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ON EXTREME VALUES IN STATIONARY SEQUENCES

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13. ABSTRACT Extreme value theory is considered for stationary sequences $\{\xi_n\}$ satisfying dependence restrictions significantly weaker than strong mixing. In particular the basic theorem of Gnedenko (developed later by Loynes for mixing sequences) is proved under the weak restrictions. The conditions for the general results are shown to apply to stationary normal sequences under the very weak covariance assumptions used previously (e.g. by S.M. Berman). Distributional limit theorems for other order statistics are also obtained. KEY WORDS: Extreme value theory; stochastic processes, dependent extremal theory.			

On Extreme Values in Stationary Sequences

by

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Summary

In this paper, extreme value theory is considered for stationary sequences $\{\xi_n\}$ satisfying dependence restrictions significantly weaker than strong mixing. The aims of the paper are:

(i) To prove the basic theorem of Gnedenko concerning the existence of three possible non-degenerate asymptotic forms for the distribution of the maximum $M_n = \max(\xi_1 \dots \xi_n)$, for such sequences.

(ii) To obtain limiting laws of the form

$$\lim_{n \rightarrow \infty} \Pr\{M_n^{(r)} \leq u_n\} = e^{-\tau} \sum_{s=0}^{r-1} \tau^s / s! \quad \text{where } M_n^{(r)}$$

is the r^{th} largest of $\xi_1 \dots \xi_n$, and

$\Pr\{\xi_1 > u_n\} \sim \tau/n$. Poisson properties (akin to those known for the upcrossings of a high level by a stationary normal process) are developed and used to obtain these results.

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- (iii) As a consequence of (ii), to show that the asymptotic distribution of $M_n^{(r)}$ (normalized) is the same as if the $\{\xi_n\}$ were i.i.d.
- (iv) To show that the assumptions used are satisfied, in particular by stationary normal sequences, under mild covariance conditions.

1. Introduction and basic framework.

There is a considerable body of theory concerning extreme values of independent and identically distributed (i.i.d) random variables (e.g. [5],[7]). Important sections of this theory have been extended to apply to certain types of stationary sequences. For example Watson ([9]) considered "m-dependent" sequences, and Loynes ([8]) (and subsequently others, e.g. [10],[4]) generalized a number of results to apply under the condition of "strong (uniform) mixing". In particular Loynes obtained a generalization of the following theorem of Gnedenko ([5]), which is central to the asymptotic theory in the i.i.d. case. (We write, here and throughout, M_n for the maximum of the first n random variables $\xi_1 \dots \xi_n$, of whatever sequence is being considered.

Theorem 1.1 (Gnedenko) : If $\xi_1, \xi_2 \dots$ are i.i.d. random variables, and if for some sequences $a_n > 0, b_n$ of

constants, the normalized maximum $a_n(M_n - b_n)$ has a non-degenerate limiting distribution function (d.f.)

$G(x)$, then $G(x)$ has one of the three following forms:

$$\text{Type I} : G(x) = \exp(-e^{-x}) \quad -\infty < x < \infty$$

$$\begin{aligned} \text{Type II} : G(x) &= 0 & x \leq 0 \\ &= \exp(-x^{-\alpha}) & (\text{for some } \alpha > 0) \quad x > 0 \end{aligned}$$

$$\begin{aligned} \text{Type III} : G(x) &= \exp(-(-x)^\alpha) & (\text{for some } \alpha > 0) \quad x \leq 0 \\ &= 1 & x > 0 \end{aligned}$$

(It is understood that each "type" of d.f. includes all d.f.'s obtainable by replacing x by $ax + b$ for any $a > 0, b$).

The strong mixing condition used by Loynes requires that

$$(1.1) \quad |P(A_n B) - P(A)P(B)| \leq g(k)$$

whenever A, B are events belonging respectively to the σ -fields generated by $\xi_1 \dots \xi_m$, and $\xi_{m+k}, \xi_{m+k+1} \dots$ for some m , and where $g(k) \rightarrow 0$ as $k \rightarrow \infty$. This is a "decay of dependence" condition of a very strong type, since it must hold (uniformly) for all such events A, B . Its verification may also be difficult since all pairs A, B must be "tested".

It is apparent that not all events A, B will be of interest in extremal theory. Indeed the event

" $\{M_n \leq u\}$ " is precisely " $\{\xi_1 \leq u, \xi_2 \leq u, \dots, \xi_n \leq u\}$ ", whose probability is just a value of a finite-dimensional d.f. of the process. Hence it is natural to look for dependence restrictions involving only these d.f.'s. An obvious such restriction (of similar but weakened form of (1.1)) would be the following, which we will refer to as "Condition D". If $F_{i_1 \dots i_r}(x_1 \dots x_r)$ denotes the joint d.f. of $\xi_{i_1} \dots \xi_{i_r}$, we shall write $F_{i_1 \dots i_r}(u)$ to denote $F_{i_1 \dots i_r}(u, u, \dots, u)$. Then the sequence $\{\xi_n\}$ will be said to satisfy the condition D if for any $i_1 < i_2 \dots < i_n < j_1 < j_2 \dots < j_m$, $j_1 - i_n \geq k$, and any real u ,

$$(1.2) \quad (D) : |F_{i_1 \dots i_n j_1 \dots j_m}(u) - F_{i_1 \dots i_n}(u) F_{j_1 \dots j_m}(u)| \leq g(k)$$

where $g(k) \rightarrow 0$ as $k \rightarrow \infty$.

While Gnedenko's Theorem will hold (as we shall see) under Condition D, we can - for this and later purposes - weaken it significantly by considering just certain sequences of u -values in (1.2). Specifically if $\{u_n\}$ is a sequence of real numbers, we shall say that the (stationary) sequence $\{\xi_n\}$ satisfies the condition $D(u_n)$ if for any integers

$$1 \leq i_1 < i_2 \dots < i_p < j_1 < \dots < j_q \leq n, \quad j_1 - i_p \geq k \quad \text{we have}$$

$$(1.3) (D(u_n)) : |F_{i_1 \dots i_p j_1 \dots j_q}(u_n) - F_{i_1 \dots i_p}(u_n) F_{j_1 \dots j_q}(u_n)| \leq \alpha_{n,l}$$

where $\lim_{l \rightarrow \infty} \lim_{n \rightarrow \infty} \alpha_{n,l} = 0$.

