[11] S. P. Gulko, *On the properties of subspaces of Σ-products*, DAN SSR 237 (1977), pp. 505–508 (in Russian).
[12] D. R. Lewis and C. Stegall, *Banach spaces whose duals are isomorphic to $l_1(\Gamma)$*, J. Func. Analysis 12 (1973), pp. 177–187.
[13] V. I. Malyhin, *On the tightness and the Souslin number of* exp *X and of a product spaces*, DAN SSSR 203 (1972), pp. 1001–1003 (in Russian).
[14] E. Michael, *Continuous selections I*, Ann. of Math. 63 (1956), pp. 361–382.
[15] S. Mrówka, *Some set-theoretic constructions in topology*, Fund. Math. 94 (1977), pp. 83–92.
[16] R. Pol, *A function space C(X) which is weakly Lindelöf but not weakly compactly generated*, Studia Math. 64 (1979), pp. 279–285.
[17] — *Concerning function spaces on separable compact spaces*, Bull. Polon. Acad. Sci. 25 (1977), pp. 993–997.
[18] H. H. Schaefer, *Topological Vector Spaces*, New York 1966.
[19] Z. Semadeni, *Banach Spaces of Continuous Functions*, Warszawa 1971.
[20] B. E. Šapirovskiĭ, *On decomposition of a perfect mapping into an irreducible mapping and a retraction*, Proc. VII Top. Conf. Minsk 1977 (in Russian).
[21] M. Talagrand, *Sur une conjecture de H. H. Corson*, Bull. Soc. Math. 99 (1975), pp. 211–212.
[22] — *Espaces de Banach faiblement K-analytiques*, Comp. Rend. Acad. Sci. 284 (1977), pp. 745–748.
[23] — *Espaces de Banach faiblement K-analytiques*, (to appear).

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On families of σ-complete ideals

by

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Abstract. Our main results are the following: Assume Martin's Axiom. Then

1. For every $\lambda<2^\omega$ and every family $\{\mu_\alpha\colon \alpha<\lambda\}$ of two-valued uniform measures on 2^ω there exists an $X\subset 2^\omega$ non-measurable with respect to any of them.

2. For every cardinal $\varkappa$ such that $2^\omega<\varkappa<$1st cardinal carrying a 2^ω-complete 2^ω-saturated ideal the following holds: if $\lambda<2^\omega$ and $\{\mu_\alpha\colon \alpha<\lambda\}$ is a family of 2^ω-additive two-valued measures on $\varkappa$, then there exists an $X\subset\varkappa$ non-measurable with respect to any of them.

0. Terminology and preliminaries. We shall use standard set-theoretical notation and terminology. Letters $\varkappa$, λ, μ will always denote uncountable cardinals. "I is an ideal on X" will mean "I is a σ-complete proper ideal of subsets of X such that $\{x\}\in I$ for all $x\in X$". An ideal I is *λ-complete* iff $\{x_\xi\colon \xi<\eta\}\subset I$ implies $\bigcup\{X_\xi\colon \xi<\eta\}\in I$ for $\eta<\lambda$. A cardinal λ is called the *character of an ideal I on $\varkappa$* ($\operatorname{ch} I=\lambda$) iff λ is the least cardinal such that $\exists X\subset\varkappa$, $|X|=\lambda$, $X\notin I$. An ideal I on $\varkappa$ is *uniform* iff $\operatorname{ch} I=\varkappa$. If I is an ideal on $\varkappa$, then I^* will denote the dual filter.

Ideals I_1 and I_2 on $\varkappa$ are called *compatible* iff there exists an ideal I_3 on $\varkappa$ such that $I_1\cup I_2\subset I_3$. It is easy to see that I_1, I_2 are compatible iff $I_1\cap I_2^*=\emptyset$ iff $I_2\cap I_1^*=\emptyset$. Otherwise I_1, I_2 are incompatible.

MA will denote Martin's Axiom. We shall use the following consequence of MA (see [4]):

1. The union of $<2^\omega$ closed nowhere dense subsets of a metric complete separable space is nowhere dense.

A subset $\mathscr{L}$ of the reals is called *strongly Lusin* if for every Lebesgue measurable set A $|\mathscr{L}\cap A|<2^\omega$ iff A has Lebesgue measure 0. It is also a consequence of MA (see [2], cf. also [1], [4], [6]) that

2. A strongly Lusin set exists.

We use the following notation:

$U(\varkappa,\lambda,\mu)$ — For every family $\{I_\alpha\colon \alpha<\lambda\}$ of μ-complete ideals on $\varkappa$ we have $\bigcup_{\alpha<\lambda}(I_\alpha\cup I_\alpha^*)\neq P(\varkappa)$.

