ON FINITE-DIMENSIONAL TERM STRUCTURE MODELS

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Abstract. In this paper we provide the characterization of all finite-dimensional Heath–Jarrow–Morton models that admit arbitrary initial yield curves. It is well known that affine term structure models with time-dependent coefficients (such as the Hull–White extension of the Vasicek short rate model) perfectly fit any initial term structure. We find that such affine models are in fact the only finite-factor term structure models with this property. We also show that there is usually an invariant singular set of initial yield curves where the affine term structure model becomes time-homogeneous. We argue that other than functional dependent volatility structures – such as local state dependent volatility structures – cannot lead to finite-dimensional realizations. Finally, our geometric point of view is illustrated by several examples.

1. Introduction

In this paper we provide the characterization of all finite-dimensional Heath–Jarrow–Morton (HJM) models that admit arbitrary initial yield curves. This is an extension and completion of a series of results obtained by Björk et al. [1, 3, 2], and [8, 9, 10, 18]. It is well known that affine term structure models with time-dependent coefficients (such as the Hull–White extension of the Vasicek short rate model [14]) perfectly fit any initial term structure. We find that such affine models are in fact the only finite-factor term structure models with this property, under some weak assumptions on the volatility structure. We also show that there is usually an invariant singular set of initial yield curves where the affine term structure model becomes time-homogeneous. This is again well known for the classical Vasicek [17] and Cox–Ingersoll–Ross (CIR) [4] short rate models, where the set of consistent inital curves is given explicitely by the model parameters.

Practitioners and academics alike have a vital interest in finite-factor term structure models, and the distinction of time-homogenous and inhomogeneous ones. According to [13] there are two groups of practitioners in the fixed income market.

Fund managers trade on the yield curve (buy and sell swaps at different maturities), trying to make money out of it. They do not believe that all the interest rate market quotes are "correct". Instead, they in general use a time-homogeneous two- or three-factor model, estimate the model parameters from long time series data, and then update the state variables (factors) each day to fit the current term structure. Hence the term structure is considered as a derivative based on more fundamental state variables (factors), such as in an equilibrium model. The discrepancies between the fitted term structure and the market prices are preceived as potential trading opportunities. For example, if the fitted curve is above the

Date: September 2001 (first draft); September 9, 2002 (this draft).

We thank Liuren Wu for his helpful comments on some practical aspects in fixed income markets.

two year and ten year swap rates, but is below the five year swap rate. Then one does a butterfly trade: receiving the five year rate (as one thinks it is high) and delivering the two year and ten year rates (as one thinks they are low compared to the five year rate). After this trade, one usually needs to wait for six months or longer for the rates to "reverse" (as predicted by the model) so that one can make money. Since this is a long term game, the model parameters must not change every day. Parameters have to be constant. If a parameter is time-varying, it is a factor and one needs to specify its dynamics so that one can make corresponding adjustments for the hedging. A state variable (factor) is time-varying, but since one has a stochastic model for its evolution, one can check on a daily basis whether its realized value lies within a statistical confidence interval or not.

Interest rate option traders, on the other hand, often take the quoted yield curve data, with minimal or no smoothing, as model input. To fit the observed yield curve perfectly, they allow some of the model parameters to be time-inhomogeneous. They intend to hedge away instantly all the risks on the yield curve and only worry about the risk in the implied volatility structure. Yet, low-dimensionality of the model is desirable, since the number of factors usually equals the number of instruments one needs to hedge in the model. And the daily adjustment of a large number of instruments becomes infeasible in practice due to transaction costs. Of course, the model factors have to represent tradable values. But this can usually be achieved by a coordinate transformation.

An HJM model for the forward curve, $x \mapsto r_t(x)$, is determined by the volatility structure, $r_t \mapsto (x \mapsto \sigma(r_t, x))$, and the market price of risk. Here $r_t(x)$ denotes the forward rate at time t for date $t+x$ (this is the Musiela [15] parameterization). That is, the price at time t of a zero-coupon bond maturing at date $T \geq t$ is given by

$$P(t,T) = e^{-\int_0^{T-t} r_t(x)\,dx}.$$

It is shown in [9] that essentially every HJM model can be realized as a stochastic equation

$$\begin{cases} dr_t = \left(\dfrac{d}{dx} r_t + \alpha_{HJM}(r_t)\right)\,dt + \sigma(r_t)\,dW_t \\ r_0 = r^*, \end{cases} \tag{1.1}$$

in a Hilbert space H_w of forward curves. We shall recapture the precise setup below in Section 2. The solution process (r_t) in general cannot be realized by a finite-dimensional state process. An HJM model is said to admit a *finite-dimensional realization (FDR)* at the initial forward curve r^* if, roughly speaking, there exists an m-dimensional diffusion state process Z (factors), for some $m \in \mathbb{N}$, and a map $\phi : \mathbb{R}^m \to H$ such that $r_t = \phi(Z_t)$. Notice that m, Z and ϕ depend on r^*. In fact, we shall be interested in those HJM models that admit an FDR of the same dimension at every initial curve, and this dimension is minimal in some sense.

One of the most basic examples for a finite-dimensional HJM model that fits any initial curve is the Hull–White extended Vasicek model [14] for the short rate $R_t := r_t(0)$,

$$dR_t = (b(t) - \beta R_t)\,dt + \rho\,dW_t, \quad R_0 = r^*(0).$$

Here W is a one-dimensional Wiener process defined on some stochastic basis $(\Omega, (\mathcal{F}_t), \mathcal{F}, \mathbb{Q})$, where $\mathbb{Q}$ is the risk-neutral measure. This example will be completely recaptured in Section 6.1, where also all the subsequent functions are explicitly given. The coefficients $\beta \geq 0$ and $\rho > 0$ are constant, and $b(t)$ is a function determined by the initial forward curve r^*. The corresponding volatility structure is constant,

$$\sigma(r, x) \equiv \sigma(x) = \rho e^{-\beta x}, \tag{1.2}$$

and the forward curve has an affine dependence on R_t,

$$r_t(x) = A_{HWV}(t, x) + B_{HWV}(x) R_t. \tag{1.3}$$

Whence a one-factor affine term structure model with time-dependent coefficients (strictly speaking, the realization is given by the two-dimensional process (t, R_t)). For those inital curves for which $b(t) \equiv b$ we obtain a time-homogeneous process (R_t) and $A_{HWV}(t, x) \equiv A_{HWV}(x)$. In this example we also can easily perform a coordinate transformation that leads to a different state process Z. Indeed (as in [2, Section 4.1]) let

$$dZ_t = -\beta Z_t\, dt + \rho\, dW_t, \quad Z_0 = 0.$$

Then we have

$$R_t = Z_t + e^{-\beta t} r^*(0) + \int_0^t e^{-\beta(t-s)} b(s)\, dt,$$

and hence

$$r_t(x) = \tilde{A}_{HWV}(t, x) + B_{HWV}(x) Z_t, \tag{1.4}$$

for some function $\tilde{A}_{HWV}$. Thus we obtained a simpler state-process Z with a similar (affine) functional form of $r_t(x)$ as in (1.3). But Z is not a tradable value and hence cannot be directly used for hedging.

