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On Factor Representations and the C^*-Algebra of Canonical Commutation Relations

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Abstract. A new C^*-algebra, $\mathscr{A}$, for canonical commutation relations, both in the case of finite and infinite number of degrees of freedom, is defined. It has the property that to each, not necessarily continuous, representation of CCR there corresponds a representation of $\mathscr{A}$. The definition of $\mathscr{A}$ is based on the existence and uniqueness of the factor type II_1 representation. Some continuity properties of separable factor representations are proved.

1. Introduction

In this paper we define and investigate a C^*-algebra for representations of canonical commutation relations (CCR). It will be natural for our considerations to start with a general abelian group $\mathscr{R}$ and a bicharacter b on $\mathscr{R}$ and then define a representation of CCR over $(\mathscr{R}, b)$ as a mapping, say W, from $\mathscr{R}$ to unitary operators on a Hilbert space such that

$$W(x)\, W(y) = b(x, y)\, W(x+y)\,. \tag{1.1}$$

The only condition we impose on b is that it be non-degenerate in a sense given later.

In applications of CCR for the description of quantum systems with infinitely many degrees of freedom one has additional structure, and only representations satisfying certain conditions are of interest. For instance for Bose systems, $\mathscr{R}$ is in fact a vector space, the bicharacter b is defined by a bilinear form and W has to be continuous on rays, i.e. for each $x \in \mathscr{R}$ the one parameter groups $\lambda \mapsto W(\lambda x)$ have to be (weakly) continuous. In statistical mechanics the representations are locally normal with respect to the Fock representations.

For representations continuous on rays, a pertinent C^*-algebra was defined by Segal [11], and the C^*-algebras of statistical mechanics are decribed in [10]. Our algebra is the minimal one. It is contained in every C^*-algebra containing unitary operators satisfying (1.1) and is defined only by $(\mathscr{R}, b)$. The results are completely analogous to those for canonical anticommutation relations [12, 13]. From our point of view this is a consequence of the fact that CAR can be also put in the form (1.1) (cf. Section 3.10).

The definition of the C^*-algebra for the representations continuous on rays rests on the Stone-von Neumann uniqueness theorem for finitely many degrees of freedom; in statistical mechanics it is based on the uniqueness of representations with the number operator. The existence and properties of the minimal C^*-algebra, on the other hand, follow from the uniqueness of the representation generating a finite von Neumann algebra.

In general representations of CCR need not have any continuity property. For instance, for the factor type II_1 representation the weak topology of the algebra of operators gives exactly the discrete topology on $\mathscr{R}$. But in the case where $\mathscr{R}$ is a vector space, with b given by a bilinear form on $\mathscr{R}$, the factor representations in separable Hilbert spaces are not far from continuous. Namely, we prove that if W is such a representation then there exists a character ξ of $\mathscr{R}$ such that the representation

$$x \mapsto \xi(x)\, W(x)$$

is continuous on rays. If $\mathscr{S}$ is a subspace of $\mathscr{R}$ equipped with a topology making it a Baire topological vector space (or, more general, a Baire topological group) and such that for each $y \in \mathscr{R}$, $x \mapsto b(y, x)$ is continuous on $\mathscr{S}$ then we have a stronger continuity property: for each $f, g \in \mathscr{H}_W$, $|(f \mid W(x)\, g)|$ is continuous on $\mathscr{S}$. This is, for instance, enough to conclude that subsets of $\mathscr{S}$ and their closures generate the same von Neumann algebras.

Whenever, in Sections 2.5–2.7, we speak about representations of CCR over finite-dimensional vector spaces, the proofs given are valid for representations of CCR over arbitrary locally compact abelian groups, satisfying the second axiom of countability. In particular Theorem 2.7 gives a description of all separable factor representations of the C^*-algebra of CCR in this case.

Apart from Theorem 2.7, where we speak about representations of C^*-algebra defined in Section 3, Sections 2 and 3 are independent. The continuity properties of the separable factor representations are discussed in Section 2 whereas Section 3 centers around the factor type II_1 representation and the definition of the C^*-algebra for CCR. In Sections 3.1–3.4 we prove the uniqueness and existence of the factor type II_1 representation and show that the tensor product of any representation of CCR over $(\mathscr{R}, b)$ with a suitable representation of the abelian group $\mathscr{R}$ gives the factor type II_1 representation. Then, in Sections 3.5 and 3.6, we prove that the C^*-algebra generated by any representation of CCR over $(\mathscr{R}, b)$ and that generated by the factor type II_1 representation are isomorphic. This allows us to give in Section 3.7 the definition of the

C^*-algebra for CCR and to prove such properties as universality and simplicity. In Sections 3.8 and 3.9 we study the effect of homomorphism and that of taking direct products. The last section is devoted to a reformulation of CAR as CCR.

Note added in proof: When this work was reported in Marseille the paper [16] was pointed out to the author, where topics similar to some of Section 3 are considered. From Theorem 3.7 above the C^*-algebras of [16] and that defined here can be easily seen to be isomorphic. The set-up, theorems and treatment of [16] and the present paper are rather different.

1.1. *Notation and some Formulas.* In applications, one often starts with two linear spaces, say ϑ_φ and ϑ_π, and a non-degenerate bilinear form $\langle , \rangle$ on $\vartheta_\varphi \times \vartheta_\pi$ and calls a representation of CCR in the Hilbert space $\mathscr{H}$ a pair U, V of mapping from ϑ_φ and ϑ_π, respectively, to the set of unitary operators on $\mathscr{H}$ such that, for all $x', y' \in \vartheta_\varphi$ and $x'', y'' \in \vartheta_\pi$,

$$U(x')\,U(y') = U(x'+y')\,, \qquad V(x'')\,V(y'') = V(x''+y'')$$
$$U(x')\,V(x'') = e^{i\langle x', x''\rangle}\,V(x'')\,U(x')\,.$$

To put this into the form (1.1) one can define $\mathscr{R} := \vartheta_\varphi \oplus \vartheta_\pi$ and $W(x) := V(x'')\,U(x)$ for $x = (x', x'')$. Then (1.1) is satisfied with

$$b(x, y) = e^{i\langle y', x''\rangle} \quad \text{for} \quad x = (x', x'')\,, \quad y = (y', y'')\,.$$

Let G be an abelian, locally compact group satisfying the second axiom of countability and let $\hat{G}$ be the dual group. A representation of CCR over G is given by a pair U and V of representations of G and $\hat{G}$ respectively such that

$$U(x)\,V(x'') = \langle x', x''\rangle\,V(x'')\,U(x')\,, \qquad x' \in G\,, \qquad x'' \in \hat{G}\,;$$

here $\langle x', x''\rangle$ denotes the value of the character $x'' \in \hat{G}$ on $x' \in G$. Defining $\mathscr{R}, W$ as in the case of vector spaces we get

$$b(x, y) = \langle y', x'') \quad \text{for} \quad x = (x', x'')\,, \quad y = (y', y'')\,.$$

The Stone-von Neumann-Mackey theorem [9], states that if $(\mathscr{R}, b)$ is as above then every continuous representation of CCR over $(\mathscr{R}, b)$ is a direct sum of representations equivalent to the following one: W acts (irreducibly) in $L^2(G)$ according to formula

$$W(x)\,f(y') = \langle y', x''\rangle\,f(y' - x'')\,, \qquad x = (x', x'')\,.$$

From (1.1) it follows that

$$W(y)^{-1}\,W(x)\,W(y) = b(x, y)\,b(y, x)^{-1}\,W(x)\,.$$

Denoting the antisymmetric bicharacter $x, y \mapsto b(x, y)\,b(y, x)^{-1}$ by β we rewrite this formula as

$$W(y)^{-1}\,W(x)\,W(y) = \beta(x, y)\,W(x)\,. \tag{1.2}$$

For $y \in \mathscr{R}$ let us define ξy to be the character $x \mapsto \beta(x, y)$ and let

$$\mathscr{R}' = \{\xi^y : y \in \mathscr{R}\};$$

the nondegeneracy condition we impose is that $\mathscr{R}'$ separates the points of $\mathscr{R}$. In other words, if $\beta(x, y) = 1$ for all $y \in \mathscr{R}$ then $x = 0$.

Throughout Section 3 we consider $\mathscr{R}$ as equipped with the discrete topology and denote by $\hat{\mathscr{R}}$ the dual group of $\mathscr{R}$, i.e., $\hat{\mathscr{R}}$ is the group of all characters of $\mathscr{R}$ equipped with the topology of pointwise convergence on $\mathscr{R}$. The non-degeneracy of b reformulates, by the duality theorem for locally compact abelian groups to condition that $\mathscr{R}'$ is a dense subgroup of $\hat{\mathscr{R}}$.

