ON FAMILIES OF BIPARTITE GRAPHS ASSOCIATED WITH SUMS OF GENERALIZED ORDER-k FIBONACCI AND LUCAS NUMBERS

EMRAH KILIÇ[1] AND DURSUN TASCI[2]

Abstract. In this paper, we consider the relationships between the sums of the generalized order-k Fibonacci and Lucas numbers and 1-factors of bipartite graphs.

1. Introduction

We consider the generalized $order-k$ Fibonacci and Lucas numbers. In [1], Er defined k sequences of the generalized $order-k$ Fibonacci numbers as shown:

$$g_n^i = \sum_{j=1}^{k} g_{n-j}^i, \text{ for } n > 0 \text{ and } 1 \leq i \leq k, \tag{1.1}$$

with boundary conditions for $1-k \leq n \leq 0$,

$$g_n^i = \begin{cases} 1 & \text{if } i = 1-n, \\ 0 & \text{otherwise,} \end{cases}$$

where g_n^i is the nth term of the ith sequence. For example, if $k = 2$, then $\{g_n^2\}$ is usual Fibonacci sequence, $\{F_n\}$, and, if $k = 4$, then the 4th sequence of the generalized $order-4$ Fibonacci numbers is

$$1, 1, 2, 4, 8, 15, 29, 56, 108, 208, 401, 773, 1490, \ldots.$$

In [9], the authors defined k sequences of the generalized $order-k$ Lucas numbers as shown:

$$l_n^i = \sum_{j=1}^{k} l_{n-j}^i \text{ , for } n > 0 \text{ and } 1 \leq i \leq k, \tag{1.2}$$

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with boundary conditions for $1-k \leq n \leq 0$,

$$l_n^i = \begin{cases} -1 & \text{if } i = 1-n, \\ 2 & \text{if } i = 2-n, \\ 0 & \text{otherwise,} \end{cases}$$

where l_n^i is the nth term of the ith sequence. For example, if $k = 2$, then $\{l_n^2\}$ is the usual Lucas sequence, $\{L_n\}$, and, if $k = 4$, then the 4th sequence of the generalized $order-4$ Lucas numbers is

$$1, 3, 4, 8, 16, 31, 59, 114, 220, 424, 817, 1575, 30636, \ldots.$$

Also, Er showed that

$$\begin{bmatrix} g_{n+1}^i \\ g_n^i \\ \vdots \\ g_{n-k+2}^i \end{bmatrix} = A \begin{bmatrix} g_n^i \\ g_{n-1}^i \\ \vdots \\ g_{n-k+1}^i \end{bmatrix}$$

where

$$A = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 & 1 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{bmatrix}$$

is a $k \times k$ companion matrix. Then he derived

$$G_{n+1} = AG_n,$$

where

$$G_n = \begin{bmatrix} g_n^1 & g_n^2 & \cdots & g_n^k \\ g_{n-1}^1 & g_{n-1}^2 & \cdots & g_{n-1}^k \\ \vdots & \vdots & \ddots & \vdots \\ g_{n-k+1}^1 & g_{n-k+1}^2 & \cdots & g_{n-k+1}^k \end{bmatrix}$$

The matrix A is said to be the generalized $order-k$ Fibonacci matrix.

In [9], we showed

$$\begin{bmatrix} l_{n+1}^i \\ l_n^i \\ \vdots \\ l_{n-k+2}^i \end{bmatrix} = A \begin{bmatrix} l_n^i \\ l_{n-1}^i \\ \vdots \\ l_{n-k+1}^i \end{bmatrix}$$

and so

$$H_{n+1} = AH_n$$

where

$$H_n = \begin{bmatrix} l_n^1 & l_n^2 & \dots & l_n^k \\ l_{n-1}^1 & l_{n-1}^2 & \dots & l_{n-1}^k \\ \vdots & \vdots & \ddots & \vdots \\ l_{n-k+1}^1 & l_{n-k+1}^2 & \dots & l_{n-k+1}^k \end{bmatrix},$$

also showed that

$$H_n = G_n K$$

where

$$K = \begin{bmatrix} -1 & 2 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ 0 & 0 & -1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & -1 & 2 \\ 0 & 0 & 0 & \dots & 0 & -1 \end{bmatrix}.$$