There is a substantial literature providing sufficient conditions for the existence of finite-dimensional HJM models (see e.g. [3, 2] for further reference). A systematic study from a geometric point of view has been made by Björk et al. [1, 3, 2], and [8, 9, 10], see also [18]. In [3] Björk and Svensson give necessary and sufficient conditions for the existence of FDRs. Their key argument is the classical Frobenius theorem. Therefore they define a Hilbert space, $\mathcal{H}$, on which d/dx is a bounded linear operator. This space consists solely of entire analytic functions. It is well known however that the forward curves implied by a CIR short rate model are of the form $r_t(x) = A_{CIR}(x) + B_{CIR}(x) R_t$, where

$$A_{CIR}(x) = d\frac{e^{ax} - 1}{e^{ax} + c} \quad \text{and} \quad B_{CIR}(x) = \frac{b e^{ax}}{(e^{ax} + c)^2}, \tag{1.5}$$

for some $a, b, c > 0$ and $d \geq 0$ (as will be recaptured in Section 6.2). But A_{CIR} and B_{CIR} are not entire analytic functions since their extensions on $\mathbb{C}$ have a singularity at $x = (\ln c + i\pi)/a$. Hence the CIR forward curves do not belong to $\mathcal{H}$, whence the Björk–Svensson [3] setting is too narrow for the HJM framework.

In [10] we recently succeeded to overcome this difficulty. The aim of the present article is to bring these results to the attention of the financial community. We shall make clear why, under fairly general assumptions, finite-dimensional HJM models are necessarily affine, and provide a deeper understanding of some important examples.

The remainder of the paper is as follows. In Sections 2 and 3 we give the precise setup and definition for a (local) HJM model and an FDR around an initial curve, respectively. In Section 4 we recapture the main results (without proofs) from [10] and [11]. We only consider functional dependent volatility structures, which includes essentially all interesting examples. This is also justified by Section 5, where we show that a local state dependent volatility structure cannot admit an FDR, unless it is constant. In Section 6 we illustrate our results by explicit calculations for two-dimensional HJM models, which turn out to be either of Hull–White extended Vasicek or CIR type.

2. HJM Models

Let $(\Omega, (\mathcal{F}_t), \mathcal{F}, \mathbb{Q})$ be a filtered probability space satisfying the usual conditions, and $W = (W^1, \dots, W^d)$ a d-dimensional Wiener process, $d \geq 1$. We suppose that $\mathbb{Q}$ is the risk neutral measure for the subsequent models. Indeed, for the existence of an FDR it is irrelevant whether we are under the physical measure $\mathbb{P}$ or under $\mathbb{Q} \sim \mathbb{P}$ (see also [9, Remark 7.1.1]).

We follow [9], where r_t is regarded as an element in the Hilbert space of forward curves H_w. This space consists of absolutely continuous functions $h : \mathbb{R}_{\geq 0} \to \mathbb{R}$, and is equipped with the norm

$$\|h\|_w^2 := |h(0)|^2 + \int_{\mathbb{R}_{\geq 0}} |\partial_x h(x)|^2 w(x)\, dx,$$

where $w : \mathbb{R}_{\geq 0} \to [1, \infty)$ is a non-decreasing C^1-function such that $w^{-1/3}$ is integrable on $\mathbb{R}_{\geq 0}$. For example, $w(x) = e^{\alpha x}$ or $w(x) = (1 + x)^\alpha$, for $\alpha > 0$ or $\alpha > 3$, respectively. The closed subspace $H_{w,0}$ of H_w, consisting of functions $h \in H_w$ with $\lim_{x\to\infty} h(x) = 0$, has the property that

$$(g, h) \mapsto g \int h,$$

where $\int h$ denotes the definite integral $x \mapsto \int_0^x h(y)\, dy$, is a continuous bilinear operator from $H_{w,0} \times H_{w,0}$ into H_w. Hence whenever $\sigma = (\sigma_1, \dots, \sigma_d)$ is a (bounded and) locally Lipschitz continuous map from H_w into $H_{w,0}^d$ then the HJM drift term,

$$\alpha_{HJM}(h) = \sum_{i=1}^d \sigma_i(h) \int \sigma_i(h), \tag{2.1}$$

is a well-defined (bounded and) locally Lipschitz continuous map from H_w into H_w.

On H_w the (unbounded) closed operator $A = d/dx$ generates the strongly continuous semigroup (S_t) of right-shifts

$$S_t h = h(\cdot + t), \quad t \geq 0.$$

The same holds for the restriction A_0 of A to $H_{w,0}$. It can be shown that the domain of A is $D(A) = \{h \in H_w \cap C^1(\mathbb{R}_{\geq 0}) \mid \partial_x h \in H_w\}$ and similarly $D(A_0) = \{h \in H_{w,0} \cap C^1(\mathbb{R}_{\geq 0}) \mid \partial_x h \in H_{w,0}\}$. We consider $A^2 = A \circ A$ and all higher order powers of A and A_0. By induction, we have

$$D(A^\infty) := \bigcap_{n\geq 0} D(A^n) = \{h \in H_w \cap C^\infty(\mathbb{R}_{\geq 0}) \mid (\partial_x)^n h \in H_w,\ \forall n \geq 1\},$$

$$D(A_0^\infty) := \bigcap_{n\geq 0} D(A_0^n) = \{h \in H_{w,0} \cap C^\infty(\mathbb{R}_{\geq 0}) \mid (\partial_x)^n h \in H_{w,0},\ \forall n \geq 1\}.$$

It is clear that $D(A_0^\infty) \subset D(A^\infty)$. Equipped with the sequence of seminorms $p_n(h) = \sum_{i=0}^n \|A^i h\|_w$, $n \geq 0$, $D(A^\infty)$ and $D(A_0^\infty)$ become Fréchet spaces, and A acts as a bounded linear operator on them. We refer to [6] for the theoretical background. In fact, $D(A^\infty)$ and $D(A_0^\infty)$ are in some sense the largest subspaces of H_w and $H_{w,0}$, respectively, with this property. The functions from (1.3), (1.4) and (1.5), $A_{HWV}(t,\cdot)$, $\tilde{A}_{HWV}(t,\cdot)$, A_{CIR}, B_{HWV}, B_{CIR}, all lie in $D(A^\infty)$, the last two even in $D(A_0^\infty)$.

By a *solution* (r_t) of (1.1) we shall always mean a *continuous mild solution* (see [5, 9]). That is, an H_w-valued continuous adapted process (r_t) which satisfies

$$r_t = S_t r^* + \int_0^t S_{t-s}\alpha_{HJM}(r_s)\,ds + \sum_{i=1}^d \int_0^t S_{t-s}\sigma_i(r_s)\,dW_s^i. \tag{2.2}$$

In the classical HJM notation, where $f(t,T) = r_t(T-t)$, this becomes the familiar expression

$$f(t,T) = f(0,T) + \int_0^t \tilde{\alpha}_{HJM}(s,T)\,ds + \sum_{i=1}^d \int_0^t \tilde{\sigma}_i(s,T)\,dW_s^i, \quad 0 \leq t \leq T,$$

where $\tilde{\alpha}_{HJM}(s,T) := \alpha_{HJM}(r_s, T-s)$ and $\tilde{\sigma}_i(s,T) := \sigma_i(r_s, T-s)$. If (2.2) only holds locally for t replaced by $t \wedge \tau$, for some stopping time $\tau > 0$, then (r_t) is a *local solution*.