Given a representation W over $(\mathscr{R}, b)$ we let $\mathscr{A}(W)^0$, $\mathscr{A}(W)$ and $\mathscr{A}(W)^-$ denote the *-algebra, the C^*-algebra and the von Neumann algebra, respectively, generated by $\{W(x)\}_{x \in \mathscr{R}}$.

Let $\mathbb{C}^{(\mathscr{R})}$ denote the linear space of complex functions on $\mathscr{R}$ of finite support. For $\alpha \in \mathbb{C}^{(\mathscr{R})}$ we define $W(\alpha)$ by

$$W(\alpha) = \sum_{\alpha \in \mathscr{R}} \alpha(x)\, W(x). \tag{1.3}$$

It follows from the commutation relations (1.1) that $\{W(\alpha)\}_{\alpha \in \mathbb{C}^{(\mathscr{R})}}$ is a *-algebra. As it is a minimal *-algebra containing $\{W(x)\}_{x \in \mathscr{R}}$, it coincides with $\mathscr{A}(W)^0$. From (1.2) and (1.3) we get

$$W(x)^{-1}\, W(\alpha)\, W(x) = \sum_{y \in \mathscr{R}} \alpha(y)\, \beta(y, x)\, W(y). \tag{1.4}$$

We also introduce the automorphisms τ_x, $x \in \mathscr{R}$, of $\mathfrak{L}(\mathscr{H}_W)$ defined by

$$\tau_x(A) = W(x)^{-1} A\, W(x), \qquad A \in \mathfrak{L}(\mathscr{H}_W). \tag{1.5}$$

A short computation using the commutation relations shows that $x \mapsto \tau_x$ is a homomorphism of the abelian group $\mathscr{R}$ into the group of all automorphisms of $\mathfrak{L}(\mathscr{H}_W)$.

We recall that a bicharacter of a group $\mathscr{R}$ is such a mapping $\mathscr{R} \times \mathscr{R} \to T$ that when one of the arguments is fixed it defines a character of $\mathscr{R}$.

T denotes the multiplicative group of the complex number of modulus one. A Borel function on topological space is a function measurable with respect to the σ-algebra generated by open subsets.

2. Continuity Properties of Separable Factor Representations

2.1. Lemma. *Let $\mathscr{A}$ be a factor in the Hilbert space $\mathscr{H}$ and let $\mathscr{A}_0$ be a weakly dense *-subalgebra of $\mathscr{A}$. Let $x \mapsto W(x)$ be a mapping from the topological group G to the set of unitary elements of $\mathscr{A}$.*

Let τ_x denote the automorphism $A \mapsto W(x)^{-1} A\, W(x)$ of $\mathfrak{L}(\mathscr{H})$. If, for each $A \in \mathscr{A}_0$, the mapping $x \mapsto \tau_x(A)$ is weakly continuous, then there exists *a function* $\varrho: G \to T$ *such that the mapping* $x \mapsto \varrho(x) W(x)$ *is Borel. If, moreover,*

i) $x \mapsto \tau_x$ *is a homomorphism of* G *into the group of automorphisms of* $\mathfrak{C}(\mathscr{H})$,

ii) G *is a Baire topological space,*

then, for all $g, h \in \mathscr{H}$, $x \mapsto |(g \mid W(x) h)|$ *are continuous functions and the function* ϱ *can be chosen in such a way that* $x \mapsto \varrho(x) W(x)$ *is (weakly) continuous on a neighbourhood of the unit element of* G.

Proof. Let $\mathscr{B}$ denote the *-algebra generated by $\mathscr{A}_0 \cup \mathscr{A}_0'$. As every element of $\mathscr{B}$ is a finite sum $\sum_\iota A_\iota A_\iota'$, the mapping $x \mapsto \tau_x(A)$ is weakly continuous for each $A \in \mathscr{B}$. As $\mathscr{A}$ is a factor, $\mathscr{B}$ acts on $\mathscr{H}$ irreducibly. Applying the Kaplansky density theorem we see that for each $A \in \mathfrak{L}(\mathscr{H})$ there exists a sequence $\{A_n\}_{n \in \mathbb{N}} \subset \mathscr{B}$ such that A is the weak limit of $\{A_n\}$. Therefore for each $g, h \in \mathscr{H}$ the function

$$x \mapsto (h \mid W(x)^{-1} A W(x) h) \, (\equiv (W(x) h \mid A W(x) h))$$

is the point-wise limit of the continuous functions

$$x \mapsto (h \mid \tau_x(A_n) h), \qquad n = 1, 2, \ldots .$$

Setting $A = P_g$, where P_g is the projection on $\mathbb{C}g$, g normalized, we get

$$(h \mid W(x)^{-1} A W(x) h) = |(g \mid W(x) h)|^2 .$$

Hence, for each $g, h \in \mathscr{H}$, the function

$$x \mapsto |(g \mid W(x) h)|, \qquad x \in G,$$

is a pointwise limit of a sequence of continuous functions.

2.2. *The Projective Group of* $\mathscr{H}$. For the proof we need some information about the projective group of $\mathscr{H}$; as the reference we choose [14].

The unitary group $\mathscr{U}$ of $\mathscr{H}$ endowed with the weak topology is a metrizable separable topological group. The topology of $\mathscr{U}$ can also be described as the coarsest for which all the mappings

$$U \mapsto (g \mid U h), \qquad g, h \in \mathscr{H},$$

are continuous.

Let $\mathscr{P}$ denote $\mathscr{U}/\mathscr{Z}$, where $\mathscr{Z}$ is the center of $\mathscr{U}$, equipped with the quotient topology and let π denote the natural homomorphism $\mathscr{U} \to \mathscr{P}$. $\mathscr{P}$ is a metrizable and separable topological group. The topology of $\mathscr{P}$ is the coarsest for which all the mappings

$$\pi(U) \mapsto d_{g,h}(\pi(U)), \qquad g, h \in \mathscr{H},$$

are continuous; here $d_{g,h}(\pi(U)) = |(g \mid U h)|$. A mapping $F: X \to \mathscr{P}$ is Borel if and only if all functions $d_{g,h} \circ F$, $g, h \in \mathscr{H}$, are Borel.

There exists a Borel map c from $\mathcal{P}$ into $\mathcal{U}$ such that

(i) $\pi \circ c = \mathrm{id}_{\mathcal{P}}$

(ii) $c \circ \pi(\mathbf{1}) = \mathbf{1}$

(iii) there is an open set containing $\pi(\mathbf{1})$ on which c is continuous.

2.3. *Proof of the First Part of the Lemma.* Let us consider the mapping $F := \pi \circ W; F: G \to \mathcal{P}$.

As $d_{g,h} \circ F = |(g \mid W(\cdot)\, h)|$, F is a Borel mapping if, for each $g, h \in \mathcal{H}$, $x \mapsto |(g \mid W(x)\, h)|$ is Borel. But $x \mapsto |(g \mid W(x)\, h)|$ being a pointwise limit of a sequence of continuous functions, cf. the end of Section 3, is Borel.

Let us now consider $c \circ F (\equiv c \circ F \circ W)$. $c \circ F(x)$ can differ from $W(x)$ only by a factor of modulus one and is a Borel map from G into $\mathcal{U}$. Therefore the function ϱ can be defined by: $\varrho(x)\, W(x) = c \circ F(x)$, $x \in G$. This proves the first part of Lemma.

2.4. *The Case of a Baire Group.* A topological space is said to be a Baire space if the complement of any set of first category is everywhere dense. For properties of Baire spaces we refer to [3].

The topology of $\mathcal{P}$ is separable and therefore, as is easy to see, one can choose from the family $d_{g,h}$, $g, h \in \mathcal{H}$, of the functions defining the topology of $\mathcal{P}$ a countable subfamily which defines the same topology. In other words, there is a sequence h_n of vectors in $\mathcal{H}$ such that a mapping F from a topological space X to $\mathcal{P}$ is continuous if $d_{h_m, h_n} \circ F$ is continuous for all $m, n \in \mathbb{N}$. We apply this to the homomorphism $F: G \to \mathcal{P}$.

By Theorem 8.5 (ii) of [3], $d_{g,h} \circ F$, $g, h \in \mathcal{H}$, being a pointwise limit of a sequence of continuous functions (functions of the zero Baire class) is a function of the first Baire class. Any function of the first Baire class on a Baire space is continuous at each point of a dense $\mathcal{G}_\delta$ set (Theorem 7.5, [3]). Let $A_{m,n}$ be such a dense $\mathcal{G}_\delta$ set for $d_{h_m, h_n} \circ F$.