It is apparent that D implies $D(u_n)$ for any sequence u_n . However for some processes (e.g. normal), $D(u_n)$ may be readily verified for sequences $\{u_n\}$ of interest, when it is not clear whether D itself holds. (In this, it turns out that $D(u_n)$ can be further "encouraged to hold" by an appropriate decrease in tail probabilities of the finite dimensional d.f.'s, over the pure "dependence decay" required for D).

In Section 2 we shall obtain Gnedenko's Theorem if the assumption that the ξ_i are i.i.d. is replaced by stationarity together with Condition $D(u_n)$ for all u_n of the form $x/a_n + b_n$ ($-\infty < x < \infty$), a_n, b_n being the particular sequence given in the theorem. (This, of course, implies that the result holds for stationary sequences satisfying D). The proof will follow the pattern of that used by Loynes in [8], with modifications to suit later applications.

Another useful (even though trivially proved) result for i.i.d. random variables is that

$$(1.4) \quad \Pr\{M_n \leq u_n\} \rightarrow e^{-\tau} \text{ as } n \rightarrow \infty$$

if $u_n = u_n(\tau)$ is chosen so that

$$(1.5) \quad \Pr\{\xi_1 > u_n\} \sim \tau/n,$$

τ being any fixed constant ($\tau \geq 0$). It is easily seen (by writing the left hand side of (1.4) as $[1 - (1 - F_1(u_n))]^n$ and taking logarithms, that (1.5) is certainly necessary for (1.4) to hold. If F_1 is discontinuous, u_n is often chosen so that $F_1(u_n^-) \leq 1 - \tau/n \leq F_1(u_n)$. However, it should be pointed out that this does not itself guarantee that (1.5) holds. (This may be seen by taking F_1 to increase only by jumps, at $1, 2, 3, \dots$, with $\Pr\{\xi_1 \leq j\} = 1 - \tau/(2^j - 1)$.) Hence, in such a case, (1.5) must be separately verified before (1.4) can be asserted.

Watson ([9]) gave conditions under which (1.4) holds for m -dependent stationary sequences, and Loynes ([8]) showed that here also, m -dependence may be replaced by strong mixing. We show in Section 3 that strong mixing may be replaced by $D(u_n)$ for the particular sequence $u_n = u_n(\tau)$ involved. The limit in (1.4) is the same as would apply if the ξ_n were i.i.d. (with the same marginal d.f. F_1 , as when dependent.) As a corollary it will follow that - under appropriate conditions -

$a_n(M_n - b_n)$ has the same asymptotic distribution as it would if the ξ_n were i.i.d. with d.f. F_1 . This is also a result shown in [8] under strong mixing.

It is well known that the Type I extreme value d.f. is the one which applies to i.i.d. normal sequences. Berman ([1]) has shown that this remains true for stationary normal sequences, under very weak conditions indeed on the covariance function (which, of course, is the most convenient type of assumption in practice, for a normal process). Specifically Berman showed that, provided the covariance sequence $\{r_n\}$ of the (zero mean, unit variance) stationary normal sequence $\{\xi_n\}$ satisfies either

$$(1.6) \quad (i) \ r_n \log n \rightarrow 0 \text{ as } n \rightarrow \infty \text{ or } (ii) \ \sum_1^{\infty} r_n^2 < \infty$$

then

$$(1.7) \quad \Pr\{a_n(M_n - b_n) \leq x\} \rightarrow \exp(-e^{-x}) \text{ as } n \rightarrow \infty$$

where

$$(1.8) \quad \begin{aligned} a_n &= (2 \log n)^{\frac{1}{2}} \\ b_n &= (2 \log n)^{\frac{1}{2}} - \frac{1}{2}(2 \log n)^{-\frac{1}{2}}[\log \log n + \log 4\pi] . \end{aligned}$$

It follows from Berman's proof that (1.4) holds (and (1.7) may then be obtained by simple transformation of (1.4)). We shall show in Section 4 that either of the conditions (1.6) (for the normal sequence) implies the general sufficient

conditions for (1.4) (including $D(u_n)$) given in Theorem 3.1. Thus for normal sequences (1.4) alternatively follows from the general result Theorem 3.1. This does not reduce the amount of calculation, but it does demonstrate the connections between arguments previously used for normal processes under covariance conditions, and those using assumptions of "mixing type" for other processes. It also indicates the satisfactory nature of Theorem 3.1, since the particular normal result is known to be a very sharp one.

It is known that for continuous parameter, stationary normal processes, the upcrossings of a high level are asymptotically Poisson in character, under quite weak conditions ([2]). Similar results are (in fact more) easily proved for normal sequences. In Section 5 we obtain an asymptotic Poisson result of this kind for (not necessarily normal) stationary sequences under assumptions which include $D(u_n)$. This result will enable us to obtain the asymptotic distribution of the r^{th} largest value $M_n^{(r)}$ say among $\xi_1 \dots \xi_n$, as well as the maximum $M_n = M_n^{(1)}$, as $n \rightarrow \infty$. It is clearly also possible to consider joint distributions of the $M_n^{(r)}$ along these lines, (cf[10]) but we do not pursue this matter here.

Finally we note that the present paper is concerned entirely with sequences. We hope, in a future paper, to consider corresponding properties in the (more complicated) continuous parameter case.

2. Basic lemmas, and Gnedenko's Theorem under $D(u_n)$.

The proof of Theorem 1.1 given in [5] may be displayed as the two following results.

Lemma 2.1 Suppose that M_n are random variables (which in this lemma may be maxima or not) and $a_n > 0$, b_n constants such that

(2.1) $\Pr\{a_n(M_n - b_n) \leq x\} \rightarrow G(x)$ (convergence in distribution) as $n \rightarrow \infty$, where G is a non-degenerate d.f., and further, for each $k = 1, 2, \dots$

(2.2) $\Pr\{a_{nk}(M_n - b_{nk}) \leq x\} \rightarrow [G(x)]^{1/k}$ as $n \rightarrow \infty$.

Then, corresponding to each $k = 1, 2, \dots$, there are constants $\alpha_k > 0$, β_k such that

(2.3) $G^k(\alpha_k x + \beta_k) = G(x)$.

This lemma follows simply from a result of Khintchine, which may be conveniently found in [6 Section 10, Theorem 1].

Lemma 2.2 Let G be a non-degenerate d.f. such that (2.3) holds for each $k = 1, 2, \dots$ (for some constants $\alpha_k > 0, \beta_k$). Then G is one of the three extreme value d.f.'s, being of Type I, II or III according as $\alpha_k = 1$ for some (and hence all) k , $\alpha_k > 1$ for some (all) k , or $\alpha_k < 1$ for some (all) k .