Since the volatility structure $\sigma = (\sigma_1, \ldots, \sigma_d)$ determines α_{HJM} by (2.1), the following terminology is justified.

Definition 2.1. *Let $\mathcal{U}$ be a convex open set in H_w. A* (local) HJM model in *$\mathcal{U}$ is a map $\sigma = (\sigma_1, \ldots, \sigma_d) : \mathcal{U} \to H_{w,0}^d$ such that* (1.1) *admits a unique $\mathcal{U}$-valued (local) solution for every initial curve $r^* \in \mathcal{U}$.*

It is shown in [9, Section 5.2] that $\sigma : \mathcal{U} \to H_{w,0}^d$ is a local HJM model in $\mathcal{U}$ if it is locally Lipschitz continuous. The restriction to $\mathcal{U}$ is convenient since it allows to incorporate important examples such as the CIR model, where $\mathcal{U}$ is the half space $\{h \in H_w \mid h(0) > 0\}$.

3. Finite-Dimensional Realizations

In this section we give the rigorous definition of an FDR as it was sketched in the introduction.

Let $\mathcal{U}$ be a convex open set in H_w and σ a local HJM model in $\mathcal{U}$. Let $r_0^* \in \mathcal{U} \cap D(A^\infty)$ and $n \in \mathbb{N}$.

Definition 3.1. *We say that σ admits an n*-dimensional realization *around r_0^* if there exists an open neighborhood V of r_0^* in $\mathcal{U} \cap D(A^\infty)$, an open set U in $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$, and a C^∞-map $\alpha : U \times V \to \mathcal{U} \cap D(A^\infty)$ such that*

i) *$r \in \alpha(U, r)$ for all $r \in V$,*
ii) *$D_z\alpha(z,r) : \mathbb{R}^n \to D(A^\infty)$ is injective for every $(z,r) \in U \times V$,*
iii) *$\alpha(z_1, r_1) = \alpha(z_2, r_2)$ implies $D_z\alpha(z_1,r_1)(\mathbb{R}^n) = D_z\alpha(z_2,r_2)(\mathbb{R}^n)$ for all $(z_i, r_i) \in U \times V$,*
iv) *for every $r^* \in V$ there exists a U-valued diffusion process Z and a stopping time $\tau > 0$ such that*

$$r_{t\wedge\tau} = \alpha(Z_{t\wedge\tau}, r^*) \tag{3.1}$$

is the (unique) local solution of (1.1) *with $r_0 = r^*$.*

Thus we only consider FDRs in $\mathcal{U} \cap D(A^\infty)$. This seems to be a restriction first, since the original local HJM model is defined on $\mathcal{U}$. However, we shall see in Proposition 4.3 below that in most interesting cases the FDRs are necessarily in $\mathcal{U} \cap D(A^\infty)$. Also we should not worry about $U \subset \mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$. It will become clear that the first component of the diffusion Z can always be chosen to be the time $t \geq 0$.

Definition 3.1 states that an n-dimensional realization around r_0^* implies the existence of an FDR at every point r^* in a neighborhood of r_0^*, and these FDRs have a smooth dependence on r^*. In fact, by i) and ii), each $\alpha(\cdot, r^*) : U \to \mathcal{U} \cap D(A^\infty)$ is (after a localization) the parametrization of an n-dimensional submanifold with boundary, say $\mathcal{M}_{r^*}$, of $\mathcal{U} \cap D(A^\infty)$, and (3.1) says that

$$r_{t\wedge\tau} \in \mathcal{M}_{r^*}, \quad \forall t \geq 0.$$

Condition iii) implies that two such leafs $\mathcal{M}_{r_1}$ and $\mathcal{M}_{r_2}$ can only intersect at points where their tangent spaces coincide. According to [10], the family $\{\mathcal{M}_r\}_{r\in V}$ is called an n-dimensional *weak foliation* on V.

The existence of FDRs around a point is assured by an extended version of the Frobenius theorem (see [10] and [12]) on the Fréchet space $D(A^\infty)$. The Frobenius theorem has also been used by Björk et al. [3, 2] on the Hilbert space $\mathcal{H}$, which however has the drawbacks mentioned in the introduction.

A striking feature of the Frobenius theorem is that it brings together an algebraic condition (dimension of a Lie-algebra) with the analytic problem of the existence of an FDR (weak foliation). The former condition can in many cases be explicitly checked, as it is exemplarily carried out in Section 5. We do not intend to go further into this theory here, but cite the main Theorem and refer the interested reader to [10] and [12]. For the calculation of Lie brackets see also Section 5.

Theorem 3.2. *Let E be a Fréchet space and $X^1, \dots, X^m$ be smooth vector fields on an open subset $U \subset E$. Assume that the distribution generated by*

$$X^1, \dots, X^m, [X^i, X^j], [X^i, [X^j, X^k]], \dots$$

for $i, j, k = 1, \dots, m$, where $[.,.]$ denotes the Lie bracket of vector fields, is locally spanned by $Y^1, \dots, Y^n$ on an open neighborhood $V \subset U$ of r^*, i.e. every (iterated) Lie bracket of the X^i is a linear combination of the vector fields $Y^1, \dots, Y^n$ on V (the rank is locally constant n).

If $Y^1, \dots, Y^{n-1}$ admit local flows and Y^n admits a local semiflow on V, then there exists a weak foliation $\{\mathcal{M}_r\}_{r\in V'}$ with $r^* \in V' \subset V$ open, of dimension n, such that $X^1, \dots, X^m$ are tangent, i.e. for all $r \in V'$ and $r_0 \in \mathcal{M}_r$ we have $X^i(r_0) \in T_{r_0}\mathcal{M}_r$.

The dimension n is the minimal dimension such that the vector fields $X^1, \dots, X^m$ are tangent to a weak foliation if there is one (which need not be the case).

Under the appropriate assumptions on the volatility structure (in particular, $\sigma : \mathcal{U} \cap D(A^\infty) \to D(A_0^\infty)^d$ has to be C^∞), we can apply Theorem 3.2. If we choose $m = d + 1$ and $X^i = \sigma^i$ for $i = 1, \dots, d$ and $X^{d+1} = \mu$ with

$$\mu(r) := Ar + \alpha_{HJM}(r) - \frac{1}{2}\sum_{i=1}^{d} D\sigma_i(r)\sigma_i(r), \tag{3.2}$$

we can prove the existence or non-existence of n-dimensional realizations at r^* for a given volatility structure σ. Here also the results of [8] are applied, where the appropriate tangent conditions are proved.

A *necessary* condition for the existence of an n-dimensional realization around r_0^* is that the distribution generated by the vector fields μ and $\sigma_1, \dots, \sigma_d$ on $\mathcal{U} \cap D(A^\infty)$ is at most n-dimensional (see [12, Section 3]).

This algebraic condition typically yields an obstruction for the examined model to admit FDRs, as it is applied in Section 5.