$A_{m,n}$ as a $\mathcal{G}_\delta$ set is a countable intersection of open sets, say $A_{m,n} = \bigcap_{k \in \mathbb{N}} A_{m,n,k}$, and each $A_{m,n,k}$ $m, n, k \in N$, is dense in G as $A_{m,n}$ are. Hence $\bigcap_{m,n \in \mathbb{N}} A_{m,n} \left(\equiv \bigcap_{m,n,k \in \mathbb{N}} A_{m,n,k}\right)$ is a countable intersection of open dense subsets of a Baire space and therefore non-empty, Proposition 7.2 of [3].

If x is an element of $\bigcap_{m,n} A_{m,n}$ then all functions $d_{h_m, hn} \circ F$ are continuous at x. It follows that F is continuous at x and, being a homomorphism, F is continuous everywhere. Therefore $|(g \mid W(x)\, h)|$ $(\equiv d_{g,h} \circ F(x))$ is continuous for each $g, h \in \mathcal{H}$. It remains only to find a ϱ making $x \mapsto \varrho(x)\, W(x)$ continuous on a neighborhood of the unit element of G. This can be done by applying c to $F(x)$ exactly as in the end of Section 5.

2.5. Theorem. *Let W be a separable factor representation of CCR over $\mathcal{R}$ and let the dimension of $\mathcal{R}$ be finite. There exists a $\xi \in \hat{\mathcal{R}}$ such that $x \mapsto \langle x, \xi \rangle\, W(x)$ is a continuous representation of CCR.*

By the Stone-von Neumann-Mackey uniqueness theorem, ξW is quasi-equivalent to the unique continuous irreducible representation.

Proof. To show that we are in a situation described in Lemma 3 let A be an element of $\mathscr{A}(W)^0$, say $A = W(\alpha)$, $\alpha \in \mathbb{C}^{(\mathscr{R})}$. It follows from (1.4) that the mapping $x \mapsto W(x)^{-1} A W(x)$ is weakly (in fact norm-)continuous.

Let ϱ be the function of Lemma 3 making $x \mapsto \varrho(x) W(x)$ Borel. Denoting $\varrho(x) W(x)$ by $\tilde{W}(x)$ we get

$$\tilde{W}(x)\, \tilde{W}(y) = b(x, y)\, \sigma(x, y)\, \tilde{W}(x+y), \qquad x, y \in \mathscr{R},$$

where

$$\sigma(x, y) = \frac{\varrho(x)\, \varrho(y)}{\varrho(x+y)}. \tag{2.1}$$

It follows from this formula that $x, y \mapsto \sigma(x, y)$ is a Borel function. It is what is called a Borel multiplier [8]. Our multiplier is symmetric. By Lemma 7.2 of [8], a symmetric multiplier on a locally compact, separable abelian group is trivial, i.e., there exists a Borel function $\lambda: \mathscr{R} \to T$ such that

$$\sigma(x, y) = \frac{\lambda(x)\, \lambda(y)}{\lambda(x+y)}, \qquad x, y \in \mathscr{R}. \tag{2.2}$$

It follows from (2.1) and (2.2) that $\xi := \lambda^{-1} \varrho$ satisfies the identity

$$\frac{\xi(x)\, \xi(y)}{\xi(x+y)} = 1, \qquad x, y \in \mathscr{R},$$

i.e., is a character of $\mathscr{R}$. On the other hand $\langle x, \xi \rangle\, W(x) = \lambda(x)^{-1}\, \tilde{W}(x)$ and therefore $x \mapsto \langle x, \xi \rangle\, W(x)$ is a measurable representation of CCR over $\mathscr{R}$.

To prove that ξW is in fact continuous, consider the decomposition $\mathscr{R} = G \oplus \hat{G}$ and the restrictions of ξW to G and $\hat{G}$. These restriction give measurable representations of locally compact groups in a separable Hilbert space. Such representations are known to be continuous. The continuity of $\xi\, W$ follows.

2.6. Theorem. *Let W be a separable factor representation of CCR over $\mathscr{R}$. Then there exists a $\xi \in \hat{\mathscr{R}}$ such that $x \mapsto \langle x, \xi \rangle\, W(x)$ is continuous on rays.*

If $\mathscr{S}$ is any subspace of $\mathscr{R}$ topologized to a Baire topological vector space such that for each $y \in \mathscr{R}$ $x \mapsto \beta(x, y)$ is continuous on $\mathscr{S}$, then for each $g, h \in \mathscr{H}$

$$x \mapsto |(g \,|\, W(x)\, h)|$$

is a continuous function on $\mathscr{S}$.

Proof. Let $\mathcal{T}$ be a finite dimensional subspace of $\mathcal{R}$ such that $\beta|\mathcal{T}$ is non-degenerate. Decomposing $\mathcal{R}$ into a sum of β-orthogonal subspaces, $\mathcal{R}=\mathcal{T}\oplus\mathcal{T}^{\perp}$, we see that the elements of the center of the von Neumann algebra generated by $\{W(x): x\in\mathcal{T}\}$ commute with $W(\mathcal{R})$. Therefore, $W|\mathcal{T}$ is also a separable factor representation and by Theorem 2.5 there exists $\xi\in\hat{\mathcal{T}}$ such that $x\mapsto\langle x,\xi\rangle\, W(x)$ is continuous on $\mathcal{T}$.

Each $x\in\mathcal{R}$ is contained in a two-dimensional subspace onto which the restriction of β is non-degenerate. Therefore for each $x\in\mathcal{R}$ there exists a $\xi_x\in\hat{\mathbb{R}}$ such that $\lambda\mapsto\langle\lambda,\xi_x\rangle\, W(\lambda x)$ is continuous.

Let $\{x_\iota\}_{\iota\in I}$ be a basis of $\mathcal{R}$ over $\mathbb{R}$ and let us choose ξ_{x_ι} as in the preceding paragraph. Let us define $\xi\in\hat{\mathcal{R}}$ by:

$$\langle x,\xi\rangle=\prod_{\iota\in J}\langle\lambda_\iota,\xi_{x_\iota}\rangle \quad\text{for}\quad x=\sum_{\iota\in J}\lambda_\iota x_\iota,\ J \text{ a finite subset of } I\,.$$

It is easy to see that ξ makes $x\mapsto\langle x,\xi\rangle\, W(x)$ continuous on rays. This proves the first part of the theorem.

As $\mathcal{S}$ is a Baire topological vector space it is also a Baire topological group. Defining $\mathcal{A}_0$ to be $\mathcal{A}(W)^0$, it follows from formulae (1.4), (1.5) and the context there that we are in position to apply the second part of Lemma 2.1. Hence, the theorem follows.

It seems that for separable $\mathcal{S}$ there always exists a ϱ which makes $x\mapsto\varrho(x)\, W(x)$ as continuous as the functions $x\mapsto|(g\,|\,W(x)\,h)|$, $g,h\in\mathcal{H}$, are. The proof that symmetric, Borel multipliers on separable, locally compact, abelian groups are trivial depends, in an essential way, on the properties of Haar measure on locally compact groups and is therefore not applicable here. Nevertheless, we have a candidate for ϱ that seems to be good in the case of separable vector spaces.

2.7. Theorem 2.5 and the Stone-von Neumann-Mackey uniqueness theorem allow us to give a complete description of the separable factor representations of the *C*-algebra* of CCR over finite-dimensional space.

The group of the automorphisms of $\mathcal{A}$ acts on the set of all representation of $\mathcal{A}$ by $\pi\mapsto\pi\circ\tau$, $\tau\in\mathrm{Aut}(\mathcal{A})$. In Section 3.8 we define an isomorphism $\xi\mapsto\tau_\xi$ of $\hat{\mathcal{R}}$ into $\mathrm{Aut}(\mathcal{A})$ such that, Formula (3.13),

$$\tau_\xi\circ\mathcal{I}(x)=\langle x,\xi\rangle\,\mathcal{I}(x)\,,\qquad x\in\mathcal{R}\,.$$

This allows us to speak about an action of $\hat{\mathcal{R}}$ on the set of representations of $\mathcal{A}$.

