Про незалежні функції (VI)

Г. Штейнгауз (Львів).

(Резюме)

Нехай Ex позначує „entier de x", нехай $[x] = x - Ex$; кажемо, що $f(t)$ $(0 < t < \infty)$ має властивість еквіпартиції (mod 1), якщо функція $[f(t)]$ має таку дистрибуанту як $[t]$. Іншими словами, для $0 \leqslant \lambda \leqslant 1$ є

$$\left| \underset{t}{E} \{[f(t)] < \lambda\} \right|_R = \lambda ;$$

при цьому через $|E|_R$ позначуємо релятивну міру множини E, як то вияснено в комунікаті (IV) [1]). Показується, що якщо $f(t)$ має властивість еквіпартиції, $[f]$ і $[g]$ є незалежні, а $[f+g]$ релятивно вимірні, то $f(t)+g(t)$ має також цю властивість (Теорема 3). Якщо $hf+kg$ має властивість еквіпартиції при довільних цілих h і k, то $[f(t)]$ і $[g(t)]$ є незалежні (Теорема 4). Як застосування одержуємо напр., що множина функцій $\{\sin 2\pi (t+a)^2\}_a$, одержана звідси, що a приймає всі дійсні значення, має парами незалежні елементи. Звідси доходимо до розв'язку певного питання Кампе де Феріє.

On extreme points of regular convex sets

by

M. KREIN and D. MILMAN (Odessa).

Let E be a Banach space (a linear normed complete space) and let $\bar{E}$ be the space of linear functionals adjoint to it.

A set $K \subset \bar{E}$ is called *regularly convex* [1]) if for every $f_0 \in \bar{E}$ not belonging to K such an element $x_0 \in E$ can be found that

$$\sup_{f \in K} f(x_0) < f_0(x_0).$$

It is obvious that every regularly convex set is convex.

Let $f_0 \in \bar{E}$, $x_i \in E$ $(|x_i| \leqslant 1,\ i = 1, 2, \ldots, n)$ and $\varepsilon > 0$; then by the neighbourhood $U(f_0;\ x_1, \ldots x_n,\ \varepsilon)$ we shall mean the set of all $f \in \bar{E}$ such that

$$|f_0(x_i) - f(x_i)| < \varepsilon \quad (i = 1, 2, \ldots, n).$$

All possible neighbourhoods $U(f_0;\ x_1, \ldots, x_n, \varepsilon)$, where $f_0 \in E$, $x_i \in \bar{E}$, $|x_i| \leqslant 1$ $(i = 1, 2, \ldots, n;\ \ n = 1, 2, \ldots,)$ and $\varepsilon > 0$, define in $\bar{E}$ a certain topology, which is called *weak topology* (*Tychonoff's topology*) [2]).

From Tychonoff's theorem on bicompactness of the topological product of segments, as it has been pointed out by Vera Gantmacher and V. Šmulyan [3]), results the following proposition:

A. *A bounded convex set $K \subset \bar{E}$ is regularly convex if and only if it is bicompact in the weak topology.*

---

[1]) This definition has been borrowed by us from the work of M. G. Krein and V. J. Šmulyan, On regularly closed sets etc. Annals of Mathematics 41 (1940).

[2]) A. Tychonoff, Über topologische Erweiterung von Raumen, Mathem. Annalen 102 (1929) 548.

[3]) V. Šmulyan, Sur les topologies différentes dans l'espace de Banach, Comptes Rendus de l'Acad. des Sc. de l'URSS, 23, 4 (1939).

A point of a convex closed set is called an *extreme point* if it is not an inner point of any segment belonging to the given set.

We now prove the theorem.

Theorem. *Let $K \subset \bar{E}$ be a bounded regularly convex set. Then the set S of extreme points of K is not empty and its regularly convex envelope* [4]) *coincides with K.*

Proof. According to proposition A, K is a bicompact set in the weak topology. To every element $x \in E$ corresponds a function $\varphi_x(f) = f(x)$ continuous on the bicompact set K.

