let $\eta_0:=4C_{17}+2C_{18}+3E_1'+2C_1+C_2+3C_{16}+2C_{15}+C_{14}$ and let $\Psi: S_3\to \boldsymbol{P}^1$ be the elliptic fibration with η_0 as a fiber. Let η_1 be the singular fiber of Ψ containing $\sum_{i=4}^{12}C_i$. Then η_1 has the type (3) in Lemma 2.5 with $n=11$. To be precise, $\eta_1=E_2'+C_4+2\sum_{i=5}^{11}C_i+C_{12}+E_3'$ where E_2', E_3' are smooth rational curves with $E_2'.C_5=E_3'.C_{11}=1$. This proves Claim (1) and also Lemma 3.7. q.e.d.

LEMMA 3.8. *Case* (α4) *of Lemma* 3.4 *does not occur.*

PROOF. Consider Case (α4). Denote by E_i' the strict transform on S_3 of E_i. Then E_i' is a nodal elliptic or type-(2.5)-cuspidal rational curve of self intersection number 2. As in Lemma 2.9, after switching the roles of E_1, Λ_1 with E_2, Λ_2 or relabelling C_i as C_{19-i} if necessary, one of the following subcases occurs, where $C_{18+j}:=E_j'$ ($j=1, 2$).

Case (α4.1) $E_i.\Pi_{18}=2$, $E_i.\Lambda_i=1$. Then $E_i'.C_{18}=2$ ($i=1, 2$) and $E_1'.E_2'=4$ for both $i=1, 2$. Now $-3n^2=\det(C_i.C_j)=-516$, which is impossible.

Case (α4.2) $E_1.\Pi_{18}=2$, $E_2.\Pi_{18}=1$, $E_1.\Lambda_1=1$, $E_2.\Lambda_2=2$. Then $E_1'C_{18}=2$, $E_2'.C_{18}=1$, $E_1'.E_2'=2$. Now $-3n^2=\det(C_i.C_j)=36$, which is impossible.

Case (α4.3) $E_i.\Pi_{18}=1$, $E_i.\Lambda_i=2$ for both $i=1, 2$. Then $E_i'.C_{18}=1$ ($i=1, 2$) and $E_1'.E_2'=1$. Now $-3n^2=\det(C_i.C_j)=93$, which is impossible. q.e.d.

LEMMA 3.9. *If Case* (α5) *of Lemma* 3.4 *occurs then Proposition* 3.1 *is true.*

PROOF. Let E_1, E_2 be as in Case (α5). As in Lemma 2.10, by calculating $\det(C_i.C_j)$ where C_{18+j} is the strict transform on S_3 of E_j, we can prove that $E_1.\Pi_{18}=E_2.\Lambda_2=1$ and $E_j.\Lambda_1=2$ for both $j=1, 2$. Moreover, $\det(C_i.C_j)=-192$.

Let $\tau: X\to X_1$ be the smooth blowing down of E_2, E_1, Π_{18}. Let $\nu_1: X_1\to Z_1$ be the contraction of $\tau(\Lambda_2)$, $\tau(\Pi_1+\Gamma_2+\Gamma_5+\Gamma_8+\Gamma_{11}+\Gamma_{14}+\Gamma_{17})$ into cyclic quotient singularities of type $\langle 2, 1\rangle$, $\langle 13, 9\rangle$, respectively. $K_X+\Gamma^*\equiv 0$ and $\rho(Z)=2$ imply that $K_{X_1}+\nu_1(\tau(\Lambda_1))/3\equiv 0$ and $\rho(Z_1)=1$. So Z_1 is a log del Pezzo surface.

