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ON FILTERS OF ORDERED SEMIGROUPS

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In the present paper we deal with a problem concerning filters of ordered semigroups which has been proposed by N. Kehayopolu [2].

1. Preliminaries

Let S be an ordered (= partially ordered) semigroup (cf. [1]). We recall two definitions from [2].

1.1. Definition. A nonempty subset F of S is said to be a filter of S if it satisfies the following conditions:

(i) Whenever $s_i \in S$ $(i = 1, 2)$ and $s_1, s_2 \in F$, then both s_1 and s_2 belong to F.
(ii) If $f \in F, s \in S$ and $f \leqslant s$, then $s \in F$.

1.2. Definition. An equivalence relation σ on S is called a semilattice congruence if the following conditions are satisfied:

(i) Whenever $(a, b) \in \sigma$ and $c \in S$, then $(ac, bc) \in \sigma$ and $(ca, cb) \in \sigma$.
(ii) For each $a, b \in S$ the relations $(a, a^2) \in \sigma$ and $(ab, ba) \in \sigma$ are valid.

For each $a \in S$ we denote by $F(a)$ the filter in S which is generated by the element a. Next, we put

$$\mathcal{N} = \{(x, y) : x, y \in S \text{ and } F(x) = F(y)\}.$$

In [2] the question was proposed whether for each ordered semigroup S the following condition is valid:

(*) If σ is a semilattice congruence on S, then $\mathcal{N} \leqslant \sigma$.

We will show that the answer to this question is negative. Namely, it will be shown that there exists a linearly ordered semigroup which does not satisfy the condition (*).

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The notion of regular semilattice congruence on S will be introduced and it will be proved that S satisfies the condition $(*)$ if and only if the least semilattice congruence on S is regular (or, equivalently, if all semilattice congruences on S are regular).

2. THE REGULARITY CONDITION

First let us consider the following example.

2.1. Example. Let S be the multiplicative semigroup of all non-negative integers. Let $\leqslant$ be the natural linear order on S; next, let $\leqslant^d$ be the linear order on S which is dual to S. Put $S^d = (S, ., \leqslant^d)$; then S^d is a linearly ordered semigroup. If F is a filter on S^d, then $F = S$. Thus $\mathcal{N} = S \times S$. Let σ be the system of all ordered pairs (x, y) of elements of S such that either $x = 0 = y$ or $x \neq 0 \neq y$. Then σ is a semilattice congruence on S^d and $\sigma < \mathcal{N}$. Therefore S^d does not satisfy the condition $(*)$.

Again, let S be an ordered semigroup. We denote by $\mathcal{S}$ the set of all semilattice congruences on S. Let σ be a fixed element of $\mathcal{S}$ and $a \in S$. We put $\overline{a}(\sigma) = \{x \in S\colon (x, a) \in \sigma\}$; if no misunderstanding can occur, then we write $\overline{a}$ instead of $\overline{a}(\sigma)$. The symbol S/σ denotes, as usual, the semigroup $\{\overline{a}\colon a \in S\}$ with the multiplication $\overline{a_1}.\overline{a_2} = \overline{a_1 a_2}$.

2.2. Definition. A semilattice congruence σ on S will be said to be regular if, whenever $x, y \in S$ and $x \leqslant y$, then $\overline{x} = \overline{xy}$.

2.3. Example. Let S^d be as in 2.1. For positive integers x and y write $x|y$ if y is divisible by x; in the opposite case we write $x|'y$. For $a_1, a_2 \in S$ we put $(a_1, a_2) \in \sigma$ if some of the following conditions is satisfied:

(i) $a_1 = a_2 = 0$;

(ii) $a_1 \neq 0 \neq a_2$ and whenever p is a positive prime, then either $p|a_i$ for $i = 1, 2$, or $p|'a_i$ for $i = 1, 2$.

Then σ si a semilattice congruence on S^d which fails to be regular.

The set $\mathcal{S}$ is partially ordered in the obvious way; then $\mathcal{S}$ is a complete lattice. We denote by σ_0 the least element of $\mathcal{S}$.

2.4. Example. Let S^d and σ be as in 2.3. Then $\sigma = \sigma_0$.

The following assertion is easy to verify.

2.5. Lemma. *σ_0 is regular if and only if all elements of $\mathcal{S}$ are regular.*

2.6. Definition. (Cf. [2].) Let $\emptyset \neq I \subseteq S$. Assume that the following conditions are satisfied:

(i) $SI \subseteq I$ and $IS \subseteq I$.

(ii) If $a \in I, b \in S$ and $b \leqslant a$, then $b \in I$.
(iii) If $a, b \in S$ and $ab \in I$, then either $a \in I$ or $b \in I$.

Under these conditions I is said to be a prime ideal of S.