Furthermore, in [3], we gave the following relationship

$$l_n^k = g_n^k + 2g_{n-1}^k \quad \text{for} \quad k \geq 2. \tag{1.3}$$

The *permanent* of an n-square matrix $A = (a_{ij})$ is defined by

$$perA = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

where the summation extends over all permutations σ of the symmetric group S_n. A matrix is said to be a $(0,1) - matrix$ if each of its entries is either 0 or 1.

In [7], Minc constructed the $n \times n$ $(0,1) - matrix$ $F(n,k)$ where, $k \leq n+1$, with 1 in the (i,j) position for $i-1 \leq j \leq i+k-1$ and 0 otherwise. That is,

$$F(n,k) = \begin{bmatrix} 1 & 1 & \dots & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & 1 & \dots & 1 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 1 & 1 & 1 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 & 1 & \dots & 1 & 1 & 0 & 0 \\ 0 & \dots & \dots & 0 & 1 & 1 & \dots & 1 & 1 & 0 \\ 0 & \dots & \dots & \dots & 0 & 1 & 1 & \dots & 1 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 & 1 & 1 & 1 \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 & 1 & 1 \end{bmatrix} \tag{1.4}$$

and he showed that

$$perF(n,k) = g_{n+1}^k \tag{1.5}$$

where g_n^k is the nth generalized order$-k$ Fibonacci number. When $k=2$, $perF\,(n,2) = F_{n+1}$.

In this paper, we find families of square matrices such that (i) each matrix is the adjacency matrix of a bipartite graph; and (ii) the permanent of the matrices are the generalized order-k Lucas numbers and a sum of consecutive generalized order$-k$ Fibonacci or Lucas numbers.

A *bipartite graph* G is a graph whose vertex set V can be partitioned into two subsets V_1 and V_2 such that every edge of G joins a vertex in V_1 and a vertex in V_2. A $1-$ factor (or $perfect\ matching$) of a graph with $2n$ vertices is a spanning subgraph of G in which every vertex has degree 1. The enumeration or actual construction of 1-factors of a bipartite graph has many applications. Let $A\,(G)$ be the adjacency matrix of the bipartite graph G, and let $\mu\,(G)$ denote the number of 1-factors of G. Then, one can find the following fact in [8]: $\mu\,(G) \leq \sqrt{perA\,(G)}$. Also, one can find more applications of permanents in [8].

Let G be a bipartite graph whose vertex set V is partitioned into two subsets V_1 and V_2 such that $|V_1| = |V_2| = n$. We construct the *bipartite adjacent matix* $B\,(G) = [b_{ij}]$ of G as following: $b_{ij} = 1$ if and only if G contains an edge from $v_i \in V_1$ to $v_j \in V_2$, and 0 otherwise. Then, in [2] and [8], the number of 1-factors of bipartite graph G equals the permanent of its bipartite adjacency matrix.

Lee defined the matrix $\mathcal{L}_n$ and gave that $per\mathcal{L}_n = L_{n-1}$ where L_n is the nth usual Lucas number (see [5]).

In [6], the authors consider the relationship between the k-generalized Fibonacci numbers and 1-factors of a bipartite graph.

Also in [4], we determine the class of bipartite graph whose number of 1-factors is the the Lucas numbers L_n. We also consider the relationships between the sums of the Fibonacci and Lucas numbers and 1-factors of bipartite graphs.

By the definitions of the generalized order-k Fibonacci and Lucas numbers for $i = k \geq 2$, we have that

$$\begin{array}{cccc} l_1^k = 1, & l_2^k = 3, & l_3^k = 2^2, & l_4^k = 2^3, \\ \ldots, & l_{k-1}^k = 2^{k-2}, & l_k^k = 2^{k-1}, & l_{k+1}^k = 2^k \end{array}$$

and

$$\begin{array}{cccc} g_1^k = 1, & g_2^k = 2^0, & g_3^k = 2^1, & g_4^k = 2^2, \\ \ldots, & g_{k-1}^k = 2^{k-3}, & g_k^k = 2^{k-2}, & g_{k+1}^k = 2^{k-1}. \end{array}$$

2. The Generalized Order-k Lucas numbers

In this section, we determine a class of bipartite graph whose number of 1-factors is the generalized order$-k$ Lucas number.