$U^*(\varkappa,\lambda,\mu)$ — For every family $\{I_\alpha\colon \alpha<\lambda\}$ of μ-complete uniform ideals on $\varkappa$ we have $\bigcup_{\alpha<\lambda}(I_\alpha\cup I_\alpha^*)\neq P(\varkappa)$.

Using this notation, we can formulate the classical problem of Ulam on sets of measures as follows:

Let $\varkappa$ be less than the first measurable cardinal. What is the minimal cardinal λ such that non $U(\varkappa, \lambda, \omega_1)$?

A particularly interesting case is $\varkappa = 2^{\omega}$. If 2^{ω} is less than the first cardinal carrying a σ-complete σ-saturated ideal, then the Erdős–Alaoglu theorem (cf. e.g. [7]) gives $U(2^{\omega}, \omega, \omega_1)$.

On the other hand, the second author proved in [5] that $U(2^{\omega}, \omega, 2^{\omega})$. In [8] A. Taylor strengthened this result to $U^*(2^{\omega}, \omega, \omega_1)$.

In the present paper we investigate the case when the family of ideals is uncountable. It turns out that under some additional set-theoretical assumptions it is possible to get information in this case as well.

1. Main results. We begin with the following

LEMMA 1.1. *Let $\varkappa$ be an uncountable cardinal. Let f be an atomless measure defined on a σ-algebra $S \subset P(\varkappa)$ and I_f the ideal of f-null sets. Then*

(i) *For every family $\{I_n: n \in \omega\}$ of ideals on $\varkappa$ which are compatible with I_f we have $\bigcup_{n\in\omega} (I_n \cup I_n^*) \neq P(\varkappa)$.*

(ii) *If, in addition, f is such that the metric space S/I_f with the metric $\varrho([A], [B]) = f(A \Delta B)$ is a separable space, then* MA *implies that for every family $\{I_\alpha: \alpha < \lambda\}$, $\lambda < 2^{\omega}$, of ideals on $\varkappa$ which are compatible with I_f we have $\bigcup_{\alpha<\lambda} (I_\alpha \cup I_\alpha^*) \neq P(\varkappa)$.*

Proof. (i) Since I_n are compatible with I_f, there exist ideals $J_n \supset I_n \cup I_f$ and it suffices to show that $\bigcup_{n\in\omega} (J_n \cup J_n^*) \neq P(\varkappa)$.

Write $S_n = (J_n \cup J_n^*) \cap S$. S_n are of course σ-algebras. Let $\tilde{S} = S/I_f$, $\tilde{S}_n = S_n/I_f$. We have $\tilde{S}_n \subset \tilde{S}$. We show that $\tilde{S}_n \neq \tilde{S}$ for $n \in \omega$. Assume that $\tilde{S}_n = \tilde{S}$. Hence for every $A \in S$ there is a $B \in S_n$ such that $A \equiv B (\mathrm{mod}\, I_f)$. But in view of the inclusion $J_n \supset I_f$ we get $S = S_n$.

Now since f is atomless, we can construct a tree of sets $A_s \in S$ for $s \in 2^{<\omega}$ such that $A_\varnothing = \varkappa$, $A_{s^\frown\langle 0\rangle}, A_{s^\frown\langle 1\rangle}$ is a disjoint partition of A_s and $f(A_s) = 2^{-\mathrm{lh}\, s}$. For every s we have $A_{s^\frown\langle 0\rangle} \in J_n^*$ or $A_{s^\frown\langle 1\rangle} \in J_n^*$ and we get a branch $\{A_{g\restriction n}: n \in \omega\}$ for a certain $g \in 2^{\omega}$ s.t. $\bigcap \{A_{g\restriction n}: n \in \omega\} \in J_n^*$. But it follows from the construction that $f(\bigcap A_{g\restriction n}: n \in \omega) = 0$ for every $g \in 2^{\omega}$, contradicting the assumption that $J_n \supset I_f$.

Let us view $\tilde{S}$ as a metric space with the distance $\varrho([A], [B]) = f(A \Delta B)$. It is well known that it is a complete metric space. $\tilde{S}_n$ is also a complete metric space with the same distance and hence it is a closed subset of $\tilde{S}$.