4. Functional Dependent Volatility

In this section we have the idea of σ being sensitive with respect to linear functionals of the forward curve. That is, we let $\sigma : \mathcal{U} \to H^d_{w,0}$ be a local HJM model in $\mathcal{U}$ and suppose in addition that $\sigma_i(r) = \phi_i(\ell_1(r), \dots, \ell_p(r))$, for some $p \in \mathbb{N}$, where $\phi_i : \mathcal{O} \subset \mathbb{R}^p \to D(A_0^\infty)$ are smooth maps and $\ell_1, \dots, \ell_p$ are continuous linear functionals on H_w. This includes essentially all interesting examples. Indeed, we may think of

$$\ell_i(r) = \frac{1}{x_i} \int_0^{x_i} r(y)\, dy \quad \text{(benchmark yields)}$$

or

$$\ell_i(r) = r(x_i) \quad \text{(benchmark forward rates)},$$

for some fixed tenor $x_1, \dots, x_p \geq 0$. For a short rate model we simply have $p = 1$ and $\ell(r) = r(0)$ (see (1.2) for the trivial constant case $\ell(r) \equiv 1$ and $\phi(z) \equiv \rho \exp(-\beta \cdot))$. This idea is formalized by the following regularity and non-degeneracy assumptions

- **(A1):** $\sigma_i = \phi_i \circ \ell$ where $\ell \in L(H_w, \mathbb{R}^p)$, for some $p \in \mathbb{N}$, $\phi_i : \mathcal{O} \to D(A_0^\infty)$ are C^∞-maps, for $1 \leq i \leq d$, and $\mathcal{O}$ is an open set in $\mathbb{R}^p$ containing $\ell(\mathcal{U})$.
- **(A2):** $\phi_1(z), \dots, \phi_d(z)$ are linearly independent, for all $z \in \mathcal{O}$.
- **(A3):** The linear map $(\ell, \ell \circ A, \dots, \ell \circ A^q) : D(A^\infty) \to \mathbb{R}^{p(q+1)}$ is open, for every finite $q \geq 0$.

Assumptions **(A1)** and **(A2)** are clear, only **(A3)** needs some further explanation. Intuitively, it says that the following *interpolation problem* is well-posed on $D(A^\infty)$: given a smooth curve $g : \mathbb{R}_{\geq 0} \to \mathbb{R}$, for any finite number of data of the form $\mathcal{I} = (\ell(g), \ell(\partial_x g), \dots, \ell((\partial_x)^q g)) \in \mathbb{R}^{p(q+1)}$ we can find an interpolating function $h \in D(A^\infty)$ with $(\ell(h), \dots, \ell \circ A^q(h)) = \mathcal{I}$. Hence degenerate examples such as $p = 3$ and $\ell(h) = (h(0), h(1), \int_0^1 h(x)\, dx)$ are excluded. Indeed, here we have $\ell \circ A(h) = (\partial_x h(0), \partial_x h(1), h(1) - h(0))$. Thus the rank of $(\ell, \ell \circ A) : D(A^\infty) \to \mathbb{R}^6$ is 5, whence this map is not open. Assumption **(A3)** is always satisfied for short rate models, since there $p = 1$ and $\ell(h) = h(0)$.

Example 4.1. *Constant direction volatility.* Let $\lambda_1, \dots, \lambda_d$ be fixed linearly independent vectors in $D(A_0^\infty)$, and

$$\sigma_i = \phi_i \circ \ell \quad \text{with} \quad \phi_i(z) = \sum_{j=1}^d \Phi_{ij}(z) \lambda_j, \quad z \in \mathcal{O}, \tag{4.1}$$

for some C^∞-map $\Phi = (\Phi_{ij})$ on $\mathcal{O}$ with values in the regular $d \times d$-matrices. This example has been extensively studied by Björk et al. [3, 2].

The vital importance of this example will be made clear by Proposition 4.4 and Theorems 4.6 and 5.1 below.

Now let σ satisfy **(A1)**–**(A3)**. This implies in particular that $\sigma : \mathcal{U} \to H^d_{w,0}$ is locally Lipschitz continuous and hence a local HJM model in $\mathcal{U}$.

Let $r_0^* \in \mathcal{U} \cap D(A^\infty)$ and $n \in \mathbb{N}$. There is a lower bound for the dimension of an FDR around r_0^*.

Proposition 4.2. *Suppose σ admits an n-dimensional realization around r_0^*. Then $n \geq d + 1$.*

Proof. This is [10, Proposition 4.8]. □

Hence there exist no one-dimensional realizations around r_0^*, even if $d = 1$. This confirms the well-known fact that a short rate model (R_t) that fits every initial curve r^* in a neighborhood of r_0^* in $D(A^\infty)$ contains necessarily some time-dependent parameters. Such that $Z_t = (t, R_t)$ yields a 2-dimensional realization around r_0^*. This also proves a conjecture in [3] (see Remark 7.1 therein).

Since low-dimensionality of the state process is preferred, it would not make much sense to look for n-dimensional realizations with $n > d + 1$ (although this can be done under some non-degeneracy assumptions, as it is carried out in [10]). Rather we restrict our attention to *minimal* (that is, $(d + 1)$-dimensional) realizations around r_0^*. In this case our focus on FDRs in $D(A^\infty)$ turns out to be no restriction at all.

Proposition 4.3. *Suppose $\mathcal{M}$ is a $(d + 1)$-dimensional C^∞-submanifold (with boundary) of $\mathcal{U}$ that is locally invariant for* (1.1)*. That is, for every $r^* \in \mathcal{M}$, the $\mathcal{U}$-valued local solution (r_t) of* (1.1) *satisfies*

$$r_{t\wedge\tau} \in \mathcal{M}, \quad \forall t \geq 0,$$

for some stopping time $\tau > 0$. Then necessarily $\mathcal{M} \subset D(A^\infty)$, and $\mathcal{M}$ is a C^∞-submanifold (with boundary) of $D(A^\infty)$.

Proof. This is [11, Theorem 3.2]. □

We now can cite [10, Theorem 4.10].

Proposition 4.4. *Suppose σ admits a $(d + 1)$-dimensional realization around r_0^*. Then σ is of the form* (4.1) *(constant direction volatility) on V, where V is given by Definition 3.1.*

So far our considerations were local. Now let U denote the set of all $r_0^* \in \mathcal{U} \cap D(A^\infty)$ around which σ admits a $(d + 1)$-dimensional realization. Is it possible that $U = \mathcal{U} \cap D(A^\infty)$? In general the answer is no. Indeed, suppose U is connected. Then, by a simple continuity argument, σ is of the form (4.1) with the same $\lambda_1, \dots, \lambda_d$ everywhere on U. It then follows from [10, Theorem 4.10] that U must not intersect with the singular set

$$\Sigma := \{h \in \mathcal{U} \cap D(A^\infty) \mid \nu(h) \in \langle\lambda_1, \dots, \lambda_d\rangle\}, \tag{4.2}$$

where

$$\nu(h) := Ah + \alpha_{HJM}(h) \tag{4.3}$$

and $\langle\lambda_1, \dots, \lambda_d\rangle$ denotes the d-dimensional subspace spanned by $\lambda_1, \dots, \lambda_d$. The set Σ is not empty in general, but it is small.

Lemma 4.5. *The set Σ is closed and lies in a subspace G of $D(A^\infty)$ with $(d+1) \leq \dim G \leq d^2 + d + 1$.*

Proof. This follows from [10, Lemmas 4.5 and 4.11]. □

We now can summarize and state the main result, which says that only affine term structure models admit a minimal FDR around any initial curve, see [10, Theorem 4.13].