Theorem. (i) *every separable factor representation of $\mathcal{A}$ is a factor type I representation.*

(ii) *$\hat{\mathcal{R}}$ acts transitively on the set of (the equivalence classes of) separable irreducible representations of $\mathcal{A}$.*

(iii) *the isotropy subgroups defined by the action of $\hat{\mathcal{R}}$ of* (ii) *are the same for all representations and equal to $\mathcal{R}'$.*

Proof. Let π be a separable factor representation of $\mathscr{A}$. Then $W := \pi \circ \mathscr{I}$ is a separable factor representation of CCR over $\mathscr{R}$. By Theorem 2.5 there exists a $\xi \in \hat{\mathscr{R}}$ such that $x \mapsto \langle x, \xi \rangle W(x)$ is continuous. By the Stone-von Neumann-Mackey uniqueness theorem, Section 1.1, ξW is quasi-equivalent to an irreducible representation and therefore it is a factor type I representation.

(ii) Follows from

$$\xi W = \pi \circ \tau_\xi \circ \mathscr{I}, \qquad \xi \in \hat{\mathscr{R}}.$$

and from the uniqueness of the continuous representation.

As $\hat{\mathscr{R}}$ is an abelian group and the action of $\hat{\mathscr{R}}$ is transitive, the isotropy subgroups are the same for all representations. Therefore it is enough to find the isotropy subgroup for one representation. Let π be such that $\pi \circ \mathscr{I}$ is continuous and irreducible. ξW is equivalent to W if, and only if, ξ is continuous. Hence, also π is equivalent to $\pi \circ \tau_\xi$ if, and only if, ξ is continuous. This proves (iii) and finishes the proof of the theorem.

2.8. *Generalizations.* The construction leading from the representation W to ξW can be generalized as follows.

Let W be a representation of CCR over $(\mathscr{R}, b)$ in the Hilbert space $\mathscr{H}$ and let $x \mapsto X(x)$ be a representation of the additive group of $\mathscr{R}$ in the Hilbert space $\mathscr{H}_X$. Then

$$x \mapsto W(x) \otimes X(x), \qquad x \in \mathscr{R}.$$

defines a representation over $(\mathscr{R}, b)$ in $\mathscr{H} \otimes \mathscr{H}_X$.

If the dimension of $\mathscr{R}$ is finite, or $\mathscr{R}$ is a locally compact group satisfying the second axiom of countability, W is continuous and the Hilbert space $\mathscr{H}_X$ is separable, then $W \otimes X$ generates a discrete von Neumann algebra: the proof will not be given here. It also follows from Theorem 2.5 and from the Stone-von Neumann-Mackey uniqueness theorem that $\mathscr{A}(W \otimes X)^-$ is a factor if, and only if, there exists $\xi \in \hat{\mathscr{R}}$ such that $x \mapsto \langle x, \xi \rangle X(x)$ is continuous. We conjecture that not all separable representations over finite-dimensional $\mathscr{R}$ are of the type $W \otimes X$, with W continuous, and, moreover, that there exist separable representations generating continuous von Neumann algebras.

In the case of non-separable $\mathscr{H}_X$, choosing X in a suitable way, one can get factors of type II and III for $\mathscr{A}(W \otimes X)^-$. For instance, the factor type II_1 representation can be constructed in such a way, and will be in the next sections. When the dimension of $\mathscr{R}$ is infinite and $\mathscr{H}_X$ is separable the situation seems to be more complicated. At least for $\mathscr{R} = \mathbb{R}^{(\mathbb{N})}$ one can get factor representations of type II_∞ and III.

3. Factor II_1 Representation and C^*-Algebra of CCR

In what follows $\mathscr{R}$ is an arbitrary abelian group, b is a bicharacter of $\mathscr{R}$, non-degenerate in the sense of Section 1.1, and W is a representation of CCR over $(\mathscr{R}, b)$.

3.1. Lemma. *If φ is a finite trace on $\mathscr{A}(W)^-$ then*

$$\varphi(W(x)) = 0\,, \qquad \forall x \neq 0\,. \tag{3.1}$$

Proof. If $x \neq 0$ then there exists $y \in \mathscr{R}$ such that $\beta(x, y) \neq 1$. Applying Formula (1.2),

$$\varphi(W(y)^{-1}\, W(x)\, W(y)) = \beta(x, y)\, \varphi(W(x))\,.$$

On the other hand, as φ is a trace,

$$\varphi(W(y)^{-1}\, W(x)\, W(y)) = \varphi(W(x))\,.$$

These two equations are compatible only if $\varphi(W(x)) = 0$. Our lemma is proved.

Taking limit $\lambda \to 0$ in the identity $\varphi(W(\lambda x)) = 0$, this lemma gives a "one-line-proof" of a result by Glimm [5], that there exists no representation continuous on rays generating a finite von Neumann algebra.

3.2. Lemma. (i) *If φ, ψ are finite, normalized (i.e., $\varphi(I) = \psi(I) = 1$), normal traces on $\mathscr{A}(W)^-$ then $\varphi = \psi$,*

(ii) *if $\mathscr{A}(W)^-$ is a finite von Neumann algebra then each finite, normal, non-zero trace on $\mathscr{A}(W)^-$ is faithful.*

Proof. Applying φ and ψ to $W(\alpha)$, using the notation of Section 1.1, we get from Lemma 3.1

$$\varphi(W(\alpha)) = \alpha(0) \quad \text{and} \quad \psi(W(\alpha)) = \alpha(0)\,, \qquad \forall \alpha \in \mathbb{C}^{(\mathscr{R})}.$$

By the von Neumann density theorem, $\mathscr{A}(W)^0$ is ultra-weakly dense in $\mathscr{A}(W)^-$ and, by Section 1.1, $\mathscr{A}(W)^\circ = \{W(\alpha)\}_{\alpha \in \mathbb{C}^{(R)}}$. On the other hand, by [4], Ch. I, § 4, Th. 1, φ and ψ are ultra-weakly continuous. Therefore $\varphi = \psi$ follows from their coincidence on $\mathscr{A}(W)^0$. This proves (i).

(ii) Follows at once from (i) and from the definition of a finite von Neumann algebra. Namely, [4], Ch. I, § 6, Def. 5, a von Neumann algebra is said to be finite if for each non-zero positive element there exists a finite, normal trace which is not zero on this element. If therefore $\psi(A) = 0$, and A is a positive element of $\mathscr{A}(W)^-$ then, by (i), every other normal trace is zero on A and $\mathscr{A}(W)^-$ is not finite. The lemma is proved.

3.3. Proposition. *Let $x \mapsto X(x)$ be the representation of the additive group of $\mathscr{R}$ in $L^2(\hat{\mathscr{R}}, d\xi)$, where $d\xi$ is the normalized Haar measure on $\hat{\mathscr{R}}$, defined by*

$$(X(x)\, \Phi)(\xi) = \langle x, \xi \rangle\, \Phi(\xi)\,, \qquad \Phi \in L^2(\hat{\mathscr{R}}, d\xi)\,.$$

Then $x \mapsto \hat{W}(x) := W(x) \otimes X(x)$ *is a factor type* II_1 *representation of CCR over* $(\mathscr{R}, b)$.

Proof. (a) Let Ω be the element of $L^2(\hat{\mathscr{R}}, d\xi)$ defined by

$$\Omega(\xi) = 1\,, \quad \forall \xi \in \hat{\mathscr{R}}\,.$$

If $\Omega_x := X(x)\,\Omega$ then $\Omega_0 = \Omega$ and from the orthogonality of characters it follows that $(\Omega_x | \Omega_y) = 0$ for $x \neq y$. Let f be a normalized vector of $\mathscr{H}_W$ and let us define φ on $\mathscr{A}(W)^-$ by

$$\varphi(A) = (f \otimes \Omega | A f \otimes \Omega)\,.$$

Then

$$\begin{aligned}\varphi(\hat{W}(x)\,\hat{W}(y)) &= (f\,|\,W(x)\,W(y) f)\,(\Omega\,|\,X(x)\,X(y)\,\Omega)\\ &= (f\,|\,W(x)\,W(y) f)\,(\Omega\,|\,\Omega_{x+y})\,.\end{aligned}$$

The last term in this chain is zero if $x + y \neq 0$. As $W(x)\,W(-x) = b(x, -x)$,

$$\varphi(W(x)\,W(y)) = \begin{cases} 0 \ \text{if} \ x + y \neq 0\,, \\ b(x, -x) \ \text{for} \ x + y = 0\,. \end{cases}$$

It follows that φ defines on $\mathscr{A}(W)^-$ a finite, normalized trace. It is obviously ultra-weakly continuous and therefore normal ([4], Ch. I, § 4, Th. 1). Applying Lemma 3.2 (i) we see that changing the vector f that appears in the definition of φ leaves the same trace; we could also use Ω_x instead of Ω.