CLAIM. *$\tilde{S}_n$ is a nowhere dense subset of $\tilde{S}$.*

Assume that it is not. Then there exist a set A and a positive real ε such that $K_\varrho([A], \varepsilon) \subset \tilde{S}_n$. Consider any $B \in S$ such that $[A] \cap [B] = 0$. Since f is atomless, there exists a partition $\{B_l: l < k\}$ of B such that $f(B_l) < \varepsilon$. Hence $[A] \cup [B_l] \in \tilde{S}_n$, and since S_n is a σ-algebra, we get $[A] \cup [B] \in \tilde{S}_n$ and $[B] \in \tilde{S}_n$. In a similar way (using $[\varkappa - A]$ instead of $[A]$) we get $[B] \in \tilde{S}_n$ for every $B \subset A$. It follows that $\tilde{S}_n = \tilde{S}$ — a contradiction.

We have shown that $\tilde{S}_n$ are closed nowhere dense subsets of $\tilde{S}$. Using the Baire Category Theorem, we get $\bigcup_{n\in\omega} \tilde{S}_n \neq \tilde{S}$ and hence there exists an $X \in S$ such that $X \notin \bigcup_{n\in\omega} (J_n \cup J_n^*)$.

(ii) In this case we use the Strong Baire Category Theorem. It is possible since we have MA and the space $\tilde{S}$ is separable. ∎

Before stating the next lemma we need another terminological convention: let $\mathscr{I}_1, \mathscr{I}_2$ be families of ideals on $\varkappa$. Then $I_i = \bigcap \mathscr{I}_i$ is also an ideal on $\varkappa$. We say that the families $\mathscr{I}_1, \mathscr{I}_2$ are *compatible* iff I_1, I_2 are compatible.

If $\mathscr{I}$ is a family of ideals, then $\mathscr{I}^* = \{I^*: I \in \mathscr{I}\}$.

LEMMA 1.2. *Let $\lambda < \mu \leqslant \varkappa$ be uncountable cardinals. For $\alpha < \lambda$ let $\mathscr{I}_\alpha$ be a family of μ-complete ideals on $\varkappa$ such that $\bigcup \mathscr{I}_\alpha \cup \bigcup \mathscr{I}_\alpha^* \neq P(\varkappa)$. Assume moreover that the families $\mathscr{I}_\alpha$: $\alpha < \lambda$ are pairwise incompatible. Then $\bigcup_{\alpha<\lambda} (\bigcup \mathscr{I}_\alpha \cup \bigcup \mathscr{I}_\alpha^*) \neq P(\varkappa)$.*

Proof. Let $I_\alpha = \bigcap \mathscr{I}_\alpha$ for $\alpha < \lambda$. We shall construct a family $\{A_\beta^\alpha: \beta \leqslant \alpha < \lambda\}$ such that the following conditions hold:

(i) $A_\alpha^\alpha \in I_\alpha^*$.

(ii) $A_\beta^\alpha \subset A_\beta^\gamma$ and $A_\beta^\gamma \setminus A_\beta^\alpha \in I_\beta$ for $\gamma < \alpha$.

(iii) $A_\beta^\alpha \cap A_\gamma^\alpha = \varnothing$ for $\beta \neq \gamma$.

First we show how our lemma follows from the existence of such a family. Let $A_\beta = \bigcap_{\beta\leqslant\alpha<\lambda} A_\beta^\alpha$. Condition (iii) implies that $A_\alpha \cap A_\beta = \varnothing$ for $\alpha \neq \beta$. Since I_α is μ-complete, it follows from (i) and (ii) that $A_\alpha \in I_\alpha^*$.

CLAIM. *For $\alpha < \lambda$ there exists a $B_\alpha \subset A_\alpha$ such that $B_\alpha \notin \bigcup \mathscr{I}_\alpha \cup \bigcup \mathscr{I}_\alpha^*$.*

Using the assumption of our lemma, we take $B \notin \bigcup \mathscr{I}_\alpha \cup \bigcup \mathscr{I}_\alpha^*$. Since $A_\alpha \in I_\alpha^*$, it is easy to see that $B \cap A_\alpha \notin \bigcup \mathscr{I}_\alpha \cup \bigcup \mathscr{I}_\alpha^*$ and we put $B_\alpha = A_\alpha \cap B$. Now $\bigcup_{\alpha<\lambda} B_\alpha \notin \bigcup_{\alpha<\lambda} (\bigcup \mathscr{I}_\alpha \cup \bigcup \mathscr{I}_\alpha^*)$ by $A_\alpha \in I_\alpha^*$ and disjointness of A_α's.