Theorem 4.6. *Suppose there exists an open connected set U in $\mathcal{U} \cap D(A^\infty)$ such that σ admits a $(d+1)$-dimensional realization around every $r_0^* \in U$. Then there exist linearly independent vectors $\lambda_1, \dots, \lambda_d$ in $D(A_0^\infty)$ such that σ is of the form (4.1) on U, and $U \cap \Sigma = \emptyset$, where Σ is given by (4.2). Moreover, for every $r^* \in U$ there exists $\epsilon > 0$ and a C^∞-map $\Psi(\cdot, r^*) : [0, \epsilon) \to \mathcal{U} \cap D(A^\infty)$ with $\Psi(0, r^*) = r^*$, an $\mathbb{R}^d$-valued diffusion process Z with $Z_0 = 0$ and a stopping time $0 < \tau \le \epsilon$ such that*

$$r_{t\wedge\tau} = \Psi(t \wedge \tau, r^*) + \sum_{i=1}^{d} Z^i_{t\wedge\tau} \lambda_i \tag{4.4}$$

is the $\mathcal{U}$-valued local solution of (1.1). Moreover, $\Psi(\cdot, r^*)$ is the unique solution to the evolution equation in $D(A^\infty)$, see (3.2),

$$\frac{d}{dt} u(t) = \mu(u(t)), \quad u(0) = r^*. \tag{4.5}$$

If, in addition, $U = \mathcal{U} \cap D(A^\infty) \setminus \Sigma$ then for every $r^* \in \Sigma$ there exists an $\mathbb{R}^d$-valued time-homogeneous diffusion process Z with $Z_0 = 0$ and a stopping time $0 < \tau \le \epsilon$ such that

$$r_{t\wedge\tau} = r^*(t \wedge \tau) + \sum_{i=1}^{d} Z^i_{t\wedge\tau} \lambda_i \tag{4.6}$$

is the $\mathcal{U}$-valued local solution of (1.1), and $r_{t\wedge\tau} \in \Sigma$ for all $t \ge 0$. Hence Σ is locally invariant for (1.1).

Thus here we have the announced decomposition of the space of initial forward curves $\mathcal{U} \cap D(A^\infty)$ into U, where the (affine) factor model becomes time-inhomogeneous, and the singular set Σ, which is invariant for the model dynamics and where the (affine) factor model becomes time-homogeneous. This phenomenon was known for some particular models, and it now is proved in full generality. We shall further illustrate this result for the case $d = 1$ in Section 6.

Remark 4.7. *The vector field μ in (4.5) can be replaced by ν and the statement about the affine form (4.4) remains true if τ is chosen such that $r_{t\wedge\tau}$ is $\overline{U} \cap \mathcal{U}$-valued. This follows since $\mu(h) - \nu(h) \in \langle \lambda_1, \dots, \lambda_d \rangle$ for all $h \in \overline{U} \cap \mathcal{U}$ (= $\mathcal{U}$ if $U = \mathcal{U} \setminus \Sigma$), which yields a straightforward modification of Ψ and Z in (4.4).*

5. Local state dependent volatilities

Continuing the discussion at the end of Section 3, we are given smooth vector fields $X_1, ..., X_m$ on $\mathcal{U} \cap D(A^\infty)$. If the generated distribution of Theorem 3.2 has rank n, then in particular for any $i, j = 1, ..., m$ there exist smooth functions $\lambda_1, ..., \lambda_n : \mathcal{U}_0 \to \mathbb{R}$, such that

$$[X_i, X_j](h) = \sum_{l=1}^{m} \lambda_l(h) Y_l(h)$$

for h in some smaller domain $\mathcal{U}_0 \subset \mathcal{U} \cap D(A^\infty)$. In this section we shall work out a simple case with $n = m = 2$.

We recapture some facts, which have been derived in [10]. The calculation of Lie brackets can be performed on the space $C^\infty(\mathbb{R}_{\geq 0}, \mathbb{R})$, since $D(A^\infty)$ is a subspace of $C^\infty(\mathbb{R}_{\geq 0}, \mathbb{R})$ and the vector fields that we consider are defined on the latter. Lie brackets are calculated by Fréchet derivatives

$$[X_i, X_j](h) = DX_i(h) \cdot X_j(h) - DX_j(h) \cdot X_i(h),$$

and calculating Fréchet derivatives is equivalent to derivation of $D(A^\infty)$-valued curves

$$DX_i(h) \cdot v = \frac{d}{dt}|_{t=0} X_i(h + tv).$$

We want to demonstrate that local volatilities cannot lead to finite-dimensional realizations, since the algebraic conditions are not satisfied. For simplicity we shall assume that $d = 1$. Given $\phi : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$, we define $\sigma(h)(x) = \phi(x, h(x))$ for $h \in \mathcal{U} \cap D(A^\infty)$ and $x \geq 0$. By cartesian closedness (see [10]) we obtain for the Fréchet derivative

$$\begin{aligned}(D\sigma(h) \cdot v)(x) &= \frac{d}{dt}|_{t=0} \phi(x, h(x) + tv(x)) \\ &= \phi'(x, h(x))v(x)\end{aligned}$$

for any $x \geq 0$. Consequently $D\sigma(h) \cdot v = (\phi' \circ h)v$, where we have an ordinary multiplication of functions. With $\circ$ we denote the composition with respect to the second variable and with $.'$ we denote derivation with respect to the second variable. Therefore

$$\mu(h) = \frac{d}{dx} h + (\phi \circ h)(\int \phi \circ h) - \frac{1}{2}((\phi)' \circ h)(\phi \circ h),$$

see (3.2). Conditions on ϕ can be found such that all parts of the drift vector field and the local volatility are smooth maps.

Theorem 5.1. *Assume that $[\mu, \sigma](h) \in \langle \sigma(h), \mu(h) \rangle$ for h in some open set $\mathcal{U}_0 \subset \mathcal{U} \cap D(A^\infty)$. Then $\sigma(h)(x) = \rho \exp(-\beta x)$ for $h \in \mathcal{U}_0$, for some constants ρ and β, which is the Vasicek volatility structure.*

Proof. We have to calculate one Lie bracket:

$$\begin{aligned}D\mu(h) \cdot v = \frac{d}{dx} v + ((\phi' \circ h)v)(\int_0^{\cdot} \phi \circ h) + (\phi \circ h)(\int_0^{\cdot} (\phi' \circ h)v) - \\ - \frac{1}{2}(\phi'' \circ h)(\phi \circ h)\nu - \frac{1}{2}(\phi' \circ h)^2 v\end{aligned}$$

for $h \in \mathcal{U} \cap D(A^\infty)$ and $v \in D(A^\infty)$. The derivative with respect to the first variable of ϕ is denoted by ∂_1. This leads to

$$\begin{aligned}[\mu, \sigma](h) = (\phi' \circ h)\frac{d}{dx} h + (\partial_1 \phi \circ h) + (\phi' \circ h)(\phi \circ h)\left(\int \phi \circ h\right) + \\ + (\phi \circ h)\left(\int (\phi' \circ h)(\phi \circ h)\right) - \frac{1}{2}(\phi'' \circ h)(\phi \circ h)(\phi \circ h) - \frac{1}{2}(\phi' \circ h)^2(\phi \circ h) - \\ - (\phi' \circ h)\frac{d}{dx} h - (\phi' \circ h)(\phi \circ h)\left(\int (\phi \circ h)\right) + \frac{1}{2}(\phi' \circ h)(\phi' \circ h)(\phi \circ h) \\ = (\partial_1 \phi \circ h) + (\phi \circ h)\left(\int (\phi' \circ h)(\phi \circ h)\right) - \frac{1}{2}(\phi'' \circ h)(\phi \circ h)^2.\end{aligned}$$