(b) Let X' be the unitary representation of the additive group of $\mathscr{R}$ defined by

$$(X'(x)\,\Phi)\,(\xi) = \varphi(\xi \xi_x),\ \Phi \in L^2(\hat{\mathscr{R}}, d\xi) \quad (\xi_x \text{ is defined in Section 2})\,.$$

Let us also define $\hat{W}'$as $W \otimes X'$. Direct calculation shows that $\hat{W}'(x)$ and $I \otimes X(x)$ commute with $\hat{W}(y)$ for all $x, y \in \mathscr{R}$. If A is in the center of $\mathscr{A}(W)^-$ then A commutes with the *-algebra $\mathscr{A}$ generated by $\mathscr{A}(\hat{W}) \cup \mathscr{A}(\hat{W}') \cup \{I \otimes X(x)\}_{x \in \mathscr{R}}$.

It is easy to see that $\mathscr{A}$ contains $I \otimes X(x)$ and $I \otimes X'(x)$ for all $x \in \mathscr{R}$. Therefore, showing that the algebra generated by $X(x)$, $X(y)$, $x, y \in \mathscr{R}$, acts in $L^2(\hat{\mathscr{R}}, d\xi)$ in an irreducible way proves that A acts in $\mathscr{H}_W$, i.e., A is of form: $A' \otimes I$.

Let B be an operator in $L^2(\hat{\mathscr{R}}, d\xi)$ commuting with all $X(x)$, $X'(y)$, $x, y \in \mathscr{R}$. As Ω is a cyclic vector for $\{X(x)\}_{x \in \mathscr{R}}$, this follows, for instance, from the duality theorem for locally compact abelian groups and from the Peter-Weyl theorem, it is enough to show that $B\Omega$ is proportional to Ω. This in turn will follow from orthogonality of $B\Omega$ to Ω_x for all $x \in \mathscr{R}$, $x \neq 0$. We compute

$$\begin{aligned}(\Omega_x | B\Omega) &= (\Omega_x | B X'(y)\,\Omega) = (\Omega_x | X'(y)\,B\Omega) = (X'(-y)\,\Omega_x | B\Omega)\\ &= (\langle x, \xi_{-y} \rangle\,\Omega_x | B\Omega) = \beta(x, y)\,(\Omega_x | B\Omega)\,.\end{aligned}$$

If $x \neq 0$ then, by the non-degeneracy of β, there exists a $y \in \mathscr{R}$ such that $\beta(x, y) \neq 1$ and therefore $(\Omega_x | B\Omega) = 0$. This finishes the proof that the center of $\mathscr{A}(\hat{W})^-$ acts in $\mathscr{H}_W$.

(c) By (a), there exists a finite, normal trace on $\mathscr{A}(\hat{W})^-$. To conclude the proof it is enough to show that $\mathscr{A}(\hat{W})^-$ is a factor. Let E' be a projection of the center of $\mathscr{A}(\hat{W})^-$, $E' \neq 0$, $E' \neq I$. It follows from (b) that $E' = E \otimes I$, where E is a projection in $\mathscr{H}_W$, $E \neq 0$, $E \neq I$. If $Ef \neq 0$, $\|f\| = 1$, then

$$\varphi(E') = (f \otimes \Omega | E \otimes I f \otimes \Omega) = (f | Ef) = \|Ef\|^2 .$$

Therefore $\varphi(E') \neq 0$ and $A \mapsto \dfrac{1}{\varphi(E')} \varphi(AE')$ is a normalized, normal trace on $\mathscr{A}(\hat{W})^-$. By Lemma 3.2 (i) this trace is equal to φ. On the other hand $\varphi((I - E')E') = 0$, whereas repeating the proof that $\varphi(E') \neq 0$ with E' replaced by $I - E'$ gives $\varphi(I - E') \neq 0$. This contradiction shows that the center of $\mathscr{A}(\hat{W})^-$ is trivial and concludes the proof of the proposition.

3.4. Proposition. *There exists a unique, up to quasi-equivalence, representation of CCR over $(\mathscr{R}, b)$ generating a finite von Neumann algebra. It is a factor type II_1 representation.*

Proof. Let us suppose that $\mathscr{A}(W)^-$ is a finite von Neumann algebra. If φ is a finite, normalized, normal trace on $\mathscr{A}(W)^-$ then, by Lemma 3.2 (ii), φ is faithful. If E is a projection from the center of $\mathscr{A}(W)^-$, $E \neq 0$, $E \neq I$, then $A \mapsto \varphi(AE)$ is again a finite, non-zero, normal trace but it gives the value zero to $I - E$, in contradiction to Lemma 3.2 (ii). This proves that $\mathscr{A}(W)^-$ is a factor.

Let φ be a finite, normalized, normal trace on $\mathscr{A}(W)^-$. It follows from [4], Ch. I, § 4, Th. 1 and Lemma 5 that we may assume, after possible passage to a quasi-equivalent representation, the existence of such a normalized vector Ω that

$$\varphi(A) = (\Omega | A\Omega), \quad \forall A \in \mathscr{A}(W)^- .$$

Denoting $W(x)\Omega$ by Ω_x, $\Omega_0 = \Omega$, we conclude from formula (3.1) and the unitarity of W that $\{\Omega_x\}_{x \in \mathscr{R}}$ is a set of orthonormal vectors. Moreover, as follows from the commutation relations,

$$W(x)\,\Omega_y = b(x, y)\,\Omega_{x+y} . \tag{3.2}$$

Therefore the subspace of $\mathscr{H}_W$ generated by $\{\Omega_x\}_{x \in \mathscr{R}}$ is W-invariant and the restriction of W to this subspace gives a representation quasi-equivalent to the original, with Ω as a cyclic vector.

Starting with another representation, say W', generating a finite von Neumann algebra, which by the first part of this proof is a factor, the same method leads to an orthonormal family $\{\Omega'_x\}_{x \in \mathscr{R}}$ and the restriction of W' to the subspace of $\mathscr{H}_{W'}$ generated by $\{\Omega'_x\}_{x \in \mathscr{R}}$ is quasi-equivalent to W'. We also have a formula corresponding to (3.2) with W replaced by W' and Ω_x by Ω'_x. It follows that the correspondence $\Omega_x \mapsto \Omega'_x$ extends to an isomorphism of these subspaces of $\mathscr{H}_W$ and $\mathscr{H}_{W'}$ bringing W and W' into quasi-equivalence.

[We indicate a shorter proof: If W and W' are two factor type II_1 representations then $x \mapsto W(x) \oplus W(x')$ is again a representation generating a finite von Neumann algebra. By what was said in the beginning of the proof of Proposition 3.4, it is a factor representation and therefore quasi-equivalent to its subrepresentations W and W'. Hence W and W' are quasi-equivalent.]

It remains to show the existence of a factor type II_1 representation for each $(\mathscr{R}, b)$. To do so, let us define in $\ell^2(\mathscr{R})$ operators $W(x)$ by (3.2), with Ω_x now denoting the function of $\ell^2(\mathscr{R})$ taking value 1 at x and 0 at other points. Direct computation shows that $x \mapsto W(x)$ is a representation of CCR over $(\mathscr{R}, b)$. It is also not hard to see that it is a factor type II_1 representation. But we need not to prove this, as Proposition 3.3 shows how, given a representation of CCR over $(\mathscr{R}, b)$, one can construct a factor type II_1 representation. This finishes the proof of Proposition 3.4.

3.5. Lemma. *There exists a unique homomorphism of $\hat{\mathscr{R}}$ into the group* $\mathrm{Aut}(\mathscr{A}(W))$ *of all automorphisms of $\mathscr{A}(W)$ such that*

$$\tau_\xi(W(x)) = \langle x, \xi \rangle\, W(x), \qquad \forall x \in \mathscr{R}. \tag{3.3}$$

If $\xi \in \mathscr{R}'$ then the automorphism τ_ξ is inner.

Proof. τ_ξ, if it exists, is unique as $\mathscr{A}(W)$ is the C^*-algebra generated by $\{W(x)\}_{x \in \mathscr{R}}$.

If $\xi \in \mathscr{R}'$, say $\xi = \xi_y$, then, formula (1.2),

$$W(y)^{-1}\, W(x)\, W(y) = \langle x, \xi_y \rangle\, W(x), \qquad \forall x \in \mathscr{R},$$

which shows that τ_ξ exists and is implementable by $W(y)$. This proves the last statement of the lemma.