By [Z3, Appendix], Z_1 fits Case No. 75 there and there is a $\boldsymbol{P}^1$-fibration $\Psi'': X_1\to \boldsymbol{P}^1$ such that the ν_1-exceptional divisor and all singular fibers of Ψ'' are precisely described in Figure (75) there. Using Lemma 3.3, we see that Ψ'' induces a $\boldsymbol{P}^1$-fibration $\Psi: X\to \boldsymbol{P}^1$ such that $\eta_0:=4E_4+2(E_2+\Lambda_2+\Gamma_2)+\Pi_1+\Gamma_5$ and $\eta_1:=2(E_3+\Gamma_{14})+E_1+\Pi_{18}+\Gamma_{17}+\Gamma_{11}$ are the only singular fibers of Ψ. Here E_3 and E_4 are (-1)-curves satisfying $E_3.\Gamma_{14}=E_3.\Lambda_1=E_4.\Gamma_2=E_4.\Lambda_2=1$. Now we are reduced to Case (α1) with E replaced by E_3. So Proposition 3.1 is true by Lemma 3.5. q.e.d.

4. Proofs of the Theorems. We first prove Theorems 1 and 3.

Let Z be a rational log Enriques surface of type D_{18} and of index I. Let $\pi: S_{\text{can}}\to Z$

be the canonical cover of Z and we denote by $\langle g\rangle \cong \boldsymbol{Z}/I\boldsymbol{Z}$ the Galois group of π. Let $\nu: S\to S_{\mathrm{can}}$ be the minimal resolution of the surface S_{can}. By the hypothesis on Z, S_{can} has a rational double point p_1 of Dynkin type D_{18}. Since rank $\mathrm{Pic}(S)\le 20$, $\mathrm{Sing}\, S_{\mathrm{can}}$ is equal to either $\{p_1\}$ or $\{p_1, p_2\}$, where p_2 is a Du Val singular point of type A_1.

Write $\Delta := \nu^{-1}(p_1)=\sum_{i=2}^{19} C_i$, which is of Dynkin type D_{18}:

$$\begin{array}{l} C_{18} \\ \quad | \\ C_{17}-C_{16}-C_{15}-\cdots-C_4-C_3-C_2\,. \\ \quad | \\ C_{19} \end{array}$$

Let us begin with the following:

LEMMA 4.1. $I=3$.

PROOF. Since g acts on S as $g^*\omega=\zeta_I\omega$ for an I-th primitive root ζ_I of unity, the Euler function $\varphi(I)$ satisfies $\varphi(I)\le \mathrm{rank}\, T_S = 22-\mathrm{rank}\,\mathrm{Pic}(S)\le 3$, where T_S is the transcendental lattice. Thus I is one of 2, 3, 4, 6, for $I\ge 2$ by the rationality of S.

Now it suffices to show that 2 is not a divisor of I. Suppose to the contrary that $2\,|\,I$. Then $S_{\mathrm{can}}/\langle g^{I/2}\rangle$ is a rational log Enriques surface of index 2 (cf. Lemma 1.7). This forces that each singular point of S_{can} must be of Dynkin type A_{2n+1} (cf. [Z1, Lemma 3.1]), a contradiction to the assumption. Thus Lemma 4.1 is proved. q.e.d.

Note that the action of $\langle g\rangle$ on S_{can} induces a faithful action on S. We want to apply Theorem 3 in [OZ1]. For this we need to show the following:

LEMMA 4.2. (1) *S^g consists of exactly six curves C_2, C_5, C_8, C_{11}, C_{14}, C_{17} in Δ and nine isolated points.*

(2) *The pair $(S, \langle g\rangle)$ is isomorphic to the pair $(S_3, \langle g_3\rangle)$ in Example* 1.1.

(3) $\mathrm{Sing}(S_{\mathrm{can}})=\{p_1\}$.

PROOF. Since the order 3 element g acts on the dual graph of $\nu^{-1}(\mathrm{Sing}(S_{\mathrm{can}}))$ as the identity, we can apply "Three Go" Lemma (Lemma 2.2 in [OZ1]) or [Z1, Table 1, p. 449] to conclude that six curves C_2, C_5, C_8, C_{11}, C_{14}, C_{17} in Δ are g-fixed curves. Now (1) and (2) follow from [OZ1, Theorem 3 and Lemma 2.3].

Suppose (3) is false. Then $\mathrm{Sing}(S_{\mathrm{can}})=\{p_1, p_2\}$. Now $\nu^{-1}(p_2)$ is a g_3-stable but not g_3-fixed curve. By [OZ1, Lemma 2.2(2)], $\nu^{-1}(p_2)$ meets one of the six g_3-fixed curves in $\Delta=\nu^{-1}(p_1)$. This is absurd. So (3) is true. q.e.d.