We now shall evaluate the equation

$$[\mu,\sigma](h)=\lambda(h)\sigma(h)+\lambda_0(h)\mu(h)$$

for $h\in\mathcal{U}_0$. Assume that $\lambda_0(h_\infty)\neq 0$ for some $h_\infty\in\mathcal{U}_0$. We can isolate $\frac{d}{dx}h$, then we take a sequence $(h_n)_{n\geq 0}$ in $\mathcal{U}_0$ with $h_n\to h_\infty$ in $D(A^m)$ as $n\to\infty$ but not in $D(A^{m+1})$, where m is chosen such that $\lambda_0(h_n)\to\lambda_0(h_\infty)$ (this is possible by applying Theorem 2.3 of [10] and adjusting $\mathcal{U}_0$). Hence the only term which does not converge up to order m is $\frac{d}{dx}h_n$. Consequently $\lambda_0=0$ on $\mathcal{U}_0$. Next we analyze the resulting equation

$$[\mu,\sigma](h)=\lambda(h)\sigma(h),$$

where we proceed by the same reasoning.

First we assume that we are given a point $h_0\in\mathcal{U}_0$ and $x_0\geq 0$ such that $\phi(x_0,h_0(x_0))\neq 0$, then we can divide by $\sigma(h)(x)$ for h in a neighborhood of h_0 and x in a neighborhood of x_0 and identify a logarithmic derivative. To this equation we apply again the operator $\frac{d}{dx}$ and obtain

$$\begin{aligned}(\partial_1^2\ln|\phi|\circ h)+(\partial_1(\ln|\phi|)'\circ h)h'+(\phi'\circ h)(\phi\circ h)-\frac{1}{2}(\partial_1\phi''\circ h)(\phi\circ h)-\\ -\frac{1}{2}(\phi'''\circ h)h'(\phi\circ h)-\frac{1}{2}(\phi''\circ h)(\partial_1\phi\circ h)-\frac{1}{2}(\phi''\circ h)(\phi'\circ h)h'=0.\end{aligned}$$

We can again isolate $h'=\frac{d}{dx}h$, which leads to a contradiction as before, therefore its coefficient has to vanish identically. This leads to the following two equations:

$$(\partial_1^2\ln|\phi|\circ h)+(\phi\circ h)(\phi'\circ h)-\frac{1}{2}(\partial_1\phi''\circ h)(\phi\circ h)-(\phi''\circ h)(\partial_1\phi\circ h)=0$$

and

$$(\partial_1(\ln|\phi|)'\circ h)-\frac{1}{2}(\phi'''\circ h)(\phi\circ h)^2-(\phi''\circ h)(\phi'\circ h)=0.$$

We take these two equations and evaluate them for h in a neighborhood of h_0 and x in a neighborhood of x_0, which leads to the equations

$$\begin{aligned}\partial_1^2\ln|\phi|+\phi\phi'-\frac{1}{2}\partial_1^2(\phi''\phi)&=0\\ \partial_1(\ln|\phi|)'-\frac{1}{2}(\phi''\phi)'&=0.\end{aligned}$$

Taking the derivative $.'$ in the first and ∂_1 in the second and finally the difference of the resulting equations we obtain

$$(\phi\phi')'=0$$

and therefore $\phi(x,y)\frac{\partial}{\partial y}\phi(x,y)=g(x)$ for (x,y) in a neighborhood of $(x_0,h_0(x_0))$ with some smooth function $g:\mathbb{R}_{\geq 0}\to\mathbb{R}$. This equation has a smooth solution if and only if $\frac{\partial}{\partial y}\phi(x,y)=0$. Therefore $\partial_1^2\ln|\phi|=0$, which leads to $\phi(x,y)=\rho\exp(-\beta x)$ for (x,y) in a neighborhood of $(x_0,h_0(x_0))$. By continuity we obtain the global result. □

Remark 5.2. *The same method can be applied to the d-dimensional case. This leads to a similar assertion, however, the calculus is rather ambitious.*

6. Characteristic Examples

In the seminal papers [3] and [2] finite-dimensional realizations, in particular the Hull-White extensions of the Vasicek and CIR-model, are considered for the first time from the geometric point of view. In addition to their excellent treatment (compare Section 5 of [2] or Section 7 of [3]), we prove that the Hull-White extensions of the Vasicek and CIR model are the only 2-dimensional local HJM models and we demonstrate the importance of the corresponding singular sets. The same type of analysis can also be performed in higher dimensional cases, which will be done elsewhere. At the end of this section we provide an example of how to embed the Svensson family as a leaf of a weak foliation associated to a functional dependent volatility structure.

We let again $d = 1$. Starting with a functional dependent volatility structure σ and $\mathcal{U}$ as in Section 4 and assuming the existence of 2-dimensional realizations on $\mathcal{U} \cap D(A^\infty) \setminus \Sigma$ (see (4.2)), we necessarily arrive by Proposition 4.4 at a constant direction volatility on $\mathcal{U}$. We shall show that this volatility is either of the Vasicek or CIR type.

In view of **(A2)** we have $\sigma \neq 0$ on $\mathcal{U}$, hence we can write $\sigma(r) = \phi(r)\lambda$ for $r \in \mathcal{U}$, for some $\lambda \in D(A_0^\infty) \setminus \{0\}$ and a smooth map $\phi : \mathcal{U} \to \mathbb{R}$, such that without loss of generality $\phi > 0$ (by a slight abuse of notation, the meaning of ϕ here is different from Section 4). We want to specify under which conditions this volatility structure admits 2-dimensional realizations and how they look like. This is already done in Section 7.3 of [3], however, their special setting does not allow to treat the CIR-case.

Writing $\psi(r) := \phi(r)(D\phi(r) \cdot \lambda)$, we obtain for $r \in \mathcal{U} \cap D(A^\infty)$

$$
\begin{aligned}
D\sigma(r) \cdot h &= (D\phi(r) \cdot h)\lambda \\
D\sigma(r) \cdot \sigma(r) &= \phi(r)(D\phi(r) \cdot \lambda)\lambda = \psi(r)\lambda \\
\mu(r) &= \frac{d}{dx} r + \phi(r)^2 \lambda \int \lambda - \frac{1}{2}\psi(r)\lambda \\
D\mu(r) \cdot h &= \frac{d}{dx} h + 2\phi(r)(D\phi(r) \cdot h)\lambda \int \lambda - \frac{1}{2}(D\psi(r) \cdot h)\lambda.
\end{aligned}
$$

Consequently we can calculate the Lie bracket

$$
\begin{aligned}
[\mu, \sigma](r) = {} & \phi(r)\frac{d}{dx}\lambda + 2\phi(r)\psi(r)\lambda \int \lambda - \frac{1}{2}\phi(r)(D\psi(r) \cdot \lambda)\lambda - \\
& - (D\phi(r) \cdot \frac{d}{dx} r)\lambda - \phi(r)^2 (D\phi(r) \cdot \lambda \int \lambda)\lambda + \frac{1}{2}\psi(r)(D\phi(r) \cdot \lambda)\lambda.
\end{aligned}
$$