Let $\{x_\alpha\}$ $(\alpha < \vartheta)$ be the set of all the elements of E with $|x| \leqslant 1$ well ordered in any way.

Correspondingly to the sequence $\{x_\alpha\}$ $(\alpha < \vartheta)$ we form a sequence of bicompact sets $\{K_\alpha\}$ $(\alpha < \vartheta)$, each one containing the following $(K_\alpha \supset K_\beta$ for $\alpha < \beta < \vartheta)$, by induction.

Define K_1 as a set of those elements $f \in K$ on which the function $\varphi_{x_1}(f)$ reaches its maximum. The set K_1 is closed in the weak topology, and consequently is bicompact. Now let all K_α be defined for $\alpha < \xi$ $(\xi < \vartheta)$. If ξ is not a limiting number, then we denote by K_ξ the set of those $f \in K_{\xi-1}$ on which the function $\varphi_{x_\xi}(f)$ considered on $K_{\xi-1}$ reaches its maximum.

If ξ is a limiting number, then we denote by K'_ξ the intersection of all K_α, with $\alpha < \xi$; since $K_1 \supset K_2 \supset \ldots K_\alpha \supset K_{\alpha+1} \supset \ldots$ are bicompact, so K'_ξ is non-empty. K_ξ will now denote the set of all the points of K'_ξ, on which the function $\varphi_{x_\xi}(f)$ $(f \in K'_\xi)$ reaches its maximum.

Denote by P the non-empty intersection of all K_α $(1 \leqslant \alpha < \vartheta)$. If $g, f \in P$ then $g, f \in K_\alpha$ and consequently

$$f(x_\alpha) = g(x_\alpha) \quad (1 \leqslant \alpha < \vartheta),$$

whence $g = f$. Thus P consists of one point g. We shall prove that this point g is an extreme point of the set K. We assume the contrary, i. e. that with some $f_1, f_2, \in K$ $(f_1 \neq f_2)$ and some t $(0 < t < 1)$

$$g = tf_1 + (1-t)f_2. \tag{1}$$

[4]) That is the smallest regularly convex set containing S.

Take the first K_ξ to which neither of the elements f_1, f_2 belong. Consider two cases. Let ξ not be a limiting number. Then $f_1, f_2 \in K_{\xi-1}$, and consequently

$$g(x_\xi) = \sup_{f \in K_{\xi-1}} f(x_\xi) \geqslant f_i(x_\xi) \qquad (i = 1, 2),$$

the equality sign being excluded at least for one of f_1, f_2 (namely for that f_i which is not included in K_ξ). Whence

$$g(x_\xi) = tg(x_\xi) + (1-t)\,g(x_\xi) > tf_1(x_\xi) + (1-t)f_2(x_\xi), \tag{2}$$

which contradicts to (1).

Let now ξ be a limiting number. Then $f_1, f_2 \in K_\alpha$ with $\alpha < \xi$, and consequently $f_1, f_2 \in K'_\xi$. Whence

$$g(x_\xi) = \sup_{f \in K'_\xi} f(x_\xi) \geqslant f_i(x_\xi) \qquad (i = 1, 2),$$

the equality sign being, as before, excluded, and consequently (2) holds, which contradicts to (1).

Thus we have proved that the point g is an extreme point of the set K, and consequently the set S is not empty.

We now prove that the regularly convex envelope K' of the set S coincides with K. It is evident that $K' \subset K$. Assuming that K' does not coincide with K, we take an element $f_0 \in K - K'$. Since K' is regularly convex, there exists an element $x_0 \in E$ $(|x_0| = 1)$ such that

$$\sup_{f \in K'} f(x_0) < f_0(x_0). \tag{3}$$

Consider then the set K_0 of those $f \in K$ on which the function $\varphi_{x_0}(f) = f(x_0)$ $(f \in K)$ reaches its maximum. Evidently the set $K_0 \subset K$ is in the weak topology a certain convex bicompact set, and consequently is a regularly convex set. Whence, in virtue of the facts already proved, K_0 has an extreme point g_0, which is an extreme point of K (for it is easily seen that every extreme point of the set K_0 is also an extreme point of the set K); on the other hand, in virtue of (3) and the definition of the set K_0, the intersection of K_0 with K', and consequently, with S is empty. We have come to a contradiction, which completes our proof.