Let $T(S)$ be the set of all prime ideals of S. For each $I \in T(S)$ we put

$$\sigma_I = \{(x, y) \in S \times S\colon \text{ either } x, y \in I \text{ or } x, y \notin I\}.$$

2.7. Proposition. (Cf. [2].) *For each $I \in T(S), \sigma_I$ is a semilattice congruence on S. Next, $\mathcal{N}$ is a semilattice congruence on S and*

$$\mathcal{N} = \bigcap_{I \in T(S)} \sigma_I.$$

2.8. Lemma. *Let $I \in T(S)$. Then σ_I is a regular semilattice congruence.*

Proof. In view of 2.7, σ_I is a semilattice congruence. For $x \in S$ we denote $\overline{x}(\sigma_I) = \overline{x}$.

Let $x, y \in S, x \leqslant y$. If $\overline{x} = \overline{y}$, then (since σ_I is a semilattice congruence) the relation $\overline{x} = \overline{xy}$ holds.

Asume that $\overline{x} \neq \overline{y}$. Hence $\{x, y\}$ fails to be a subset of I and $\{x, y\} \cap I \neq \emptyset$. If $y \in I$, then x belongs to I as well, which is a contradiction. Hence $x \in I$ and so $xy \in I$; therefore $\overline{x} = I = \overline{xy}$. □

Let S_2 be a two-element semilattice $\{0, 1\}$ with $0 \wedge 1 = 0$; we view S_2 as a semigroup where the multiplication coincides with the operation $\wedge$.

2.9. Lemma. *Assume that σ_0 is regular. Then the condition $(*)$ is satisfied.*

Proof. Put $S/\sigma_0 = \overline{S}$. From the fact that σ_0 is an element of $\mathcal{S}$ we infer that $\overline{S}$ is a semilattice.

If $\operatorname{card} \overline{S} = 1$, then the condition $(*)$ obviously holds. Suppose that $\operatorname{card} \overline{S} > 1$. Then $\overline{S}$ is a subdirect product of semigroups S_j, where j runs over an appropriately chosen set J, and for each $j \in J$, S_j is isomorphic to S_2. For $\overline{x} \in \overline{S}$ and $j \in J$ we denote by $\overline{x}(j)$ the component of $\overline{x}$ in S_j.

Let $j \in J$; we put
$A_j = \{\overline{x} \in \overline{S}\colon \overline{x}(j) = 0\}, \quad B_j = \{\overline{x} \in \overline{S}\colon \overline{x}(j) = 1\},$
$\overline{\sigma_j} = \{(\overline{x}, \overline{y}) \in \overline{S} \times \overline{S}\colon \overline{x}, \overline{y} \in A_j \quad \text{or} \quad \overline{x}, \overline{y} \notin A_j\}.$

In view of the subdirect decomposition of $\overline{S}$ under consideration we infer that

$$\bigcap_{j \in J} \overline{\sigma}_j = \overline{\sigma}_{\min} 1$$

holds, where $\overline{\sigma}_{\min}$ is the minimal equivalence on $\overline{S}$.

For $j \in J$ we denote

$$A'_j = \{x \in S : \overline{x} \in A_j\}, \quad B'_j = \{x \in S : \overline{x} \in B_j\}.$$

Then A'_j is a nonempty subset of S and it satisfies the conditions (i), (iii) of 2.6. Let $a \in A'_j$, $b \in S$ and $b \leqslant a$. Since σ_0 is regular, we obtain $\overline{b} = \overline{a}\overline{b}$. Further we have $\overline{a} \in A_j$ and hence $\overline{a}\overline{b} \in A_j$. Therefore $\overline{b} \in A_j$ and so $b \in A'_j$. Thus the condition (ii) from 2.6 holds as well; we have verified that A'_j is an ideal of S. By similar steps we can verify that B'_j is a filter of S. Clearly $B'_j = S \setminus A'_j$.

Put $\sigma_j = \sigma_I$, where $I = A'_j$. In view of (1) we get

$$\bigcap_{j \in J} \sigma_j = \sigma_0 .$$

This yields that

$$\bigcap_{I \in T(S)} \sigma_I = \sigma_0 .2$$

Thus by virtue of 2.7 the condition (*) is satisfied. □

2.10. Lemma. *Assume that the condition (*) is satisfied. Then σ_0 is regular.*

Proof. Since (*) holds, in view of 2.7 the relation (2) is valid. For $x \in S$ and $I \in T(S)$ we put $\overline{x}^I = \{y \in S : (x, y) \in \sigma_I\}$.

Let $x, y \in S$ such that $x \leqslant y$. Then according to 2.8, $\overline{x}^I = \overline{x}^I \overline{y}^I = \overline{xy}^I$. By applying (2) we obtain $\overline{x} = \overline{xy}$ (these symbols concern the semilattice congruence σ_0), whence σ_0 is regular. □

2.11. Theorem. *Let S be an ordered semigroup. Then the condition (*) holds if and only if the least semilattice congruence on S is regular (or, equivalently, if all semilattice congruences are regular).*

Proof. This is a consequence of 2.9, 2.10 and 2.5. □

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