The derivation of this latter result constitutes the major part of Gnedenko's proof. It is easily seen that if $M_n = \max(\xi_1 \dots \xi_n)$ where the ξ_j are i.i.d., then if (2.1) holds, so does (2.2) and hence (2.3). Thus the conclusion of Theorem 1.1 follows for the i.i.d. case. Our task here will be to show that (2.1) implies (2.2) if $\{\xi_n\}$ is a stationary sequence satisfying $D(u_n)$ for $u_n = x/a_n + b_n$, (for all real x). This will be done by means of several lemmas, and will enable us to obtain the desired generalization of Theorem 1.1.

First, write $M(E) = \max\{\xi_j : j \in E\}$ for any set E of integers. It will be convenient to talk of an "interval" to mean any finite set E of consecutive integers (j_1, j_1+1, \dots, j_2) say. We shall then say that E has "length" $j_2 - j_1 + 1$. If $F = (k_1, k_1+1, \dots, k_2)$ with $k_1 > j_2$, we shall say that E and F are separated by $k_1 - j_2$. The following result will have several

applications :

Lemma 2.3 Suppose that $D(u_n)$ holds for some sequence (u_n) . Let N, r, k be fixed integers and E_1, E_2, \dots, E_r subintervals of $(1, 2, \dots, N)$, such that E_i and E_j are separated by at least k when $i \neq j$. Then

$$\left| P\left\{ \bigcap_{j=1}^r (M(E_j) \leq u_N) \right\} - \prod_{j=1}^r \Pr\{M(E_j) \leq u_N\} \right| \leq (r-1)\alpha_{N,k}$$

Proof: Let $E_j = (k_j, k_j+1, \dots, \ell_j)$ where (by renumbering if necessary) $k_1 \leq \ell_1 < k_2 \leq \ell_2 \dots$. For brevity write $A_j = \{M(E_j) \leq u_N\}$. Then

$$|P(A_1 \cap A_2) - P(A_1)P(A_2)| =$$

$$\left| F_{k_1, k_1+1, \dots, \ell_1, k_2, \dots, \ell_2}(u_N) - F_{k_1, \dots, \ell_1}(u_N)F_{k_2, \dots, \ell_2}(u_N) \right| \leq \alpha_{N,k}$$

since $k_2 - \ell_1 \geq k$. Similarly

$$\begin{aligned} |P(A_1 \cap A_2 \cap A_3) - P(A_1)P(A_2)P(A_3)| &\leq |P(A_1 \cap A_2 \cap A_3) - P(A_1 \cap A_2)P(A_3)| \\ &\quad + |P(A_1 \cap A_2) - P(A_1)P(A_2)| P(A_3) \\ &\leq 2\alpha_{N,k} \end{aligned}$$

since $E_1 \cup E_2 \subset (k_1, k_1+1, \dots, \ell_2)$ and $k_3 - \ell_2 \geq k$.

Proceeding in this way, we obtain the result.

Now let k be a fixed positive integer, and write $N = nk$, $n = 1, 2, \dots$. In the following we shall approximate $\Pr\{M_N \leq u_N\}$ by $[\Pr\{M_n \leq u_N\}]^k$ (which might be expected intuitively), when $D(u_n)$ holds. This will enable us to obtain various results (including the extended Gnedenko Theorem) quite simply. The method used follows that of Loynes ([8]) in many of its essential features. Specifically, we divide the first $N = nk$ integers into $2k$ consecutive intervals as follows. Let m be a fixed integer and write $I_1 = (1, 2, \dots, n-m)$, $I_1^* = (n-m+1, \dots, n)$, $I_2 = (n+1, \dots, 2n-m)$, $I_2^* = (2n-m+1, \dots, 2n)$, and so on. Thus $I_1, I_1^*, I_2, I_2^*, \dots, I_k, I_k^*$ alternately have length $(n-m) (\rightarrow \infty)$ and m (fixed). The main steps of the approximation are displayed in the following lemma.

Lemma 2.4 With the above notation, and assuming $D(u_n)$ holds,

- (i) $0 \leq P\{\bigcap_{j=1}^k (M(I_j) \leq u_N)\} - \Pr\{M_N \leq u_N\} \leq k \Pr\{M(I_1) \leq u_N < M(I_1^*)\}$
- (ii) $|P\{\bigcap_{j=1}^k (M(I_j) \leq u_N)\} - P^k\{M(I_1) \leq u_N\}| \leq k \alpha_{N,m}$
- (iii) $|P^k\{M(I_1) \leq u_N\} - P^k\{M_n \leq u_N\}| \leq K \Pr\{M(I_1) \leq u_N < M(I_1^*)\}$

for some constant K . Hence, by combining (i), (ii), (iii),

$$(2.4) \quad |\Pr\{M_N \leq u_N\} - p^k\{M_n \leq u_N\}|$$

$$\leq (k+K)\Pr\{M(I_1) \leq u_N \leq M(I_1^*)\} + k\alpha_{N,m}$$

Proof: (i) follows at once since $\bigcap_{j=1}^k (M(I_j) \leq u_N) \supseteq (M_N \leq u_N)$, and their difference implies $M(I_j) \leq u_N < M(I_j^*)$ for some j , the probabilities of these latter events being independent of j by stationarity.

(ii) follows from Lemma 2.3 with I_j for E_j , noting that $\Pr\{M(I_j) \leq u_N\}$ is independent of j .

To obtain (iii) we note that

$$0 \leq \Pr\{M(I_1) \leq u_N\} - \Pr\{M_n \leq u_N\} = \Pr\{M(I_1) \leq u_N < M(I_1^*)\}.$$

The result then follows (writing $y = \Pr\{M(I_1) \leq u_N\}$, $x = \Pr\{M_n \leq u_N\}$) from the obvious inequalities

$$0 \leq y^k - x^k \leq K(y-x) \quad \text{for some } K > 0, \quad 0 \leq x, y \leq 1.$$

We now dominate the right hand side of (2.4), to obtain the desired approximation.