It follows from (3.3) that if, for each $\xi \in \hat{\mathscr{R}}$, there exists an automorphism τ_ξ of $\mathscr{A}(W)$ satisfying (3.3) then $\xi \mapsto \tau_\xi$ is a homomorphism. On the other hand, the existence of τ_ξ is equivalent to equations

$$\Big\| \sum_{x \in \mathscr{R}} \alpha(x) \langle x, \xi \rangle\, W(x) \Big\| = \Big\| \sum_{x \in \mathscr{R}} \alpha(x)\, W(x) \Big\|, \qquad \forall \alpha \in \mathbb{C}^{(\mathscr{R})}. \tag{3.4}$$

Defining the mapping $\alpha \mapsto \alpha^\xi$ by:

$$\alpha^\xi(x) = \langle x, \xi \rangle\, \alpha(x),$$

(3.4) can be rewritten as

$$\| W(\alpha^\xi) \| = \| W(\alpha) \|, \qquad \forall \alpha \in \mathbb{C}^{(\mathscr{R})}. \tag{3.5}$$

To prove (3.5) for all $\xi \in \hat{\mathscr{R}}$ let us first remark that this equation holds for $\xi \in \mathscr{R}'$. This follows from the existence of τ_ξ for such ξ.

The topology of $\hat{\mathscr{R}}$ is just the topology of pointwise convergence on $\mathscr{R}$. Therefore, for each $x \in \mathscr{R}$, $\xi \mapsto \langle x, \xi \rangle W(x)$ is a continuous map from $\hat{\mathscr{R}}$ to $\mathfrak{L}(\mathscr{H}_W)$, $\mathfrak{L}(\mathscr{H}_W)$ equipped with the norm topology. It follows that $\xi \mapsto \| W(\alpha^\xi)\|$ is also continuous for each $\alpha \in \mathbb{C}^{(\mathscr{R})}$. But, as was remarked in Section 1.1, $\mathscr{R}'$ is dense in $\hat{\mathscr{R}}$. Therefore the function $\xi \mapsto \| W(\alpha^\xi)\|$, being continuous and taking the value $\| W(\alpha)\|$ on a dense subset of $\hat{\mathscr{R}}$, must have this value everywhere. This proves (3.5) and the lemma.

3.6. Theorem. *Given any two representations W and W', of CCR over $(\mathscr{R}, b)$, there exists a unique homomorphism $\tau_{W'W}$ of $\mathscr{A}(W)$ onto $\mathscr{A}(W')$ such that*

$$\tau_{W'W}: W(x) \mapsto W'(x), \qquad \forall x \in \mathscr{R}. \tag{3.6}$$

Before proving the theorem we give another description of the representation $\hat{W}$ of Proposition 3.3. Some readers will recognize in the following elements of the theory of direct integrals of constant families of Hilbert spaces.

Let $\mathscr{C}(\hat{\mathscr{R}}, \mathscr{H})$ denote the linear space of continuous mappings from $\hat{\mathscr{R}}$ to $\mathscr{H}$. $\mathscr{C}(\hat{\mathscr{R}}, \mathscr{H})$ with the scalar product

$$\Phi, \Psi \mapsto \int_{\hat{\mathscr{R}}} (\Phi(\xi) | \Psi(\xi))\, d\xi$$

becomes a Hausdorff, pre-Hilbert space whose completion we denote by $L^2(\hat{\mathscr{R}}, \mathscr{H})$.

For $h \in \mathscr{H}$, $\hat{h}$ will denote the element of $L^2(\hat{\mathscr{R}}, \mathscr{H})$ which takes value h at every $\xi \in \hat{\mathscr{R}}$; for $h \in \mathscr{H}$, $\varphi \in \mathscr{C}(\hat{\mathscr{R}}, \mathbb{C})$, $h\varphi$ is the element of $L^2(\hat{\mathscr{R}}, \mathscr{H})$ taking value $\varphi(\xi) h$ at ξ. From the bi-linearity of $h, \varphi \mapsto h\varphi$ and from

$$(h\varphi | h'\varphi') = (h|h') \int \overline{\varphi(\xi)}\, \varphi'(\xi)\, d\xi\,,$$

by well known properties of the tensor product, there exists an isomorphism of $\mathscr{H} \otimes L^2(\hat{\mathscr{R}}, d\xi)$ onto $L^2(\hat{\mathscr{R}}, \mathscr{H})$ such that $h\varphi$ corresponds to $h \otimes \varphi$.

If A is a continuous mapping from $\hat{\mathscr{R}}$ to $\mathfrak{L}(\mathscr{H})$, $\mathfrak{L}(\mathscr{H})$ equipped, for instance, with the norm topology, then

$$\Psi \mapsto A\Psi\,, \qquad A\Psi(\xi) = A(\xi)\, \Psi(\xi)$$

defines a linear mapping $\mathscr{C}(\hat{\mathscr{R}}, \mathscr{H}) \to \mathscr{C}(\hat{\mathscr{R}}, \mathscr{H})$ and A extends to a bounded operator on $L^2(\hat{\mathscr{R}}, \mathscr{H})$. We want to prove the following formula for the norm of A:

$$\|A\| = \sup_{\xi \in \hat{\mathscr{R}}} \|A(\xi)\|\,. \tag{3.7}$$

As $\|A\Psi\| = (\int \|A(\xi)\Psi(\xi)\|^2 d\xi)^{\frac{1}{2}}$, the inequality $\|A\| \leqq \sup_{\xi \in \hat{\mathscr{R}}} \|A(\xi)\|$ holds. To prove that $\|A\| \geqq \sup \|A(\xi)\|$, fix $\xi_0 \in \hat{\mathscr{R}}$ and for $\varepsilon > 0$ choose $h \in \mathscr{H}$,

$\|h\| = 1$, such that

$$\|A(\xi_0)\,h\| \geqq \|A(\xi_0)\| - \frac{\varepsilon}{2}\,.$$

As $\xi \mapsto \|A(\xi)h\|$ is continuous, there exists an open neighborhood U of ξ_0 such that $|\,\|A(\xi)\,h\| - \|A(\xi)\,h\|\,| < \frac{\varepsilon}{2}$ for $\xi \in U$ and therefore

$$\|A(\xi)\,h\| > \|A(\xi_0)\| - \varepsilon\,.$$

By a well known property of the Haar measure, the measure of U is non-zero. As the Haar measure is regular there exists a compact K, $K \subset U$, also of a non-zero measure. By the normalcy of compact spaces, there exists a continous function which is 1 on K and 0 on $\hat{\mathscr{R}} \setminus U$. Therefore there is a continuous function φ taking the value 0 on $\hat{\mathscr{R}} \setminus U$ such that

$$\int_{\hat{\mathscr{R}}} |\varphi(\xi)|^2\,d\xi = 1\,.$$

Then

$$\begin{aligned}\|Ah\varphi\|^2 &= \int_{\hat{\mathscr{R}}} \|\varphi(\xi)\,A(\xi)\,h\|^2\,d\xi = \int_U \|A(\xi)\,h\|^2\,|\varphi(\xi)|^2\,d\xi \\ &\geqq (\|A(\xi_0)\| - \varepsilon)^2 \int_{\hat{\mathscr{R}}} |\varphi(\xi)|^2\,d\xi = (\|A(\xi_0)\| - \varepsilon)^2\,.\end{aligned}$$

As $\|h\varphi\| = 1$ and ε is arbitrary positive, it follows that $\|A\| \geqq \|A(\xi_0)\|$ for all $\xi_0 \in \hat{\mathscr{R}}$. Thus $\|A\| = \sup_{\xi \in \hat{\mathscr{R}}} \|A(\xi)\|$.

Proof of Theorem. Under the isomorphism of $\mathscr{H}_W \otimes L^2(\hat{\mathscr{R}}, d\xi)$ and $L^2(\hat{\mathscr{R}}, \mathscr{H}_W)$ the representation $x \mapsto \hat{W}(x)$ goes onto $x \mapsto \tilde{W}(x)$, where

$$(\tilde{W}(x)\,\Psi)(\xi) = \langle x, \xi \rangle\, W(\xi)\,\Psi(\xi)\,, \qquad \Psi \in \mathscr{C}(\hat{\mathscr{R}}, \mathscr{H}_W) \tag{3.8}$$

and $\tilde{W}(x)$ leaves $\mathscr{C}(\hat{\mathscr{R}}, \mathscr{H}_W)$ invariant for all $x \in \mathscr{R}$.