We assume $[\mu, \sigma](r) \in \langle \lambda \rangle$ on $\mathcal{U} \cap D(A^\infty)$, which follows from the Frobenius condition and is justified by Lemmas 2.12 and 3.4 of [10]. We can divide by $\phi(r)$ and obtain an equation

$$
\frac{d}{dx}\lambda + 2\psi(r)\lambda \int \lambda - \theta(r)\lambda = 0
$$

with some smooth function $\theta : \mathcal{U} \cap D(A^\infty) \to \mathbb{R}$. There are consequently two cases:

i) If λ and $\lambda \int \lambda$ are linearly independent in $D(A^\infty)$, then by derivation with respect to r we obtain that ψ and θ are constant, say $2\psi(r) = a$ and $\theta(r) = b$ with real numbers a and b. Defining $\Lambda := \int \lambda$ we obtain finally a Riccati equation for Λ, which yields the CIR-type if $a \neq 0$ or the Vasicek-type if

$a = 0$:

$$\frac{d}{dx}\Lambda + \frac{a}{2}\Lambda^2 + b\Lambda = \lambda(0), \quad \Lambda(0) = 0. \tag{6.1}$$

The Ho-Lee model is considered as particular case of the Vasicek model for $b = 0$.

ii) If λ and $\lambda \int \lambda$ are linearly dependent in $D(A^\infty)$, then we necessarily obtain an equation of the type

$$\frac{d}{dx}\lambda + b\lambda = 0,$$

which yields that λ is vanishes identically, since otherwise λ and $\lambda \int \lambda$ are linearly independent. This case was excluded at the beginning.

Notice that by (6.1), $\lambda(0) = 0$ if and only if $\lambda = 0$, which is not possible. Hence a fortiori we have $\lambda(0) \neq 0$, such that by rescaling we always can assume that $\lambda(0) = 1$. This observation slightly improves the discussion in Section 7.3 in [3].

By the definition of ψ we have $D\phi^2(r) \cdot \lambda = a$, hence we obtain the following representation for ϕ. We split $D(A^\infty)$ into $\mathbb{R}\lambda + E$, where $E := \ker ev_0$. We denote by $pr : D(A^\infty) \to E$ the corresponding projection. Then

$$\phi(h) = \sqrt{a ev_0(h) + \eta(pr(h))}, \tag{6.2}$$

where $\eta : pr(\mathcal{U} \cap D(A^\infty)) \subset E \to \mathbb{R}$ is a smooth function (compare with Proposition 7.3 of [3]).

Recalling (4.2), we have

$$\Sigma = \left\{ h \in \mathcal{U} \cap D(A^\infty) \mid \nu(h) = Ah + \phi(h)^2 \lambda \int \lambda \in \langle \lambda \rangle \right\}.$$

Thus, if λ and $\lambda \int \lambda$ are linearly independent in $D(A^\infty)$ then any $h \in \Sigma$ is necessarily of the form

$$h = a_1 + a_2\Lambda^2 + a_3\Lambda$$

in all cases for some real numbers a_i. By the particular representation of ϕ we obtain that $a_2 = aa_1 + g(a_3)$, where g is some smooth real function derived from

$$aa_1 + \eta(a_2\Lambda^2 + a_3\Lambda) = a_2.$$

By Fl^X we denote the local (semi-)flow of a vector field X on $\mathcal{U} \cap D(A^\infty)$. The leaves through r^* of the weak foliation are given by the local parametrization

$$(u_0, u_1) \mapsto Fl^\nu_{u_0}(r^*) + u_1 \frac{d}{dx}\Lambda$$

if r^* does not lie in the singular set Σ. If $r^* \in \Sigma$, then the leaf is a one dimensional immersed submanifold of $\langle 1, \Lambda, \Lambda^2 \rangle$. Notice that the stochastic evolution of the factor process takes place in the u_1-component, see Theorem 4.6 and Remark 4.7.

We summarize the preceding results in the following theorem.

Theorem 6.1. *Let σ and $\mathcal{U}$ be as above. Assume that σ admits a 2-dimensional realization around any initial curve $r^* \in \mathcal{U} \cap D(A^\infty) \setminus \Sigma$. Then there exists $\lambda \in D(A_0^\infty)$ and a function $\phi : \mathcal{U} \to \mathbb{R}_{>0}$ of the form* (6.2) *such that $\sigma(h) = \phi(h)\lambda$. The singular set Σ is a (possibly empty) subset of $\langle 1, \Lambda, \Lambda^2 \rangle$, where $\Lambda = \int \lambda$ satisfies the Riccati equation* (6.1)*. The local HJM model is an affine short rate model. That* *is, for every initial curve $r^* \in \mathcal{U} \cap D(A^\infty)$ there exist functions $b : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$, $\theta : \mathbb{R}_{\geq 0} \to \mathbb{R}$, $A : [0, \epsilon) \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ and a stopping time $\tau > 0$ such that*

$$r_{t\wedge\tau}(x) = A(t \wedge \tau, x) + \lambda(x) R_{t\wedge\tau} \tag{6.3}$$

is the unique $\mathcal{U}$-valued local solution to (1.1) and the short rates $R_t = r_t(0)$ follow, locally for $t \leq \tau$, a time-inhomogeneous diffusion process

$$dR_t = b(t, R_t)\, dt + \sqrt{aR_t + \theta(t)}\, dW_t.$$

This process becomes time-homogeneous if and only if $r^* \in \Sigma$, and then $r_{t\wedge\tau} \in \Sigma$ for all $t \geq 0$.

Proof. We know that $\lambda(0) \neq 0$. Hence (6.3) follows from (4.4). The rest of the theorem is a consequence of Theorem 4.6 and the preceding discussion. □

6.1. **The Hull-White extension of the Vasicek model.** We consider the volatility structure of the Vasicek model: $\sigma(r)(x) = \rho \exp(-\beta x) = \rho\lambda$ with $\rho > 0$ and $\beta > 0$, for $r \in \mathcal{U} \cap D(A^\infty) = D(A^\infty)$ and $x \geq 0$. Then by the above formulas

$$[\mu, \sigma] = -\beta\rho\lambda.$$

The singular set Σ is characterized by

$$\frac{d}{dx} h + \frac{\rho^2}{\beta} \exp(-\beta x)(1 - \exp(-\beta x)) = c \exp(-\beta x)$$

for some real c. Therefore a_2 is some fixed value, namely $a_2 = \frac{\rho^2}{2}$ and a_1, a_3 are arbitrary. Consequently the singular Σ set is an affine subspace for the fixed values ρ, β:

$$h = a_1 - \frac{\rho^2}{2} \Lambda^2 + a_3 \Lambda.$$

Going back to traditional notations for the Vasicek model we write

$$\begin{aligned} \Lambda(x) &= \frac{1}{\beta}(1 - \exp(-\beta x)) \\ B_V(x) &= \Lambda'(x) = e^{-\beta x} \\ A_V(x) &= b\Lambda(x) - \frac{\rho^2}{2} \Lambda(x)^2, \end{aligned}$$

then h lies in the singular set Σ if and only if

$$h \in A_V + \langle B_V \rangle$$

for some value b (which becomes an additional parameter in the short rate equation). The solution for r^* in the singular set reads as follows

$$\begin{aligned} r_t &= A_V + B_V R_t \\ dR_t &= (b - \beta R_t)\, dt + \rho\, dW_t, \end{aligned}$$

where $R_t = ev_0(r_t)$ denotes the short rate, which is the Vasicek short rate model.