Lemma 2.5 If $D(u_n)$ holds, $r \geq 1$ is any integer, and if n is sufficiently large, then

$$(2.5) \quad \Pr\{M(I_1) \leq u_N < M(I_1^*)\} \leq \frac{1}{r} + 2r \alpha_{N,m}.$$

It then follows from Lemma 2.4 that

$$(2.6) \quad \Pr\{M_N \leq u_N\} - p^k \{M_n \leq u_N\} \rightarrow 0 \quad \text{as } n \rightarrow \infty .$$

Proof : If n is sufficiently large, we may choose r intervals $E_1 \dots E_r$, each of length m from $1, 2 \dots n-m$, so that they are separated from each other and from I_1^* , by at least m . Then

$$\begin{aligned} \Pr\{M(I_1) \leq u_N < M(I_1^*)\} &\leq \Pr\left\{\bigcap_{s=1}^r (M(E_s) \leq u_N), M(I_1^*) > u_N\right\} \\ &= P\left\{\bigcap_{s=1}^r (M(E_s) \leq u_N)\right\} - P\left\{\bigcap_{s=1}^r (M(E_s) \leq u_N), M(I_1^*) \leq u_N\right\} . \end{aligned}$$

By stationarity, $\Pr\{M(E_s) \leq u_N\} = \Pr\{M(I_1^*) \leq u_N\} = p$, say, and by Lemma 2.3, the two terms on the right differ from p^r , p^{r+1} (in absolute magnitude) by no more than $(r-1) \alpha_{N,m}$, $r \alpha_{N,m}$ respectively. Hence

$$\Pr\{M(I_1) \leq u_N < M(I_1^*)\} \leq p^r - p^{r+1} + 2r \alpha_{N,m} ,$$

from which (2.5) follows since $p^r - p^{r+1} \leq 1/(r+1)$.

Finally by (2.4) and (2.5),

$$\limsup_{n \rightarrow \infty} |\Pr\{M_N \leq u_N\} - p^k \{M_n \leq u_N\}| \leq \frac{k+K}{r} + [k+2r(k+K)] \limsup_{n \rightarrow \infty} \alpha_{N,m}$$

from which it follows (by letting $m \rightarrow \infty$ and then $r \rightarrow \infty$ on the right), that the left hand side is zero. Thus (2.6) is proved.

The Gnedenko Theorem now follows easily under general conditions.

Theorem 2.1 Let ξ_n be a stationary sequence and $a_n > 0$, b_n , given constants such that $\Pr\{a_n(M_n - b_n) \leq x\}$ converges in distribution to a non-degenerate d.f. $G(x)$. Suppose that $D(u_n)$ is satisfied for $u_n = \frac{x}{a_n} + b_n$, for each real x . Then $G(x)$ has one of the three extreme value forms listed in Theorem 1.1.

Proof: By Lemma 2.5, with $u_n = x/a_n + b_n$ we have

$$P^k\{a_{nk}(M_n - b_{nk}) \leq x\} = P^k\{M_n \leq u_n\} = \Pr\{M_N \leq u_N\} + o(1)$$

by (2.6), and this may be rewritten as $\Pr\{a_N(M_N - b_N) \leq x\} + o(1)$ which converges to $G(x)$ by assumption. Thus (2.2) holds (as well as (2.1)) and the conclusion follows as in the i.i.d. case from Lemmas 2.1 and 2.2.

Corollary : The result remains true if the condition that $D(u_n)$ be satisfied for each $u_n = x/a_n + b_n$, is replaced by the requirement that the condition D holds. (For then $D(u_n)$ is satisfied by any sequence at all, and in particular by $u_n = x/a_n + b_n$ for each x).

Finally we remark that the condition $D(u_n)$ may be changed in various ways and still serve the present purposes. Our choice of $D(u_n)$ is just one of a number of possible ones, but is a choice which seemed convenient and natural for our purposes.

3. Convergence of $\Pr\{M_n \leq u_n\}$ and its consequences.

In this section we first consider conditions under which (1.4) holds (i.e. $\Pr\{M_n \leq u_n\} \rightarrow e^{-\tau}$) for a stationary sequence $\{\xi_n\}$ where the $\{u_n\}$ are assumed to satisfy

(3.1) $1 - F_1(u_n) = \Pr\{\xi_1 > u_n\} = \tau/n + o(1/n)$ as $n \rightarrow \infty$, where $\tau \geq 0$. We can then show as a corollary that $a_n(M_n - b_n)$ has the same limiting distribution as it would if the ξ_n were i.i.d.

As may be seen from the derivation below, if (3.1) holds, Condition $D(u_n)$ is then sufficient to guarantee that $\liminf \Pr\{M_n \leq u_n\} \geq e^{-\tau}$. However we need a further assumption to obtain the opposite inequality for the upper limit. Various forms of such an assumption may be used. Here we content ourselves with the following simple variant of conditions used in [9],[8]. (We refer to this as $D'(u_n)$):

Condition $D'(u_n)$ will be said to hold for the sequence (u_n) if

$$(3.2) (D'(u_n)) : \limsup_{n \rightarrow \infty} \left[n \sum_{j=2}^n \Pr\{\xi_1 > u_{nk}, \xi_j > u_{nk}\} \right] = o(1/k) \text{ as } k \rightarrow \infty.$$

Note that if (3.1) holds, then a sufficient condition for (3.2) to hold is

$$(3.3) \quad n \sum_{j=2}^n |\Pr\{\xi_1 > u_n, \xi_j > u_n\} - \Pr\{\xi_1 > u_n\}\Pr\{\xi_j > u_n\}| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

For if (3.3) holds it is readily checked that

$$\begin{aligned} n \sum_{j=2}^n \Pr\{\xi_1 > u_{nk}, \xi_j > u_{nk}\} &\leq n \sum_{j=1}^n P^2\{\xi_1 > u_{nk}\} \\ &\quad + N \sum_{j=2}^N |\Pr\{\xi_1 > u_N, \xi_j > u_N\} - P^2\{\xi_1 > u_N\}| \end{aligned}$$

(with $N = nk$). The second term on the right tends to zero as $n \rightarrow \infty$, and the first to $\tau^2/k^2 = o(1/k)$, yielding (3.2).

Theorem 3.1 Suppose that $D(u_n), D'(u_n)$ hold

(i.e. (1.3), (3.2)), for the stationary sequence $\{\xi_n\}$,

where the u_n satisfy (3.1). Then $\Pr\{M_n \leq u_n\} \rightarrow e^{-\tau}$ as $n \rightarrow \infty$.