We now show that

$$\|\tilde{W}(\alpha)\| = \|W(\alpha)\|\,, \qquad \forall \alpha \in \mathbb{C}^{(\mathscr{R})}\,. \tag{3.9}$$

As remarked in the proof of Lemma 3.5, the mapping $\xi \mapsto W(\alpha^\xi)$ is continuous for each $\alpha \in \mathbb{C}^{(\mathscr{R})}$. On the other hand, it follows from (3.8) that

$$(\tilde{W}(\alpha)\,\Psi)(\xi) = W(\alpha^\xi)\,\Psi(\xi)\,, \qquad \Psi \in \mathscr{C}(\hat{\mathscr{R}}, \mathscr{H}_W)\,.$$

Therefore the norm of $\tilde{W}(\alpha)$ can be computed using formula (3.7), which gives

$$\|\tilde{W}(\alpha)\| = \sup_{\xi \in \hat{\mathscr{R}}} \|W(\alpha^\xi)\|\,.$$

But this implies (3.9) by (3.5).

It follows from (3.9) that there exists an isomorphism τ_W of $\mathscr{A}(W)$ and $\mathscr{A}(\hat{W})$ such that

$$\tau_W : W(x) \mapsto \tilde{W}(x), \qquad \forall x \in \mathscr{R}.$$

In the same way, starting with the representation W' we arrive at $\tilde{W}'$ and $\tau_{W'}$. The representations $\tilde{W}$ and $\tilde{W}'$ being equivalent to $\hat{W}$ and $\hat{W}'$, respectively, are both factor type II_1 representations over $(\mathscr{R}, b)$, Proposition 3.3. By Proposition 3.4 a factor II_1 representation over $\mathscr{R}$ is unique up to quasi-equivalence. Therefore there exists an isomorphism $\tau_{\tilde{W}'\tilde{W}}$ of $\mathscr{A}(\tilde{W}')$ and $\mathscr{A}(\tilde{W})$ (even of $\mathscr{A}(\tilde{W}')^-$ and $\mathscr{A}(\tilde{W})^-$) such that

$$\tau_{\tilde{W}'\tilde{W}} : \tilde{W}(x) \mapsto \tilde{W}'(x), \qquad \forall x \in \mathscr{R}.$$

Now it is easy to see that $\tau_{W'}^{-1} \circ \tau_{\tilde{W}'\tilde{W}} \circ \tau_W$ is the required isomorphism of $\mathscr{A}(W)$ and $\mathscr{A}(W')$. As $\mathscr{A}(W)$ is just the C^*-algebra generated by $\{W(x)\}_{x\in\mathscr{R}}$ homomorphism $\tau_{W'W}$ satisfying (3.6) is obviousely unique. The theorem is proved.

3.7. After Theorem 9.7 we can formulate the following

Theorem. *Given* $(\mathscr{R}, b)$, *there exists a* C^*-*algebra* $\mathscr{A}$ *and an injection* $\mathscr{I} : \mathscr{R} \to \mathscr{A}$ *such that*

(i) $\mathscr{I}(x)$ *is unitary for all* $x \in \mathscr{R}$ *and* $\{\mathscr{I}(x)\}_{x\in\mathscr{R}}$ *generates the* C^*-*algebra* $\mathscr{A}$.

(ii) $\mathscr{I}(x)\,\mathscr{I}(y) = b(x, y)\,\mathscr{I}(x + y)$, $\forall x, y \in \mathscr{R}$.

It follows from (i) *and* (ii) *that*

(iii) *the pair* $(\mathscr{A}, \mathscr{I})$ *is unique up to isomorphism and for each representation* W *of CCR over* $(\mathscr{R}, b)$ *there exists a representation* W *of* $\mathscr{A}$ *such that*

$$\tilde{W} \circ \mathscr{I} = W,$$

(iv) *the* C^*-*algebra* $\mathscr{A}$ *is simple.*

Proof. Given $(\mathscr{R}, b)$ and a representation π of $\mathscr{A}$, $\pi \circ \mathscr{I}$ defines a representation of CCR over $(\mathscr{R}, b)$. Let π' be a faithful representation of $\mathscr{A}$, i.e., π' gives an isomorphism of $\mathscr{A}$ onto $\pi'(\mathscr{A})$. Then, by Theorem 3.6 and the condition (i), $\mathscr{A}(\pi \circ \mathscr{I})$ is isomorphic to $\mathscr{A}(\pi' \circ \mathscr{I})$ and therefore also to $\mathscr{A}$. This shows that each representation of $\mathscr{A}$ is an isomorphism. Thus $\mathscr{A}$ is simple. Almost the same argument proves (iii), and, that given a representation W, $(\mathscr{A}(W), W)$ satisfies (i)–(iv). Therefore, the existence of $(\mathscr{A}, \mathscr{I})$ for each $(\mathscr{R}, b)$ follows from the existence of at least one representation of CCR over $(\mathscr{R}, b)$ which is assured by Proposition 3.4. The theorem is proved.

A pair $(\mathscr{A}, \mathscr{I})$ will be called the enveloping C^*-algebra for representations of CCR over $(\mathscr{R}, b)$, or the C^*-algebra of $(\mathscr{R}, b)$. We allowed similar abuses of language speaking about "the" factor type II_1 representation. The same applies to the next sections.

3.8. *Homomorphisms.* In most of the preceding considerations the bicharacter b enters through the anti-symmetric bicharacter β. The importance of β will be even more evident in the discussion of homomorphisms. What follows can be partly reformulated in terms of the group $H^2(\mathscr{R}, T)$, as the mapping $b \mapsto \beta$ factorizes to an isomorphism $[b] \mapsto \beta$, where $[b]$ is the image of b in $H^2(\mathscr{R}, T)$. Our treatment is influenced by that in [15].

Given $(\mathscr{R}, b)$ and $(\mathscr{R}', b')$, with the corresponding bicharacters β and β' a homomorphism $h: \mathscr{R} \Rightarrow \mathscr{R}'$ will be called (β, β')-symplectic if

$$\beta'(h(x), h(y)) = \beta(x, y) . \tag{3.10}$$

In terms of b and b' (3.10) reads

$$b'(h(x), h(y))\, b'(h(y), h(x))^{-1} = b(x, y)\, b(y, x)^{-1}$$

or

$$b'(h(x), h(y))\, b(x, y)^{-1} = b'(h(y), h(x))\, b(y, x)^{-1} .$$

This shows in particular that the bicharacter

$$x, y \mapsto b'(h(x), h(y))\, b(x, y)^{-1}$$

is symmetric and therefore, Lemma 7.2 of [8], trivial. That is, there exists a function $f: \mathscr{R} \to T$ such that

$$b(x, y) = f(x) f(y) f(x + y)^{-1}\, b'(h(x), h(y)) . \tag{3.11}$$

We now show that given (h, f) connected by (3.11) there exists a unique homomorphism $\tau_{(h,f)}: \mathscr{A} \to \mathscr{A}'$ such that

$$\tau_{(h,f)} \circ \mathscr{I}(x) = f(x)\, \mathscr{I}'(h(x)) , \qquad x \in \mathscr{R} . \tag{3.12}$$

The uniqueness is obvious; to prove the existence let $f(x)\, \mathscr{I}(h(x))$ be denoted by $W(x)$. Then, as follows from (3.11),

$$W(x)\, W(y) = b(x, y)\, W(x + y) .$$

Therefore the existence of $\tau_{(h,f)}$ follows from Theorem 3.7.

[Theorem 3.7 is stated for representation of CCR in Hilbert spaces and representations of C^*-algebras. But, obviously, all remains true when instead suitable mapping into C^*-algebras and homomorphisms of C^*-algebras are considered.]

Putting $\mathscr{R}' = \mathscr{R}$, $b' = b$ and $h = \mathrm{id}_{\mathscr{R}}$ we see that to each $\xi \in \hat{\mathscr{R}}$ there corresponds an automorphism τ_ξ of $\mathscr{A}$ such that

$$\tau_\xi \circ \mathscr{I}(x) = \langle x, \xi \rangle\, \mathscr{I}(x) ; \tag{3.13}$$

$\xi \mapsto \tau_\xi$ is an isomorphism of $\hat{\mathscr{R}}$ into $\mathrm{Aut}(\mathscr{A})$.