Firstly, let n and k be positive integers such that $n > k \geq 2$ and let $M(n,k) = [m_{ij}]$ be the $n \times n$ $(0,1)$ −matrix with $M(n,k) = F(n,k) + U(n,k)$ where $U(n,k) = [u_{ij}]$ be the $n \times n$ $(0,1)$ −matrix with $u_{n-k-1,n-1} = u_{n-k,n} = 1$ and 0 otherwise, and the matrix $F(n,k)$ is given by (1.4). Clearly

$$M(n,k) = \begin{bmatrix} 1 & \dots & 1 & 1 & 0 & 0 & 0 & \dots & 0 & \dots & 0 \\ 1 & 1 & \dots & 1 & 1 & 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & 1 & \dots & 1 & 1 & 0 & \dots & 0 & \dots & 0 \\ \dots & \ddots & \ddots & \dots & \ddots & \ddots & \ddots & \ddots & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 & 1 & \dots & 1 & 1 & 0 & 0 & 0 \\ 0 & \dots & \dots & 0 & 1 & 1 & \dots & 1 & 1 & 1 & 0 \\ 0 & \dots & \dots & \dots & 0 & 1 & 1 & \dots & 1 & 1 & 1 \\ 0 & \dots & \dots & \dots & \dots & 0 & 1 & 1 & \dots & 1 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 & 1 & 1 & 1 \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 & 1 & 1 \end{bmatrix}.$$

Then we have following Theorem.

Theorem 1. *Let $G(M(n,k))$ be the bipartite graph with bipartite adjacency matrix $M(n,k)$, $n \geq 3$. Then the number of 1-factors of $G(M(n,k))$ is the nth generalized order-k Lucas number, l_n^k.*

Proof. It is easy to see that expanding $per M(n,k)$ by the elements of the last row and if we consider the definition of the matrix $F(n,k)$, then we obtain

$$per M(n,k) = per \begin{bmatrix} 1 & 1 & \dots & 1 & 0 & 0 & 0 & \dots & \dots & 0 \\ 1 & 1 & \dots & 1 & 1 & 0 & 0 & \dots & \dots & 0 \\ 0 & 1 & \dots & 1 & 1 & 1 & 0 & \dots & \dots & 0 \\ \dots & \ddots & \ddots & \dots & \dots & \dots & \ddots & \ddots & \dots & \dots \\ 0 & \dots & 0 & 1 & \dots & 1 & 1 & 1 & 0 & 0 \\ 0 & \dots & \dots & 0 & 1 & \dots & 1 & 1 & 1 & 1 \\ 0 & \dots & \dots & \dots & 0 & 1 & \dots & 1 & 1 & 1 \\ \dots & \dots & \dots & \dots & \dots & \ddots & \ddots & \dots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 & 1 & 1 & 1 \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 & 1 & 1 \end{bmatrix} + per F(n-1,k). \quad (2.1)$$

Also if we again compute the above permanent by the elements of the last row, then we have