Hence it suffices to construct the sets A_β^α. We proceed by induction. As A_0^0 we take an arbitrary element of I_0^*. Assume that A_γ^α are already constructed for $\alpha < \beta$. We put $B_\gamma^\beta = \bigcap_{\gamma\leqslant\alpha<\beta} A_\gamma^\alpha$. Then we take $U_\gamma \in I_\gamma \cap I_\beta^*$ for $\gamma < \beta$. It can be done since $\{I_\alpha: \alpha < \lambda\}$ are pairwise incompatible. Clearly, $\bigcap_{\gamma<\beta} U_\gamma \in I_\gamma \cap I_\beta^*$ for $\gamma < \beta$ (by μ-completeness). We put $A_\beta^\beta = \bigcap_{\gamma<\beta} U_\gamma$ and $A_\gamma^\beta = B_\gamma^\beta \setminus A_\beta^\beta$ for $\gamma < \beta$. It is easy to see that conditions (i), (ii) and (iii) are satisfied. ∎

We are now ready to prove

THEOREM 1.3. *Assume* MA. *Then $U^*(2^{\omega}, \lambda, \omega_1)$ for $\lambda < 2^{\omega}$.*

Proof. Let $\mathscr{L}$ be a strongly Lusin set. Denote by $\mathscr{B}$ the family $\{B \cap \mathscr{L}: B$ is a Borel subset of the reals$\}$. Clearly $\mathscr{B}$ is a σ-algebra on $\mathscr{L}$. Denote by m the Lebesgue measure. We define for $A \in \mathscr{B}$

$$f(A) = m(B) \quad \text{if} \quad A = B \cap \mathscr{L}.$$

Since $\mathscr{L}$ is a strongly Lusin set, the function f is well defined. Indeed, take B_1, B_2 s.t. $A = B_1 \cap \mathscr{L} = B_2 \cap \mathscr{L}$. Then $(B_1 \Delta B_2) \cap \mathscr{L} = \emptyset$; hence $m(B_1 \Delta B_2) = 0$ and finally $m(B_1) = m(B_2)$.

The function f is an atomless measure on $\mathscr{B}$, and $\mathscr{B}/I_f$ with the usual distance is a separable space. Applying Lemma 1.1, we conclude that for every family $\{I_\alpha:\ \alpha<\lambda\}$, $\lambda<2^\omega$, of ideals on $\mathscr{L}$ compatible with I_f we have $\bigcup_{\alpha<\lambda}(I_\alpha \cup I_\alpha^*) \neq P(\mathscr{L})$. Since $|\mathscr{L}| = 2^\omega$, it is sufficient to show that every uniform ideal on $\mathscr{L}$ is compatible with I_f.

It turns out that every uniform ideal I on $\mathscr{L}$ contains I_f. Indeed, if $f(X) = 0$ for $X \subset \mathscr{L}$, then $|X \cap \mathscr{L}| = |X| < 2^\omega$. Since I is uniform, $X \in I$. This completes the proof. ■

Our next theorem gives some information about sets of ideals on cardinals greater than 2^ω.

THEOREM 1.4. *Let θ be the first cardinal carrying a 2^ω-complete 2^ω-saturated ideal. Assume* MA. *Then $2^\omega < \varkappa < \theta$ implies $U(\varkappa, \lambda, 2^\omega)$ for $\lambda < 2^\omega$.*

Proof. Let I be a 2^ω-complete ideal in $\varkappa$. Since $\varkappa < \theta$, there exists a pairwise disjoint partition $\{A_g:\ g \in 2^\omega\}$ of $\varkappa$ such that $A_g \notin I$. Let $A_B = \bigcup_{g \in B} A_g$ for Borel $B \subset 2^\omega$. $\mathscr{B}' = \{A_B:\ B$ — Borel subset of $2^\omega\}$ is a σ-algebra on $\varkappa$. Let $\mathscr{B} = \{X \Delta N:\ X \in \mathscr{B}',\ N \in I\}$. We define for $Y \in \mathscr{B}$, $Y = A_B \Delta N$,

$$f(Y) = m(B).$$

It is easy to see that f is an atomless measure on $\mathscr{B}$ and $I_f \supset I$. Also $\mathscr{B}/I_f$ is separable.