Proof: Fix a positive integer k . We use (2.6) to consider first convergence of $\Pr\{M_N \leq u_N\}$ where $N = nk$, $n = 1, 2, \dots$. For this we note that

$$\Pr\{M_n \leq u_N\} = 1 - P\left\{\bigcup_{j=1}^n (\xi_j > u_N)\right\} \geq 1 - n \Pr\{\xi_1 > u_N\}$$

and hence by (3.1),

$$(3.4) \quad \liminf_{n \rightarrow \infty} \Pr\{M_n \leq u_N\} \geq 1 - \tau/k.$$

Corresponding we also have

$$(3.5) \Pr\{M_n \leq u_N\} \leq 1 - n \Pr\{\xi_1 > u_N\} + \sum_{1 \leq i < j \leq n} \Pr\{\xi_i > u_N, \xi_j > u_N\}$$

$$\leq 1 - n \Pr\{\xi_1 > u_N\} + n \sum_{j=2}^n \Pr\{\xi_1 > u_N, \xi_j > u_N\}$$

by stationarity. It follows from (3.2) (and the fact that $n \Pr\{\xi_1 > u_N\} \rightarrow \tau/k$) that

$$\limsup_{n \rightarrow \infty} \Pr\{M_n \leq u_N\} \leq 1 - \tau/k + o(1/k),$$

and hence by

(3.4) and (2.6) that

$$(3.6) \left(1 - \frac{\tau}{k}\right)^k \leq \liminf_{n \rightarrow \infty} \Pr\{M_N \leq u_N\} \leq \limsup_{n \rightarrow \infty} \Pr\{M_N \leq u_N\}$$

$$\leq \left[1 - \frac{\tau}{k} + o\left(\frac{1}{k}\right)\right]^k$$

To complete the proof we show that $N (= nk)$ may be replaced by n in (3.6) and the result then clearly follows by letting $k \rightarrow \infty$. Choose r , depending on n , so that $rk \leq n < (r+1)k$. It is easily checked (since $M_n \leq M_{(r+1)k}$) that

$$\Pr\{M_n \leq u_n\} \geq \Pr\{M_{(r+1)k} \leq u_{(r+1)k}\} - \Pr\{u_n < M_{(r+1)k} \leq u_{(r+1)k}\}.$$

Now $\Pr\{u_n < M_{(r+1)k} \leq u_{(r+1)k}\}$ (which is zero if $u_n > u_{(r+1)k}$) is dominated by

$$\begin{aligned}
P\left\{\bigcup_{j=1}^{(r+1)k} (u_n < \xi_j \leq u_{(r+1)k})\right\} &\leq (r+1)k \Pr\{u_n < \xi_1 \leq u_{(r+1)k}\} \\
&\leq (r+1)k |F_1(u_{(r+1)k}) - F_1(u_n)| \\
&= (r+1)k \left| \frac{\tau}{n} - \frac{\tau}{(r+1)k} + o\left(\frac{1}{n}\right) \right|
\end{aligned}$$

which is easily seen to tend to zero as $n \rightarrow \infty$, by the choice of r . Thus

$$\liminf \Pr\{M_n \leq u_n\} \geq \liminf \Pr\{M_{(r+1)k} \leq u_{(r+1)k}\} \geq (1 - \frac{\tau}{k})^k$$

by the first inequality of (3.6) (with $(r+1)k$ for N).

Similarly,

$$\Pr\{M_n \leq u_n\} \leq \Pr\{M_{rk} \leq u_{rk}\} + \Pr\{u_{rk} < M_{rk} \leq u_n\}$$

from which it follows that $\limsup \Pr\{M_n \leq u_n\} \leq [1 - \frac{\tau}{k} + o(\frac{1}{k})]^k$, as desired.

We now write \hat{M}_n for the maximum of n i.i.d. random variables with the same marginal d.f. F_1 as each ξ_n ; (following [8] we may call these the "independent sequence associated with $\{\xi_n\}$ "). The following result is then a corollary of the theorem.

Corollary : Suppose that for some sequence

$$\{u_n\}, \Pr\{\hat{M}_n \leq u_n\} \rightarrow \theta > 0.$$

If $D(u_n)$ and $D'(u_n)$ are satisfied, then also

$$\Pr\{M_n \leq u_n\} \rightarrow \theta .$$

Proof : The condition $\Pr\{\hat{M}_n \leq u_n\} \rightarrow \theta$ may be rewritten as

$$F_1^n(u_n) \rightarrow \theta \text{ or } n \log[1 - (1 - F_1(u_n))] \rightarrow \log \theta , \text{ which}$$

implies $n(1 - F_1(u_n)) \rightarrow -\log \theta$. Thus (3.1) holds with $\tau = -\log \theta$, and hence the result follows.

We may also deduce at once that the limiting distribution of $a_n(M_n - b_n)$ is the same as that which would apply if the ξ_i were i.i.d. , i.e. of $a_n(\hat{M}_n - b_n)$, under conditions $D(u_n)$ and $D'(u_n)$. This result was proved in [8] under conditions including strong mixing,

Theorem 3.2 Suppose that with the above notation,

$\Pr\{a_n(\hat{M}_n - b_n) \leq x\} \rightarrow G(x)$ for some constants $a_n > 0, b_n$, and some (Type I, II or III) d.f. G . Suppose also that $D(u_n)$, $(D'(u_n))$ are satisfied when $u_n = \frac{x}{a_n} + b_n$, for each x . Then $\Pr\{a_n(M_n - b_n) \leq x\} \rightarrow G(x)$.

Proof : Since $\Pr\{a_n(M_n - b_n) \leq x\} = \Pr\{M_n \leq u_n\}$, the result follows from the previous corollary if $G(x) > 0$.

If $G(x) = 0$ we have, for any $y > x$ with $G(y) > 0$,

$$\Pr\{a_n(M_n - b_n) \leq x\} \leq \Pr\{a_n(M_n - b_n) \leq y\} + G(y).$$

Since G is continuous, we may take $G(y)$ arbitrarily small, and the result easily follows.

4. Stationary normal sequences.

Suppose now that $\{\xi_n\}$ is a stationary normal sequence, with zero means, unit variances, and covariances $r_p = E \xi_n \xi_{n+p}$, where $r_p \rightarrow 0$ as $p \rightarrow \infty$. As noted in Section 1, Berman ([1]) has shown that if either $r_n \log n \rightarrow 0$ or $\sum r_n^2 < \infty$, then the Type I asymptotic law (1.7) holds, with the constants given by (1.8). We propose to show how this result, and the technique for obtaining it, fit in with the theory of the previous section. We shall show in particular, that either of the above covariance conditions implies our conditions $D(u_n), D'(u_n)$, for appropriately chosen $\{u_n\}$ (satisfying (3.1)). It has, incidentally, been pointed out in [4] that $\sum r_n^2 < \infty$ for any strongly mixing stationary normal sequence. Thus strong mixing implies $\sum r_n^2 < \infty$, which in turn implies our assumptions $D(u_n), D'(u_n)$, in this normal case.