$$
\begin{aligned}
perM\left(n,k\right) &= 2per\left[\begin{array}{ccccccccccc}
1 & 1 & \dots & 1 & 1 & 0 & 0 & 0 & \dots & \dots & 0 \\
1 & 1 & \dots & \dots & 1 & 1 & 0 & 0 & \dots & \dots & 0 \\
0 & 1 & \dots & \dots & 1 & 1 & 1 & 0 & \dots & \dots & 0 \\
\vdots & \ddots & \ddots & \dots & \dots & \ddots & \ddots & \ddots & \ddots & \dots & \vdots \\
0 & \dots & 0 & 1 & \dots & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & \dots & \dots & 0 & 1 & \dots & 1 & 1 & 1 & 1 & 0 \\
0 & \dots & \dots & \dots & 0 & 1 & \dots & 1 & 1 & 1 & 1 \\
0 & \dots & \dots & \dots & \dots & 0 & 1 & \dots & 1 & 1 & 1 \\
\dots & \dots & \dots & \dots & \dots & \dots & \ddots & \ddots & \dots & \vdots & \vdots \\
0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 & 1 & 1 & 1 \\
0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 & 1 & 1
\end{array}\right] \\
&\quad + perF(n-1,k).
\end{aligned}
$$

which satisfy, by the definition of the matrix $F\left(n,k\right)$

$$perM\left(n,k\right) = 2perF(n-2,k) + perF(n-1,k).$$

Using the Eq. (1.5), we can write the last equation as

$$perM\left(n,k\right) = 2g_{n-1}^{k} + g_{n}^{k}$$

and by the Eq. (1.3)

$$perM\left(n,k\right) = 2g_{n-1}^{k} + g_{n}^{k} = l_{n}^{k}.$$

So the proof is complete. □

For eaxample, if we take $k=2$, then the matrix $M(n,k)$ is reduced to the matrix

$$
M(n,2) = \left[\begin{array}{cccccccc}
1 & 1 & 0 & 0 & \dots & \dots & \dots & 0 \\
1 & 1 & 1 & 0 & \dots & \dots & \dots & 0 \\
0 & 1 & 1 & 1 & \ddots & \dots & \dots & \dots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & 0 & 0 \\
0 & \dots & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & \dots & \dots & 0 & 1 & 1 & 1 & 1 \\
\vdots & \dots & \dots & \dots & 0 & 1 & 1 & 1 \\
0 & \dots & \dots & \dots & \dots & 0 & 1 & 1
\end{array}\right]
$$

and by Theorem 1, $perM(n,2) = L_n$ where L_n is the nth usual Lucas number. In [4], we define the matrix C_n and show that $perC_n = L_n$. However, the matrix C_n is different from the matrix $M(n,2)$.

3. On the sums of generalized order-k Fibonacci and Lucas numbers

In this section, we determine two classes of bipartite graphs whose number of 1-factors are sums the generalized order$-k$ Fibonacci and Lucas numbers, $\sum_{j=1}^{n} g_j^k$ and $\sum_{j=1}^{n} l_j^k$, respectively.

Let n and k be positive integers such that $n > k \geq 2$ and let $T(n,k) = [t_{ij}]$ be the $n \times n$ $(0,1)-$matrix with $T(n,k) = F(n,k) + V(n,k)$ where $V(n,k) = [v_{ij}]$ be the $n \times n$ $(0,1)-$matrix with $v_{1j} = 1$ for $k+1 \leq j \leq n$ and 0 otherwise, and the matrix $F(n,k)$ is given by (1.4). That is,

$$T(n,k) = \left[\begin{array}{cccccccccccc}
1 & 1 & 1 & \dots & \dots & 1 & 1 & 1 & 1 & \dots & \dots & 1 \\
1 & 1 & 1 & \dots & \dots & 1 & 0 & 0 & 0 & \dots & \dots & 0 \\
0 & 1 & 1 & \dots & \dots & 1 & 1 & 0 & 0 & \dots & \dots & 0 \\
0 & 0 & 1 & \dots & \dots & 1 & 1 & 1 & 0 & \dots & \dots & 0 \\
\vdots & \dots & \ddots & \ddots & \dots & \dots & \dots & \dots & \ddots & \ddots & \dots & \vdots \\
0 & \dots & \dots & 0 & 1 & \dots & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & \dots & \dots & \dots & 0 & 1 & \dots & 1 & 1 & 1 & 1 & 0 \\
0 & \dots & \dots & \dots & \dots & 0 & 1 & \dots & 1 & 1 & 1 & 1 \\
0 & \dots & \dots & \dots & \dots & \dots & 0 & 1 & \dots & 1 & 1 & 1 \\
\vdots & \dots & \dots & \dots & \dots & \dots & \dots & \ddots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 & 1 & 1 & 1 \\
0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 & 1 & 1
\end{array}\right].$$

Then we have following Theorem.