Corollary. *If a space E is regular (reflective), then any bounded convex closed set is the convex closed envelope of the set of its extreme points.*

M. Krein and V. Šmulyan [5]) have proved that if $S < \bar{E}$ is a bounded set, then its regularly convex envelope consists of those and only those $g \in \bar{E}$ that admit the representation

$$g(x) = M\{\varphi_x(f)\} \quad (x \in E,\ \varphi_x(f) = f(x)),$$

where $M\{\varphi\}$ is a certain mean value defined on the space of all bounded and continuous in the weak topology functions $\varphi(f)$ $(f \in S)$.

As it has been shown by A. Markoff [6]), to every mean value $M\{\varphi\}$ corresponds in a unique way an additive non-negative function $\mu(e)$ of sets $e \subset S$ $(\mu(S) = 1)$ possessing a number of properties and such that

$$M\{\varphi\} = \int \varphi(f)\, d\mu(e),$$

where the integral is understood in the sense of Fréchet-Stieltjes [7]).

Owing to all this, our theorem permits us to say that every point of a regularly convex space is, in a certain sense, the centre of gravity of masses, distributed on the extreme points of this set.

Notice that the unit sphere $|f| \leqslant 1$ of the adjoint space is regularly convex and therefore if E is infinite-dimensional, then the sphere has an infinite set of extreme points. Hence:

If the unit sphere of an infinite-dimensional space E has a finite number of extreme points, then E is not adjoint to any Banach space.

We shall now give two examples to which this remark is applicable.

1. Let Q be a topological space and let C_Q be a linear set of all the bounded continuous functions $\varphi(q)$ $(q \in Q)$, with the definition of the norm:

$$\|\varphi\| = \sup_{q \in Q} |\varphi(q)|.$$

---

[5]) See their work quoted in footnote [1]).

[6]) A. Markoff, On mean values and exterior densities, Recueil Mathématique 4 (46), 1 (1938).

[7]) For more details see A. Markoff [6]) loc. cit.

It is easily seen that in this case the point $\varphi(q)$ of the unit sphere K $(\|\varphi\| \leqslant 1)$ of the space C_Q is an extreme point for K if and only if $|\varphi(q)| \equiv 1$ $(q \in Q)$. Therefore, if the space Q is decomposed on α components [8]) then K has exactly 2^α extreme points.

In virtue of this, if α is a finite number and C_Q is infinite-dimensional (the latter, for instance, is carried out if Q contains an infinite number of points and is completely regular) [9]), then E is not adjoint to any Banach space.

2. Let Q be an arbitrary abstract set, and let $\mu(e)$ be an additive function of the subsets $e \in Q$, forming a certain Borel-corpus B. Let the corpus B besides that possess in respect to $\mu(e)$ the following property: if $\mu(e) > 0$ for a set $e \in B$, then there exists a sub-division of e: $e = e_1 + e_2$ $(e_1, e_2 \in B)$ such that $\mu(e_1) > 0$ and $\mu(e_2) > 0$.

Denote by L_Q^μ a linear set of all the functions $\varphi(q)$ $(q \in Q)$ measurable and absolutely integrable in respect to the function $\mu(e)$, the norm φ being defined by the equality:

$$\|\varphi\| = \int_Q |\varphi(q)|\, d\mu(e).$$

It is easily seen that the unit sphere $\|\varphi\| \leqslant 1$ in the space L_Q^μ does not have extreme points and therefore L_Q^μ is not adjoint to any Banach space.

For the space (L) this result (in a more considerable general form) has been obtained by I. M. Gelfand [10]).

---

[8]) i. e. on disjointed closed connected parts.