Define, now, u_n by $1 - \phi(u_n) = \tau/n$, where ϕ

denotes the standard normal d.f. (with density $\phi(x)$).

If we can show that (1.4) holds, it will (as previously noted) follow that so does (1.7). The steps involved in this are :

(i) Write $\tau = e^{-x}$, and use the fact that $1 - \Phi(u_n) \sim \phi(u_n)/u_n$ to show that

$$(4.1) \quad u_n^2/2 = x + \log n - \frac{1}{2} \log 2\pi - \log u_n + o(1)$$

(ii) Deduce that $u_n^2/(2 \log n) \rightarrow 1$ and hence by taking logarithms and combining with (4.1), that

$$u_n = (2 \log n)^{\frac{1}{2}} [1 + (x - \frac{1}{2} \log 4\pi - \frac{1}{2} \log \log n) / (2 \log n) + o(1/\log n)].$$

Insertion of this into (1.4) and rearrangement yields (1.7).

Thus we confine attention to (1.4). First we state a useful technical lemma, proved in [1].

Lemma 4.1 Suppose that either $r_n \log n \rightarrow 0$ or $\sum_{n=1}^{\infty} r_n^2 < \infty$.

Then with u_n as defined above, we have

$$(4.2) \quad n \sum_{j=1}^n |r_j| \exp\{-u_n^2/(1+|r_j|)\} \rightarrow 0 \text{ as } n \rightarrow \infty$$

(It is perhaps worth noting that the condition $\sum r_n^2 < \infty$ may be replaced by $\sum |r_n|^p < \infty$ for some $p > 0$, in this lemma).

In order to obtain (1.4) it is simplest to show (as in [1]) that $\Pr\{M_n \leq u_n\}$ can be approximated by $[\Phi(u_n)]^n$. Practically the same calculations are involved in verifying $D(u_n)$ and $D'(u_n)$, and hence we do this in order to show the relationship between normal sequences, and those of the previous sections. These will follow from the lemma below (which is virtually the same as Lemma 3.1 of [1], from which the details of proof may be obtained by slight changes. The technique of proof develops an argument originally due to Slepian, and which has been extended in many ways by a number of authors (especially S.M. Berman), to the extent that it may now be regarded as part of the basic machinery of the subject.

Lemma 4.2 Let ξ_1, ξ_2, \dots be the stationary normal sequence as defined at the start of this section, and let $1 \leq \ell_1 \leq \ell_2 \leq \dots \leq \ell_s$. Then, for any u ,

$$(4.3) \left| \Pr\{\xi_{\ell_j} \leq u \text{ for } j = 1, 2, \dots, s\} - \Phi^s(u) \right| \\ \leq K \sum_{1 \leq i < j \leq s} |\rho_{ij}| e^{-u^2/(1+|\rho_{ij}|)}$$

where $\rho_{ij} = r_{\ell_i - \ell_j}$ is the correlation between ξ_{ℓ_i} and ξ_{ℓ_j} , and K is constant.

The conditions $D(u_n), D'(u_n)$ now follow easily :

Lemma 4.3 Suppose that the covariances r_n of the (standard) stationary normal sequence $\{\xi_n\}$ satisfy either $r_n \log n \rightarrow 0$ or $\sum r_n^2 < \infty$. Then $D(u_n), D'(u_n)$ are satisfied for u_n defined by $1 - \Phi(u_n) = \tau/n$.

Proof : From (4.3), with $s = 2, \ell_1 = 1, \ell_2 = j, N = nk$ we have

$$|\Pr\{\xi_1 \leq u_N, \xi_j \leq u_N\} - \Phi^2(u_N)| \leq K|r_{j-1}|e^{-u_N^2/(1+|r_{j-1}|)}$$

and thus, by simple manipulation,

$$|\Pr\{\xi_1 > u_N, \xi_j > u_N\} - (1-\Phi(u_N))^2| \leq K|r_{j-1}|e^{-u_N^2/(1+|r_{j-1}|)}$$

Hence

$$n \sum_{j=2}^n \Pr\{\xi_1 > u_N, \xi_j > u_N\} \leq \frac{\tau^2}{k^2} + KN \sum_{j=1}^N |r_j| e^{-u_N^2/(1+|r_j|)}$$

from which $D'(u_n)$ follows, by Lemma 4.1.

It follows also from (4.3) that if $1 \leq \ell_1 \leq \ell_2 \dots \leq \ell_s \leq n$, then

$$|F_{\ell_1 \ell_2 \dots \ell_s}(u_n) - \Phi^s(u_n)| \leq Kn \sum_{j=1}^n |r_j| e^{-u_n^2/(1+|r_j|)}$$

Suppose now that $1 \leq i_1 < i_2 \dots < i_p < j_1 < j_2 \dots < j_q \leq n$.

Identifying $(l_1 \dots l_s)$ in turn with

$(i_1, \dots, i_p, j_1, \dots, j_q), (i_1 \dots i_p), (j_1 \dots j_q)$ we thus have

$$\begin{aligned} & |F_{i_1 \dots i_p j_1 \dots j_q}(u_n) - F_{i_1 \dots i_p}(u_n) F_{j_1 \dots j_q}(u_n)| \\ & \leq 3Kn \sum_{j=1}^n |r_j| e^{-u_n^2/(1+|r_j|)} \end{aligned}$$

which tends to zero by Lemma 4.1. Thus $D(u_n)$ is satisfied (and indeed $\lim_n \alpha_{n\ell} = 0$ for each ℓ).

5. Poisson results, and asymptotic distribution of r^{th} largest values.

As noted in Section 1, for stationary normal processes in continuous time, the upcrossings of a high level tend to behave asymptotically like a Poisson process ([3] [2]). An analogous result can be shown to hold in discrete time (an upcrossing of the level u being said to occur between $t = r$ and $t = r+1$ if $\xi_r < u \leq \xi_{r+1}$.) For example if C_n denotes the number of the points $1, 2, \dots, n$ which are upcrossings of the level u_n ($u_n = u_n(\tau)$, $1 - \Phi(u_n) = \tau/n$), then under (1.6) (i) or (ii) C_n can be shown to be asymptotically Poisson.