Theorem 2. *Let $G(T(n,k))$ be the bipartite graph with bipartite adjacency matrix $T(n,k) = F(n,k) + V(n,k)$, $n \geq 2$. Then the number of 1-factors of $G(T(n,k))$ is the sums of generalized order-k Fibonacci numbers, $\sum_{j=1}^{n} g_j^k$.*

Proof. We will use the induction method. If $n = 3$, then we have

$$T(3,k) = \left[\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{array}\right],$$

and hence $perT(3,k) = 4$. From also the definition of the generalized order-k Fibonacci numbers, we have $g_1^k = g_2^k = 1,\ g_3^k = 2^1$. Thus $perT(3,k) = \sum_{j=1}^{3} g_j^k$. Let we suppose that the equality holds for n, then we have

$$perT(n,k) = \sum_{j=1}^{n} g_j^k. \tag{3.1}$$

Now we show that the equality holds for $n+1$. If we compute the $perT(n+1,k)$ by the Laplace expansion of permanent on the elements of the first column and consider the definition of the matrix $F(n,k)$, then we obtain

$$\begin{aligned} perT(n+1,k) &= per\left[\begin{array}{cccccccccc} 1 & \dots & 1 & 1 & 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & \dots & 1 & 1 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 & 1 & \dots & 1 & 1 & 0 & 0 \\ 0 & \dots & \dots & 0 & 1 & 1 & \dots & 1 & 1 & 0 \\ 0 & \dots & \dots & \dots & 0 & 1 & \dots & 1 & 1 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 & 1 & 1 & 1 \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 & 1 & 1 \end{array}\right] \\ &\quad +perF(n,k). \end{aligned}$$

Furthermore, from the definition of the matrix $T(n,k)$, we can write the last equation as

$$perT(n+1,k) = perT(n,k) + perF(n,k). \tag{3.2}$$

By the Eqs. (1.5) and (3.1), we write the Eq. (3.2) as follow

$$\begin{aligned} perT(n+1,k) &= \sum_{j=1}^{n} g_j^k + g_{n+1}^k \\ &= \sum_{j=1}^{n+1} g_j^k. \end{aligned}$$

So the proof is complete. □

Let n be positive integer such that $n > k \geq 2$ and let $E(n,k) = [e_{ij}]$ be the $n \times n$ $(0,1)$ −matrix with $E(n,k) = M(n,k) + D(n,k)$ where $D(n,k) = [d_{ij}]$ be the $n \times n$ $(0,1)$ −matrix with $d_{1j} = 1$ for $k+1 \leq j \leq n$ and 0 otherwise, and the matrix $M(n,k)$ be as in the section 2 . That is,

$$E\left(n,k\right)=\left[\begin{array}{cccccccccccc}
1 & 1 & \dots & 1 & 1 & 1 & 1 & 1 & 1 & \dots & \dots & 1\\
1 & 1 & \dots & 1 & 1 & 0 & 0 & 0 & 0 & \dots & \dots & 0\\
0 & 1 & \dots & 1 & 1 & 1 & 0 & 0 & 0 & \dots & \dots & 0\\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\
0 & \dots & 0 & 1 & 1 & \dots & 1 & 1 & 0 & 0 & 0 & 0\\
0 & \dots & \dots & 0 & 1 & 1 & \dots & 1 & 1 & 0 & 0 & 0\\
0 & \dots & \dots & \dots & 0 & 1 & 1 & \dots & 1 & 1 & 1 & 0\\
0 & \dots & \dots & \dots & \dots & 0 & 1 & 1 & \dots & 1 & 1 & 1\\
0 & \dots & \dots & \dots & \dots & \dots & 0 & 1 & 1 & \dots & 1 & 1\\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\
0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 & 1 & 1 & 1\\
0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 & 1 & 1
\end{array}\right]. \quad (3.3)$$

Then we have following Theorem.