If (h, f) satisfies (3.11) h is symplectic. Once more setting $\mathscr{R} = \mathscr{R}'$, $\beta' = \beta$, it follows that

(i) for each pair (h,f) where h is a β-symplectic automorphism of $\mathscr{R}$ and $f:\mathscr{R}\to T$ is such that $x, y\mapsto f(x)f(y)f(x+y)^{-1}$ is a bicharacter (thus f is a character of $\mathscr{R}$ of second kind, in the terminology of [15]), there is a unique automorphism of $\mathscr{A}$ such that

$$\tau_{(h,f)}\circ\mathscr{I}(x)=f(x)\,\mathscr{I}\circ h(x)$$

[In fact, allowing b to be an arbitrary multiplier, instead of bicharacter, we get a generalization to arbitrary f.]

(ii) The mapping

$$(h,f),(h',f')\mapsto(hh',f\circ h'f')$$

defines a group structure in the set of all pairs of (i) and $(h,f)\mapsto\tau_{(h,f)}$ is an isomorphism of this group into $\mathrm{Aut}(\mathscr{A})$.

We can enlarge the group $\{(h,f)\}$ by including for h anti-symplectic automorphism of $\mathscr{R}$, i.e., such that

$$\beta(h(x),h(y))=\beta(x,y)^{-1};$$

examples of anti-symplectic h appear in physical applications as linear transformations changing sign of the corresponding anti-symmetric form. To this end, let us first remark that if $h:\mathscr{R}\to\mathscr{R}'$ is such that

$$\beta'(h(x),h(y))=\beta(x,y)^{-1}$$

then h is $(\bar{\beta},\beta')$-symplectic; here $z\mapsto\bar{z}$, $z\in\mathbb{C}$, denotes the conjugation in $\mathbb{C}$ and we used the identity $z^{-1}=\bar{z}$ for $|z|=1$.

Let $\bar{\mathscr{A}}$ denote the C^*-algebra conjugated to $\mathscr{A}$, i.e., there exists a one-to-one mapping $A\mapsto\bar{A}$ of $\mathscr{A}$ onto $\bar{\mathscr{A}}$ which preserves multiplication, norm and conjugation but is anti-linear. Denoting by $\bar{b}$ the bicharacter $x,y\mapsto\overline{b(x,y)}$ and by $\bar{\mathscr{I}}$ the injection $x\mapsto\overline{\mathscr{I}(x)}$ we see that $(\bar{\mathscr{A}},\bar{\mathscr{I}})$ is the enveloping C^*-algebra over $(\mathscr{R},\bar{b})$. As $\bar{\beta}$ is the anti-symmetric bicharacter associated with $\bar{b}$ and h is $(\bar{\beta},\beta')$-symplectic, for each $f:\mathscr{R}\to T$ there exists a homomorphism $\bar{\tau}_{(h,f)}:\mathscr{A}\to\mathscr{A}'$ such that

$$\bar{\tau}_{(h,f)}\circ\bar{\mathscr{I}}(x)=f(x)\,\mathscr{I}'(h(x)),\qquad x\in\mathscr{R}.$$

Denoting the superposition of $\bar{\tau}_{(h,f)}$ with the mapping inverse to $A\mapsto\bar{A}$ by $\tau_{(h,f)}$ we get an anti-homomorphism $\tau_{(h,f)}:\mathscr{A}\to\mathscr{A}'$ such that (3.12) is satisfied. If $\mathscr{R}'=\mathscr{R}$ and h is a β-anti-symplectic automorphism the group described in (i), (ii) can be enlarged by (h,f).

The last thing we want to prove in this section is the converse to (i) in the following form.

(iii) For every homomorphism $\tau:\mathscr{A}\to\mathscr{A}'$ such that

$$\tau(\mathscr{I}(\mathscr{R}))\subset\mathscr{I}'(\mathscr{R}')$$

there is an (h,f) such that $\tau=\tau_{(h,f)}$.

For a proof, let us first observe that the proportionality of $\mathscr{I}'(x')$ and $\mathscr{I}'(y')$ implies that $x'=y'$. For, in such a case $\mathscr{I}'(x')^{-1}\mathscr{I}'(z')\mathscr{I}'(x') = \mathscr{I}'(y')^{-1}\mathscr{I}'(z')\mathscr{I}'(y')$, $\forall z'\in\mathscr{R}'$, and this implies that $\beta'(z',x')=\beta'(z',y')$ for all $z'\in\mathscr{R}'$ from which the identity of x' and y' follows by the non-degeneracy of β.

This observation and assumption about τ allows us to define the mappings $h:\mathscr{R}\to\mathscr{R}'$ and $f:\mathscr{R}\to T$ by

$$\tau(\mathscr{I}(x)) = f(x)\,\mathscr{I}'(h(x)).$$

We compute

$$\begin{aligned} f(x+y)\,\mathscr{I}'(h(x+y)) &= \tau(\mathscr{I}(x+y)) = \tau(b(x,y)^{-1}\mathscr{I}(x)\,\mathscr{I}(y)) \\ &= b(x,y)^{-1} f(x) f(y)\,\mathscr{I}'(h(x))\,\mathscr{I}'(h(y)) \\ &= b(x,y)^{-1} f(x) f(y)\, b'(h(x),h(y))\,\mathscr{I}'(h(x)+h(y)). \end{aligned}$$

This implies, according to the observation, both that $h(x+y)=h(x)+h(y)$ and that the Eq. (3.11) holds. As $h(0)=0$, h is a homomorphism $\mathscr{R}\to\mathscr{R}'$. On the other hand, it follows from (3.11) that h is (β,β')-symplectic and that $x,y\mapsto f(x)f(y)f(x+y)^{-1}$ is a bicharacter. This finishes the proof of (iii).

3.9. *Direct Sums and Tensor Products.* As a reference to the tensor products of C^*-algebras we choose [6, 7].

Given a family $\{(\mathscr{R}_\iota, b_\iota)\}_{\iota\in I}$ of abelian groups $\mathscr{R}_\iota$ and non-degenerate bicharacters b_ι of $\mathscr{R}_\iota$ we denote by $(\mathscr{A}_\iota, \mathscr{I}_\iota)$ the corresponding enveloping C*-algebras. It follows from Theorem 3.7 that the tensor products $\overset{v}{\bigotimes_{\iota\in I}}\mathscr{A}_\iota$ and $\overset{*}{\bigotimes_{\iota\in I}}\mathscr{A}_\iota$ coincide (in fact, all the C^*-subcross norms on the algebraic tensor product of $\{\mathscr{A}_\iota\}_{\iota\in I}$ coincide); we denote them by $\bigotimes_{\iota\in I}\mathscr{A}_\iota$.

Let $(\mathscr{R},b)$ be the direct sum of $\{(\mathscr{R}_\iota,b_\iota)\}_{\iota\in I}$; i.e., $\mathscr{R}=\sum_{\iota\in I}\mathscr{R}_\iota$ and $b=\sum_{\iota\in I} b_\iota$; b is again a non-degenerate bicharacter of $\mathscr{R}$. Then $\left(\bigotimes_{\iota\in I}\mathscr{A}_\iota, \bigotimes_{\iota\in I}\mathscr{I}_\iota\right)$, where $\bigotimes_{\iota\in I}\mathscr{I}_\iota$ is defined as it should be, is the enveloping C^*-algebra for $(\mathscr{R},b)$.

3.10. CAR. We give the formulation of CAR in the form (1.1). In fact corresponding definition of Clifford algebras appear in [1, 2].

We recall that a representation of CAR over a real Hilbert space H in the Hilbert space $\mathscr{H}$ is given by a linear mapping a from H into the set of bounded hermitian operators on $\mathscr{H}$ such that

$$a(f)\,a(g)+a(g)\,a(f) = 2(f\,|\,g). \tag{3.13}$$

Let $\{f_\iota\}_{\iota\in I}$ be an orthonormal basis of H, let the indexing set I be linearily ordered and let us denote by $\mathscr{R}$ the set of all finite subsets of I. If $x=\{\iota_1,\dots,\iota_n\}$ is an element of $\mathscr{R}$, $\iota_1<\dots<\iota_n$, then we define $W(x)$ to be $a(f_{i_1}) \dots a(f_{i_n})$. It follows from (3.13) that $W(x)$ is unitary and that $W(x)W(y)$ is proportional to $W(x+y)$, where by $x+y$ we denote the symmetric difference of the sets x and y. Denoting this coefficient by $b(x, y)$ we have

$$W(x)W(y) = b(x, y)W(x+y). \tag{3.14}$$

$(\mathscr{R}, +)$ is an abelian group, with the empty subset of I as 0. That b is a bicharacter is proved in [2], p. 62, and that b is non-degenerate follows from the following formula of [1]

$$\beta(x, y) = (-1)^{[x][y]-[x \cap y]}$$

where $[x]$ denotes the number of elements of x.

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