For $\lambda < 2^\omega$, let $\{I_\alpha:\ \alpha<\lambda\}$ be an arbitrary family of 2^ω-complete ideals on $\varkappa$. For every I_α let f_α be the above atomless measure. It suffices to show that $\bigcup_{\alpha<\lambda}(I_{f_\alpha} \cup I_{f_\alpha}^*) \neq P(\varkappa)$.

We construct the following sequence of families of ideals:

Consider the sequence $\{I_{f_\alpha}:\ \alpha<\lambda\}$. Put $I^0 = I_{f_0}$. Let $J_0 = \{I_{f_\alpha}:\ I_{f_\alpha}$ is compatible with $I^0\}$. If I^ξ, J_ξ are defined for $\xi<\eta$, let I^η be the first I_{f_α} which is incompatible with any I^ξ for $\xi<\eta$ and

$$J_\eta = \{I_{f_\alpha}:\ I_{f_\alpha} \notin \bigcup_{\xi<\eta} J_\xi \text{ and } I_{f_\alpha} \text{ is compatible with } I^\eta\}.$$

We proceed in this way for all $\eta<\lambda$. Then we define $\tilde{J}_\xi = \{I^\xi \cup I_{f_\alpha}:\ I_{f_\alpha} \in J_\xi\}$ for $\xi<\lambda$. Clearly, $I^\xi \subset \bigcap \tilde{J}_\xi$ and it follows from the construction that the families $\tilde{J}_\xi$: $\xi<\lambda$ are pairwise incompatible. Now it follows from Lemma 1.1 that, for every $\xi<\lambda$, $\bigcup \tilde{J}_\xi \cup \bigcup \tilde{J}_\xi^* \neq P(\varkappa)$. The assumptions of Lemma 1.2 are fulfilled and hence $\bigcup_{\xi<\lambda}(\bigcup \tilde{J}_\xi \cup \bigcup \tilde{J}_\xi^*) \neq P(\varkappa)$ and we get $\bigcup_{\alpha<\lambda}(I_\alpha \cup I_\alpha^*) \neq P(\varkappa)$. ■

2. Applications to the countable case. We begin with the following fact connecting properties U and U^*:

PROPOSITION 2.1. *Let $\lambda<\mu\leqslant\varkappa$ be uncountable cardinals. Then $U(\varkappa, \lambda, \mu)$ iff $\forall\alpha[\mu\leqslant\alpha\leqslant\varkappa \to U^*(\alpha, \lambda, \mu)]$.*

Proof. In both cases we argue by contradiction:

$\Rightarrow$ Assume that $\alpha<\varkappa$ is such that $\{I_\xi:\ \xi<\lambda\}$ are uniform μ-complete ideals on α such that $\bigcup_{\xi<\lambda}(I_\xi \cup I_\xi^*) = P(\alpha)$. Let $J_\xi = \{A \subset \varkappa:\ A \cap \alpha \in I_\xi\}$. Clearly, $\bigcup_{\xi<\lambda}(J_\xi \cup J_\xi^*) = P(\varkappa)$ and J_ξ are μ-complete, a contradiction.

$\Leftarrow$ Let $\{I_\xi:\ \xi<\lambda\}$ be μ-complete ideals on $\varkappa$ such that $\bigcup_{\xi<\lambda}(I_\xi \cup I_\xi^*) = P(\varkappa)$. Let $\mathscr{A} = \{\mathrm{ch}(I_\xi):\ \xi<\lambda\}$. We enumerate the set $\mathscr{A}$: $\{a_\eta:\ \eta<\gamma\}$, where $\gamma\leqslant\lambda$. Let $\{I_\xi^\eta:\ \xi<\lambda\}$ be the family of those ideals which have character a_η (it is possible that some of them appear in the enumeration several times).

For I_ξ^η let A_ξ^η be a set of cardinality a_η such that $A_\xi^\eta \notin I_\xi^\eta$ and let $A^\eta = \bigcup_{\xi<\lambda} A_\xi^\eta$. Hence $|A^\eta| = a_\eta$ and $A^\eta \notin I_\xi^\eta$ for $\xi<\lambda$.