[9]) Selim Krein has called our attention to the fact that in order that the space C_Q (with a finite α) should be infinite-dimensional, it is necessary and sufficient that Q should contain an infinite number of points and that the number of dimensions should be greater than α. The necessity of the conditions is obvious. To prove their sufficiency we show that if C_Q is finitely-dimensional, then the number of dimensions of C_Q exactly equals α. In fact, in this case the unit sphere K in C_Q contains exactly m (m is the number of dimensions of C_Q) linearly independent extreme points φ. But as we know, for every such point $\varphi(q) \equiv \pm 1$ and consequently there is exactly α of such linearly independent points, and accordingly $m = \alpha$.

[10]) I. Gelfand, Abstrakte Funktionen und lineare Operatoren, Recueil Mathématique 4 (46), 2 (1938) p. 265.

(Received 10-th 4. 1940).

Про екстремальні точки регулярно конвексних множин

М. Крейн і Д. Мільман (Одеса).

(Резюмо)

Нехай E є простір Банаха (тобто лінійний, нормований та повний простір) і $\bar{E}$ спряжений до нього простір лінійних функціоналів.

Множина $K \subset \bar{E}$ зветься регулярно конвексною[1]), якщо для кожного $f_0 \in \bar{E}$, не належного до множини K $(f \in K)$, знайдеться такий елемент $x \in E$, що

$$\sup_{f \in K} f(x_0) < f_0(x_0).$$

В цій статті встановлюється така

Теорема. *Якщо* $K \subset \bar{E}$ є обмежена регулярно конвексна множина, то множина S екстремальних точок K не є пуста і, більш того, наймєнша регулярно конвексна множина, що містить S, співпадає з K.

При цьому точка x даної конвексної множини C зветься екстремальною точкою C, якщо вона не є внутрішня точка жодного сегмента, що увіходить до C.

З теореми безпосередньо випливає

Висновок. *Якщо* простір E є регулярний, то кожна обмежена, конвексна, замкнена множина є конвексна замкнена оболонка множини своїх екстремальних точок.

Доведена теорема дозволяє вказати одну достатню ознаку того, щоб даний простір Банаха не був спряженим до жодного іншого простору Банаха.

Sur la divergence des séries orthogonales

par

S. BANACH (Léopol).

Introduction.

Soit $\mathfrak{F}$ l'ensemble formé par toutes les suites $\{\varphi_i(t)\}$ orthogonales et normées dans l'intervalle $(0,1)$. La distance de deux suites $\{\varphi_i(t)\}$, $\{\psi_i(t)\}$ appartenant à l'ensemble $\mathfrak{F}$ sera définie par

$$(\{\varphi_i(t)\}, \{\psi_i(t)\}) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\|\varphi_i(t) - \psi_i(t)\|}{1 + \|\varphi_i(t) - \psi_i(t)\|}$$

où $\|\varphi(t)\| = \left(\int_0^1 \varphi^2(t)\, dt\right)^{1/2}$. L'ensemble $\mathfrak{F}$ est alors un espace métrique, complet et séparable.

Dans ce Mémoire, nous démontrons les théorèmes suivants:

Théorème I. *L'ensemble P des suites complètes $\{\varphi_n(t)\} \varepsilon \mathfrak{F}$ est un ensemble G_δ partout de la seconde catégorie dans $\mathfrak{F}$.*

Par conséquent, l'ensemble des suites incomplètes est un ensemble F_σ de la première catégorie.

Théorème II. *Si $\{c_i\}$ est une suite numérique donnée, telle que $\sum c_i^2 < \infty$, alors deux cas seulement sont possibles:*

1) la série $\sum c_i \varphi_i(t)$ est presque partout convergente pour chaque suite $\{\varphi_i(t)\} \varepsilon \mathfrak{F}$;

2) l'ensemble Q des suites $\{\varphi_i(t)\} \varepsilon \mathfrak{F}$ pour chacune des lesquelles on a presque partout

$$\overline{\lim_{n \to \infty}} \left| \sum_{i=1}^{n} c_i \varphi_i(t) \right| = +\infty$$

est un ensemble G_δ partout de la seconde catégorie dans $\mathfrak{F}$.