By Lemma 4.2, we shall, from now on, identify $(S, \langle g\rangle)$ with $(S_3, \langle g_3\rangle)$.

By Proposition 2.1, we can find a smooth rational curve C_1 on S_3 such that $C_1+\Delta$ has Dynkin type D_{19}. Let $S_3\to S'_{3,\mathrm{can}}$ be the contraction of $C_1+\Delta$. Then $\langle g_3\rangle$ acts on $S'_{3,\mathrm{can}}$ with no fixed curves and $S'_{3,\mathrm{can}}/\langle g\rangle$ is a rational log Enriques surface of type D_{19}

and index 3 (cf. Lemmas 4.2 and 1.4). Thus by [OZ1, Theorem 1], $S'_{3,\mathrm{can}}/\langle g\rangle \cong Z_3$, $S'_{3,\mathrm{can}} \cong S_{3,\mathrm{can}}$ and there exists an automorphism φ of S_3 such that $\varphi(C_1+\Delta)=\Delta_3$ and $g_3\circ\varphi=\varphi\circ g_3$. This implies Theorem 3.

Clearly, $\varphi(\Delta)=\Delta_3-C_1$ and hence φ induces an isomorphism $Z=S_{\mathrm{can}}/\langle g_3\rangle \cong S_\delta/\langle g_3\rangle=Z_\delta$ (see Example 1.2 for the notation). This proves Theorem 1.

We now prove Theorems 2 and 4.

Let Z be a rational log Enriques surface of type A_{18} and of index I. Let $\pi: S_{\mathrm{can}}\to Z$ be the canonical cover of Z and we denote by $\langle g\rangle \cong \boldsymbol{Z}/I\boldsymbol{Z}$ the Galois group of π. Let $\nu: S\to S_{\mathrm{can}}$ be the minimal resolution of the surface S_{can} and Δ the inverse by ν, of the singular point on S_{can} of Dynkin type A_{18}. Write $\Delta=\sum_{i=1}^{18} C_i$, where $C_i.C_{i+1}=1$ $(1\le i\le 17)$.

The following lemma can be proved similarly as in Lemmas 4.1 and 4.2.

LEMMA 4.3. (1) $I=3$.

(2) *S^g consists of exactly six curves C_2, C_5, C_8, C_{11}, C_{14}, C_{17} in Δ and nine isolated points.*

(3) *The pair $(S, \langle g\rangle)$ is isomorphic to the pair $(S_3, \langle g_3\rangle)$ in Example* 1.1.

(4) Sing(S_{can}) *consists of a single point, which is of Dynkin type A_{18}.*

In view of Lemma 4.3, we shall, from now on, identify $(S, \langle g\rangle)$ with $(S_3, \langle g_3\rangle)$.

By Proposition 3.1, we can find a smooth rational curve F on S_3 such that $\Delta+F$ has Dynkin type D_{19}. Let $S_3\to S'_{3,\mathrm{can}}$ be the contraction of $\Delta+F$. Then $\langle g_3\rangle$ acts on $S'_{3,\mathrm{can}}$ with no fixed curves and $S'_{3,\mathrm{can}}/\langle g\rangle$ is a rational log Enriques surface of type D_{19} and index 3 (cf. Lemmas 4.3 and 1.4). Thus by [OZ1, Theorem 1], $S'_{3,\mathrm{can}}/\langle g\rangle \cong Z_3$, $S'_{3,\mathrm{can}}\cong S_{3,\mathrm{can}}$ and there exists an automorphism φ of S_3 such that $\varphi(\Delta+F)=\Delta_3$ and $g_3\circ\varphi=\varphi\circ g_3$. This implies Theorem 4.

Clearly, $\varphi(\Delta)$ is equal to either Δ_3-C_{18} or Δ_3-C_{19}. Hence we get $Z=S_{\mathrm{can}}/\langle g_3\rangle \cong S_{\alpha_i}/\langle g_3\rangle=Z_{\alpha_i}$ for $i=1$ or $i=2$ (see Example 1.2 for the notation). Now Theorem 2 follows from Theorem 1.6.

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