Consider $J_\xi^\eta = \{X \subset \varkappa:\ X \cap A_\eta \in I_\xi^\eta\}$. J_ξ^η are μ-complete ideals. Write $J^\eta = \{J_\xi^\eta:\ \xi<\lambda\}$. The families J^η: $\eta<\lambda$ are pairwise incompatible and by the assumption $\bigcup J^\eta \cup \bigcup J^{\eta *} \neq P(\varkappa)$. Hence by Lemma 1.2 we get $\bigcup_{\xi,\eta}(J_\xi^\eta \cup J_\xi^{\eta *}) \neq P(\varkappa)$, and thus $\bigcup_{\xi,\eta}(I_\xi^\eta \cup I_\xi^{\eta *}) \neq P(\varkappa)$, contrary to our assumption. ■

THEOREM 2.2. *Assume that 2^ω is the first cardinal carrying a σ-complete σ-saturated ideal. Then $U(2^\omega, \omega, \omega_1)$.*

Proof. The theorem follows from the Erdös–Alaoglu theorem, and $U^*(2^\omega, \omega, \omega_1)$ by the above proposition where $\varkappa = 2^\omega$, $\lambda = \omega$, $\mu = \omega_1$. ■

Ulam's problem in the countable case (i.e. $U(\varkappa, \omega, \mu)$) is closely connected with the existence of ideals I on $\varkappa$ such that $P(\varkappa)/I$ has a countable dense set. Such ideals are called separable. Actually it is proved in [7] that $U^*(\varkappa, \omega, \omega_1)$ iff no uniform ideal on $\varkappa$ is separable. A closer inspection of this proof gives, for every $\mu\leqslant\varkappa$, $U(\varkappa, \omega, \mu)$ iff no μ-complete ideal on $\varkappa$ is separable. Thus the investigation of ideals I such that $P(\varkappa)/I$ has a dense set of a given cardinality seems to be interesting.

PROPOSITION 2.3. *Let $2^\lambda<\varkappa<1$st measurable cardinal. Then $P(\varkappa)/I$ does not have dense sets of cardinality λ for any $(2^\lambda)^+$-complete ideal I on $\varkappa$.*

Proof. Let I be a $(2^\lambda)^+$-complete ideal on $\varkappa$. Let $s:\ \lambda \to P(\varkappa)$; a function $t:\ \lambda \to P(\varkappa)$ will be called a flip of s iff, for all $\alpha<\lambda$, $t(\alpha) = s(\alpha)$ or $t(\alpha) = \varkappa - s(\alpha)$. Let $F(s)$ denote the family of all flips of s. Clearly, $|F(s)| = 2^\lambda$. By definition $\bigcup_{u \in F(s)} \bigcap_{\xi<\lambda} u(\xi) = \varkappa$; hence there exists a $u \in F(s)$ such that $\bigcap_{\xi<\lambda} u(\xi) \notin I$.

Assume that $s = \{s(\xi):\ \xi<\lambda\}$ is such that $\{[s(\xi)]:\ \xi<\lambda\}$ is dense in $P(\varkappa)/I$. Take a flip u of s such that $\bigcap_{\xi<\lambda} u(\xi) \notin I$. Hence there is an $s(\eta)$ such that $s(\eta) \subset \bigcap_{\xi<\lambda} u(\xi) (\mathrm{mod}\, I)$, but $P(\varkappa)/I$ is atomless, a contradiction. ■

COROLLARY 2.4. *Let $2^\omega<\varkappa<1$st measurable cardinal. Then $U(\varkappa, \omega, (2^\omega)^+)$.* ■

We conclude this section with the following proposition, pointed out by R. Sztencel.

PROPOSITION 2.5. *Let $\{f_n: n \in \omega\}$ be a family of atomless measures on 2^ω such that $\mathrm{Dom} f_n \neq 2^\omega$. Then there exists a subset of 2^ω non-measurable with respect to any of them.*

Proof. Use the proof of Lemma 1.1. Notice that each atomless measure f_n can be extended to an outer atomless measure f_n^*. Consider $h = \sum_{n=1}^{\infty} (1/2^n) f_n^*$. Then h is an atomless outer measure and if $I = \{A \subset 2^\omega: h(A) = 0\}$, then $P(2^\omega)/I$ with the metric $\varrho([A],[B]) = h(A \Delta B)$ forms a complete space. ■

3. Some remarks on consistency. Our Theorems 1.3 and 1.4 give some information about the consistency of sentences $U(\varkappa, \lambda, \mu)$ for uncountable λ.