Theorem 3. *Let $G(E\left(n,k\right))$ be the bipartite graph with bipartite adjacency matrix $E\left(n,k\right)$, $n\geq 3$. Then the number of 1-factors of $G(E\left(n,k\right))$ is the sums of generalized order-k Lucas number, $\sum_{j=1}^{n-1} l_j^k$.*

Proof. We will use the induction method to prove that $perE\left(n,k\right)=\sum_{j=1}^{n-1} l_j^k$. If $n=3$, then we have

$$E\left(3,k\right)=\left[\begin{array}{ccc} 1 & 1 & 1\\ 1 & 1 & 1\\ 0 & 1 & 1\end{array}\right],$$

and hence $perE\left(3,k\right)=4$. Since the definition of the generalized order-k Lucas numbers, we have that $l_1^k=1$ and $l_2^k=3$, $perE\left(3,k\right)=\sum_{j=1}^{2} l_j^k=4$. We suppose that the equality holds for n. Then we have

$$perE\left(n,k\right)=\sum_{j=1}^{n-1} l_j^k. \quad (3.4)$$

Now we show that the equality holds for $n+1$. It is easy to see that expanding $perE\left(n+1,k\right)$ by the elements of the first column and if we consider the definition of the matrix $M(n,k)$, then we obtain

$$\begin{aligned} perE(n+1,k) &= per\begin{bmatrix} 1 & 1 & \dots & 1 & 1 & 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & \dots & 1 & 1 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \dots & \dots & \dots & \ddots & \ddots & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 & 1 & \dots & 1 & 1 & 0 & 0 & 0 \\ 0 & \dots & \dots & 0 & 1 & 1 & \dots & 1 & 1 & 1 & 0 \\ 0 & \dots & \dots & \dots & 0 & 1 & 1 & \dots & 1 & 1 & 1 \\ 0 & \dots & \dots & \dots & \dots & 0 & 1 & 1 & \dots & 1 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 & 1 & 1 & 1 \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 & 1 & 1 \end{bmatrix} \\ &\quad +perM(n,k). \end{aligned}$$

From also the definition of the matrix $E(n,k)$, we can write the last equation as

$$perE(n+1,k) = perE(n,k) + M(n,k). \tag{3.5}$$

By the Eq.(3.4) and Theorem 1, we can write the Eq. (3.5) as follow

$$\begin{aligned} perE(n+1,k) &= \sum_{j=1}^{n-1} l_j^k + l_n^k \\ &= \sum_{j=1}^{n} l_j^k. \end{aligned}$$

So the proof is complete. □

Furthermore, a matrix A is called *convertible* if there is an $n \times n$ $(1,-1)$ −matrix H such that $perA = \det(A \circ H)$, where $A \circ H$ denotes the Hadamard product of A and H. Such a matrix H is called a *converter* of A.

Let W be a $(1,-1)$ −matrix of order n, defined by

$$W = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ -1 & 1 & 1 & \dots & 1 & 1 \\ 1 & -1 & 1 & \dots & 1 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & -1 & 1 \end{bmatrix}.$$

Combining the above result and Thorems 1, 2 and 3, following Theorems hold.

Theorem 4. *Let l_n^k be the nth generalized order-k Lucas number. Then, for $n \geq 3$*

$$l_n^k = \det(M(n,k) \circ W).$$

Theorem 5. *Let g_n^k be the nth generalized order-k Fibonacci number. Then, for $n \geq 3$*

$$\sum_{j=1}^{n} g_j^k = \det\left(T\left(n,k\right) \circ W\right).$$

Theorem 6. *Let l_n^k be the nth generalized order-k Lucas number. Then, for $n \geq 3$*

$$\sum_{j=1}^{n} l_j^k = \det\left(E\left(n,k\right) \circ W\right).$$

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[1]TOBB Economics and Technology University Mathematics Department 06560 Ankara Turkey

E-mail address: ekilic@etu.edu.tr

[2]Department of Mathematics, Gazi University 06500 Ankara Turkey