We propose here to generalize this result to non-normal stationary sequences satisfying conditions $D(u_n)$, $D'(u_n)$, and thereby obtain, as a corollary, asymptotic distributions for the r^{th} largest among

$\xi_1, \xi_2, \dots, \xi_n$. In doing so, it seems slightly more natural in the discrete case to consider not the number C_n of upcrossings, but rather the number (L_n , say) of points r of $1, 2, \dots, n$ for which $\xi_r > u_n$. That is

$$L_n = \sum_{i=1}^n \chi_i \quad \text{where} \quad \chi_i = 1 \quad \text{if} \quad \xi_i > u_n, \quad \chi_i = 0 \quad \text{otherwise}$$

(L_n and C_n are essentially the same asymptotically; we may regard L_n as the "total time out of n which ξ_r spends above u_n ").

The connection between L_n and the r^{th} largest value ($M_n^{(r)}$ say) of ξ_1, \dots, ξ_n is clear, viz.,

$$(5.1) \quad \Pr\{M_n^{(r)} \leq u_n\} = \Pr\{L_n < r\}.$$

In the following, then, we shall assume that $\{\xi_n\}$ form a stationary sequence satisfying $D(u_n)$ and $D'(u_n)$, where $\{u_n\}$ satisfies (3.1), for a fixed $\tau \geq 0$. We shall obtain a limiting Poisson distributing for L_n , by the lemmas below. The same notation will be used as in Section 2 (stated prior to Lemma 2.4) concerning the intervals $I_1, I_1^*, \dots, I_k, I_k^*$, k again being a fixed integer. It will also be convenient to write $L_n(E)$ for the number of integers $r \in E$ such that $\xi_r > u_n$, if E is any set of integers.

Lemma 5.1 For any $r = 1, 2, \dots$, let A_r be the event that $L_N(I_j) \geq 1$ for at least r values of $j = 1, \dots, k$ ($N = nk$). Then

$$(i) \quad 0 \leq \Pr\{L_N \geq r\} - \Pr\{L_N(\bigcup_{j=1}^k I_j) \geq r\} \leq mk \Pr\{\xi_1 > u_N\}$$

$$(ii) \quad 0 \leq \Pr\{L_N(\bigcup_{j=1}^k I_j) \geq r\} - P(A_r) \leq k \Pr\{L_N(I_1) \geq 2\}$$

and hence

$$(iii) \quad 0 \leq \Pr\{L_N \geq r\} - P(A_r) \leq mk \Pr\{\xi_1 > u_N\} + k \Pr\{L_N(I_1) \geq 2\}$$

Further, (and uniformly in m)

$$(iv) \quad \limsup_{n \rightarrow \infty} \Pr\{L_N(I_1) \geq 2\} = o(1/k) \quad \text{as } k \rightarrow \infty.$$

Proof: (i) follows since if $L_N \geq r$ but $L_N(\bigcup_{j=1}^k I_j) < r$, then $\xi_j > u_N$ for some j in one of the km points of the I_j^* intervals. (ii) is equally obvious and (iii) follows by adding (i) and (ii). Also

$$\begin{aligned} \Pr\{L_N(I_1) \geq 2\} &\leq \sum_{1 \leq i < j \leq n-m} \Pr\{\xi_i > u_N, \xi_j > u_N\} \\ &\leq n \sum_{j=2}^n \Pr\{\xi_1 > u_N, \xi_j > u_N\} \end{aligned}$$

and hence (iv) follows at once by $D'(u_N)$ (Eqn.(3.2)).

Lemma 5.2 Let $B_r = A_r - A_{r+1}$; i.e. B_r is the event that $L_N(I_j) \geq 1$ for exactly r of the intervals I_j .

Then

$$\limsup_{n \rightarrow \infty} |\Pr\{L_N = r\} - P(B_r)| \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

(uniformly in m).

Proof : By subtracting the inequalities (iii) of Lemma 5.1 with $(r+1)$ replacing r , from those with r itself, we obtain

$$|\Pr\{L_N = r\} - P(B_r)| \leq mk \Pr\{\xi_1 > u_N\} + k \Pr\{L_N(I_1) \geq 2\}.$$

The first term on the right tends to zero as $n \rightarrow \infty$ by choice of u_N , and the result follows from (iv) of Lemma 5.1.

Lemma 5.3 Write $p(= p_n) = \Pr\{M(I_1) \leq u_N\}$. Then

$$|P(B_r) - \binom{k}{r} p^{k-r} (1-p)^r| \leq k \binom{k}{r} 2^{k-r} \alpha_{N,m}.$$

Proof : Write e_j for the event $M(I_j) \leq u_N$. Then

$$B_r = \bigcup e'_{i_1} e'_{i_2} \dots e'_{i_r} e_{i_{r+1}} \dots e_{i_k}$$

where the primes denote complements, the intersection signs are omitted, and the union is over sets of distinct integers $i_1 \dots i_k$ with $i_1 < i_2 \dots < i_r$; $i_{r+1} < i_{r+2} \dots < i_k$.

Now by Lemma 2.3 if $1 \leq j_1 < j_2 \dots < j_s \leq k$,

$|P(e_{j_1} e_{j_2} \dots e_{j_s}) - p^s| \leq k \alpha_{N,m}$, and by induction, if $i_1 < i_2 \dots < i_t$, $j_1 < j_2 \dots < j_s$, $i_\ell \neq j_{\ell'}$, for any ℓ, ℓ' ,

$$|P(e'_{i_1} e'_{i_2} \dots e'_{i_t} e_{j_1} e_{j_2} \dots e_{j_s}) - p^s (1-p)^t| \leq k \cdot 2^t \alpha_{N,m}$$

(the left hand side does not exceed

$$|P(e'_{i_1} \dots e'_{i_{t-1}} e_{j_1} \dots e_{j_s}) - p^s (1-p)^{t-1}| \\ + |p^{s+1} (1-p)^{t-1} - P(e'_{i_1} \dots e'_{i_{t-1}} e_{i_t} e_{j_1} \dots e_{j_s})|$$

which does not exceed $2k \cdot 2^{t-1} \alpha_{N,m}$ if the inductive hypothesis holds for $t-1$. It holds for $t=0$ by the above). Thus

$$|P(e'_{i_1} \dots e'_{i_r} e_{i_{r+1}} \dots e_{i_k}) - p^{k-r} (1-p)^r| \leq k \cdot 2^{k-r} \alpha_{N,m}$$

from which the desired result follows.