PROPOSITION 3.1. (i) *Let $\varkappa$ be a regular uncountable cardinal and $\lambda < \varkappa$. Then*

$$\mathrm{Con(ZFC)} \to \mathrm{Con}(\mathrm{ZFC} + U^*(\varkappa, \lambda, \omega_1)).$$

(ii) *Let θ be the first cardinal carrying a 2^ω-complete 2^ω-saturated ideal and $\lambda < \mu = \mathrm{cf}\,\mu \leqslant \varkappa$. Then*

$$\mathrm{Con(ZFC)} \to \mathrm{Con}(\mathrm{ZFC} + U(\varkappa, \lambda, \mu)).$$

Proof. (i) We force $\mathrm{MA} + 2^\omega = \varkappa$ and apply Theorem 1.3.

(ii) We force $\mathrm{MA} + 2^\omega = \mu$ and apply Theorem 1.4. ■

Our next remark refers to the countable case. It is a consequence of a result of R. Laver (cf. [3]), namely, that Con(ZFC + a measurable cardinal exists) implies Con(ZFC + 2^ω carries a 2^ω-complete σ-saturated ideal + $U(2^\omega, \lambda, \omega_1)$ for $\lambda < 2^\omega$). The proof, however, does not seem to generalize so as to allow the real-valued measurability of 2^ω even if $\lambda = \omega$. In view of that, consider

PROPOSITION 3.2. Con(ZFC + *a measurable cardinal exists*) $\to$ Con(ZFC + 2^ω *is real-valued measurable* + $U(2^\omega, \omega, \omega_1)$).

Proof. It is possible to make 2^ω real-valued measurable without any cardinal carrying a σ-complete σ-saturated ideal below. Hence our proposition follows from Theorem 2.2. ■

4. Problems. We close our paper with a list of open problems.

A. Is it possible to prove in ZFC that $U(2^\omega, \omega, \omega_1)$?

In view of Proposition 2.1 Problem A is equivalent to the question whether $\forall \alpha < 2^\omega\, U^*(\alpha, \omega, \omega_1)$? On the other hand, by Taylor's result and Corollary 2.4, we have $U(\varkappa, \omega, \varkappa)$ for all $\varkappa$ s.t. $2^\omega < \varkappa <$ 1st measurable cardinal. This yields the following problem, less general than A:

B. Is it possible to prove in ZFC that $U(\varkappa, \omega, \varkappa)$ for all $\varkappa < 2^\omega$?

Of course, in view of the Erdös–Alaoglu theorem, problems A and B are interesting only in the case where 2^ω is large.

What about results of the type: non $U^*(\varkappa, \varkappa, \omega_1)$? The only one known is due to Magidor (cf. [7]): if there exists a huge cardinal, then Con(non $U^*(\omega_3, \omega_3, \omega_1)$).

C. Is non $U^*(2^\omega, 2^\omega, \omega_1)$ consistent with ZFC?

Notice that if we change non $U^*(2^\omega, 2^\omega, \omega_1)$ into non $U(2^\omega, 2^\omega, \omega_1)$, problem C has an easy affirmative answer. Finally, notice that for all $\varkappa$: non $U(\varkappa, 2^\varkappa, \omega_1)$. This yields the following problem:

D. Is $U(\varkappa, \varkappa^+, \omega_1)$ consistent with ZFC for some $\varkappa$ (e.g. $\varkappa = 2^\omega$)?

References

[1] J. Brzuchowski, J. Cichoń and B. Węglorz, *Some applications of strongly Lusin sets*, to appear.
[2] T. G. McLaughlin, *Martin's Axiom and some classical constructions*, Bull. Austral. Math. Soc. 12 (1975).
[3] R. Laver, *A saturation property of ideals*, Comp. Math. 36.
[4] D. A. Martin and R. M. Solovay, *Internal Cohen extensions*, Ann. Math. Logic 2 (1970), pp. 143–178.
[5] A. Pelc, *Ideals on the real line and Ulam's problem*, Fund. Math. (to appear).
[6] W. Sierpiński, *Hypothèse du Continu*, Warszawa–Lwów 1935.
[7] A. D. Taylor, *On saturated sets of ideals and Ulam's problem*, Fund. Math. 109 (1980), pp. 37–53.
[8] — *On the cardinality of reduced products and the Boolean Algebra $\mathfrak{P}(\varkappa)/I$*, to appear.
[9] — *On the cardinality of $\mathfrak{P}(\varkappa)/I$*, handwritten pages.

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