Lemma 5.4 Again write $p = p_n = \Pr\{M(I_1) \leq u_N\}$. Then

$$p \rightarrow e^{-\tau/k} \text{ as } n \rightarrow \infty.$$

Proof : Let $p' = p'_n = \Pr\{M_n \leq u_N\}$. Then

$$0 \leq p_n - p'_n \leq \Pr\{M(I_1^*) > u_N\} \leq m \Pr\{\xi_1 > u_N\}$$

and hence $p_n - p'_n \rightarrow 0$ as $n \rightarrow \infty$. But by Theorem 3.1, $\Pr\{M_n \leq u_N\} \rightarrow e^{-\tau}$ and hence by (2.6),

$$p'_n = \Pr\{M_n \leq u_N\} \rightarrow e^{-\tau/k}$$

from which the desired result follows.

By Lemmas 5.2, 5.3, 5.4 we at once obtain

$$\limsup_{n \rightarrow \infty} |\Pr\{L_N = r\} - \binom{k}{r} e^{-\frac{r}{k}(k-r)} (1 - e^{-\frac{r}{k}})^r| \leq$$

$$k \binom{k}{r} 2^{k-r} \limsup_{n \rightarrow \infty} \alpha_{N,m} + o(1) \text{ as } k \rightarrow \infty,$$

and hence, letting $m \rightarrow \infty$ (since the $o(1)$ is uniform in m),

$$(5.2) \quad \limsup_{n \rightarrow \infty} |\Pr\{L_N = r\} - \binom{k}{r} (1 - e^{-\frac{r}{k}})^r e^{-\frac{r}{k}(k-r)}| \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Our final lemma shows that N may be replaced by n in (5.2).

Lemma 5.5

$$(5.3) \quad \limsup_{n \rightarrow \infty} |\Pr\{L_n = r\} - \binom{k}{r} (1 - e^{-\frac{r}{k}})^r e^{-\frac{r}{k}(k-r)}| \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Proof : It follows easily from (5.2) that

$$(5.4) \quad \limsup_{n \rightarrow \infty} |\Pr\{L_N \leq r\} - \sum_{\ell=0}^r \binom{k}{\ell} (1 - e^{-\frac{\ell}{k}})^{\ell} e^{-\frac{\ell}{k}(k-\ell)}| \rightarrow 0 \text{ as } k \rightarrow \infty.$$

We will show that

$$(5.5) \quad \limsup_{n \rightarrow \infty} |\Pr\{L_n \leq r\} - \sum_{\ell=0}^r \binom{k}{\ell} (1 - e^{-\frac{\ell}{k}})^{\ell} e^{-\frac{\ell}{k}(k-\ell)}| \rightarrow 0 \text{ as } k \rightarrow \infty$$

from which (5.3) follows easily by expressing $\Pr\{L_n = r\}$ as $\Pr\{L_n \leq r\} - \Pr\{L_n \leq r-1\}$.

As in the proof of Theorem 3.1 we choose s so that $sk \leq n < (s+1)k$. Then it follows without difficulty that

$$\begin{aligned} \Pr\{L_{(s+1)k} \leq r\} - n \Pr\{u_n < \xi_1 \leq u_{(s+1)k}\} &\leq \Pr\{L_n \leq r\} \\ &\leq \Pr\{L_{sk} \leq r\} + sk \Pr\{u_{sk} < \xi_1 \leq u_n\}. \end{aligned}$$

(5.4) may now be applied with $N = (s+1)k$, $N = sk$ and, as in the proof of Theorem 3.1, terms such as $n \Pr\{u_n < \xi_1 \leq u_{(s+1)k}\}$ tend to zero. (5.3) then follows simply.

We may now obtain the main result.

Theorem 5.1 Suppose that $D(u_n)$, $D'(u_n)$ are satisfied, for the stationary sequence $\{\xi_n\}$, where u_n satisfies (3.1). Then

$$(5.6) \quad \Pr\{L_n = r\} \rightarrow e^{-\tau} \tau^r / r! \quad \text{as } n \rightarrow \infty.$$

Proof : If $\theta = 1 - e^{-\tau/k}$ then $k\theta \rightarrow \tau$ as $k \rightarrow \infty$ and

$$\binom{k}{r} \left(1 - e^{-\frac{\tau}{k}}\right)^r e^{-\frac{\tau}{k}(k-r)} = \binom{k}{r} \theta^r (1-\theta)^{k-r} \rightarrow e^{-\tau} \tau^r / r!$$

The result then follows from (5.3).

Two theorems now follow as corollaries.

Theorem 5.2 Under the conditions of Theorem 5.1, if $M_n^{(r)}$ denotes the r^{th} largest of $\xi_1 \dots \xi_n$, then

$$\Pr\{M_n^{(r)} \leq u_n\} \rightarrow \sum_{s=0}^{r-1} e^{-\tau} \tau^s / s!$$

Proof : $\Pr\{M_n^{(r)} \leq u_n\} = \Pr\{L_n < r\}$ and the result follows from (5.6).

Theorem 5.3 Let $\{\xi_n\}$ be a stationary sequence. Suppose that \hat{M}_n (the maximum of the "associated independent sequence" - cf. Theorem 3.2) satisfies

$$\Pr\{a_n(\hat{M}_n - b_n) \leq x\} \rightarrow G(x)$$

where G is non-degenerate (and hence Type I, II or III).

Let $D(u_n), D'(u_n)$ be satisfied by $\{\xi_n\}$ where $u_n = x/a_n + b_n$, for each x . Then the limiting distribution of $M_n^{(r)}$, the r^{th} largest of $\xi_1 \dots \xi_n$, is given by

$$\Pr\{a_n(M_n^{(r)} - b_n) \leq x\} \rightarrow G(x) \sum_{s=0}^{r-1} (-\log G(x))^s / r!$$

(zero if $G(x) = 0$).

Proof : The condition $\Pr\{a_n(\hat{M}_n - b_n) \leq x\} \rightarrow G(x)$ may be written as $F_1^n(u_n) \rightarrow G(x)$ from which we have (if $G(x) > 0$), $1 - F_1(u_n) \sim -\log G(x)/n$. Theorem 5.2 now applies with $\tau = -\log G(x)$.

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