Outside the singular set Σ we have a 2-dimensional realization. First we calculate the deterministic part of the dynamics

$$\begin{aligned} Fl^{\nu}_{u_0}(r^*)(x) &= S_{u_0} r^*(x) + \int_0^{u_0} S_{u_0-s}(\frac{\rho^2}{\beta}\exp(-\beta x)(1-\exp(-\beta x))\, ds \\ &= S_{u_0} r^*(x) + \frac{\rho^2}{2}\int_0^{u_0} \frac{d}{dx}(\Lambda)^2(x+u_0-s)\, ds \\ &= r^*(x+u_0) + \frac{\rho^2}{2}\Lambda(x+u_0)^2 - \frac{\rho^2}{2}\Lambda(x)^2. \end{aligned}$$

If we identify u_0 with the time variable t, which is possible since the stochastics only occurs in direction of B_{HWV} (see Remark 4.7), we obtain by direct calculations for (4.4)

$$\begin{aligned} r_t(x) &= r^*(x+t) + \frac{\rho^2}{2}\Lambda(x+t)^2 - \frac{\rho^2}{2}\Lambda(x)^2 + \Lambda'(x) Z_t \\ dZ_t &= -\beta Z_t\, dt + \rho\, dW_t. \end{aligned}$$

A parameter transformation yields the customary form, namely

$$R_t = e^{-\beta t} r^*(0) + \int_0^t e^{-\beta(t-s)} b(s)\, ds + Z_t.$$

This yields the following expressions:

$$\begin{aligned} A_{HWV}(t,x) &= r^*(x+t) + \frac{\rho^2}{2}\Lambda(x+t)^2 - \frac{\rho^2}{2}\Lambda(x)^2 - \\ &\quad - (\Lambda'(x))^2 r^*(0) - \Lambda'(x)\int_0^t e^{-\beta(t-s)} b(s)\, ds \\ B_{HWV}(x) &= B_V(x) = \Lambda'(x) \\ dR_t &= (b(t) - \beta R_t)\, dt + \rho\, dW_t \\ r_t &= A_{HWV}(t) + B_{HWV} R_t \\ b(t) &= \frac{d}{dt} r^*(t) + \beta r^*(t) + \frac{\rho^2}{2\beta}(1-\exp(-2\beta t)). \end{aligned}$$

The functions A_{HWV} and B_{HWV} are solutions of time-dependent Riccati equations constructed by geometric methods. The equation for b follows from the fact that $A_{HWV}(t,0) = 0$.

6.2. **The Hull-White extension of the CIR model.** We proceed in the same spirit: $\sigma(r) := \rho\sqrt{ev_0(r)}\lambda$ for $\rho > 0$. The volatility structure is defined on the convex open set $\mathcal{U} = \{ev_0(r) > \epsilon\}$ for some $\epsilon > 0$. The function $\Lambda := \int \lambda$ satisfies (in certain normalization) a Riccati equation, namely

$$\frac{d}{dx}\Lambda + \frac{\rho^2}{2}\Lambda^2 + \beta\Lambda = 1, \quad \Lambda(0) = 0.$$

We obtain the solution (see e.g. [9, Section 7.4.1])

$$\Lambda(x) = \frac{2\exp(x\sqrt{\beta^2+2\rho^2}) - 1}{(\sqrt{\beta^2+2\rho^2}-\beta)(\exp(x\sqrt{\beta^2+2\rho^2})-1) + 2\sqrt{\beta^2+2\rho^2}}.$$

Under this assumption we can proceed as above: The singular set Σ is determined by the equation

$$\nu(h) = \frac{d}{dx}h + \rho^2 ev_0(h)\Lambda\Lambda' \in \langle\lambda\rangle$$

hence

$$h = a_1 + \frac{\rho^2}{2}a_1\Lambda^2 + a_3\Lambda.$$

Again a_1 and a_3 can be chosen freely, which completely determines Σ. Traditionally one writes the singular set in the following form:

$$\begin{aligned} A_{CIR} &= b\Lambda \\ B_{CIR} &= 1 - \beta\Lambda - \frac{\rho^2}{2}\Lambda^2 = \Lambda' \end{aligned}$$

with some additional parameter b and we obtain equally that h lies in Σ if and only if

$$h \in A_{CIR} + \langle B_{CIR}\rangle\,.$$

The short rate dynamics follows the known pattern:

$$\begin{aligned} r_t &= A_{CIR} + B_{CIR}R_t \\ dR_t &= (b - \beta R_t)\,dt + \rho\sqrt{R_t}\,dW_t \end{aligned}$$

for $r^* \in \Sigma$. Outside the singular set we have a 2-dimensional realization. First we calculate the deterministic part, by the variation of constants formula,

$$Fl^{\nu}_{u_0}(r^*)(x) = S_{u_0}r^*(x) + \rho^2\int_0^{u_0} Fl^{\nu}_s(r^*)(0)(S_{u_0-s}(\Lambda'\Lambda))(x)\,ds.$$

Identifying u_0 with the time parameter yields the following formula 2-dimensional realization, which is derived by direct calculations,

$$\begin{aligned} r_t &= Fl^{\nu}_t(r^*) + \Lambda' Z_t \\ dZ_t &= -\beta Z_t\,dt + \rho\sqrt{c(t) + Z_t}\,dW_t, \end{aligned}$$

where $c(t) = Fl^{\nu}_t(r^*)(0)$. The short rate is given through $R_t = c(t) + Z_t$ and

$$\begin{aligned} dR_t &= (\beta c'(t) - \beta Z_t)\,dt + \rho\sqrt{R_t}\,dW_t \\ &= (b(t) - \beta R_t)\,dt + \rho\sqrt{R_t}\,dW_t. \end{aligned}$$

Notice that $\lambda(0) = \Lambda'(0) = 1$ by the Riccati equation and $b(t) = c'(t) + \beta c(t)$.

This formula closes the circle with the classical Hull-White extension of the CIR-model:

$$\begin{aligned} A_{HWCIR}(t,x) &= Fl^{\nu}_t(r^*)(x) - c(t)\Lambda'(x) \\ B_{HWCIR} &= B_{CIR} = \Lambda' \\ dR_t &= (b(t) - \beta R_t)\,dt + \rho\sqrt{R_t}\,dW_t \\ r_t &= A_{HWCIR}(t) + B_{HWCIR}R_t \\ b(t) &= \beta c(t) + \frac{d}{dt}c(t) \\ c(t) &= r^*(t) + \rho^2\int_0^t c(s)(\Lambda\Lambda')(t-s)\,ds. \end{aligned}$$

Again this is a geometrical construction of solutions of time-dependent Riccati equations.

6.3. **Fitting procedures as leaves of foliations.** A popular forward curve-fitting method is the Svensson [16] family

$$G_S(x,z) = z_1 + z_2 e^{-z_5 x} + z_3 x e^{-z_5 x} + z_4 x e^{-z_6 x}.$$

It is shown in [7] that the only non-trivial interest rate model that is consistent with the Svensson family is of the form

$$r_t = Z_t^1 g_1 + \cdots + Z_t^4 g_4, \tag{6.4}$$

where

$$g_1(x) \equiv 1, \quad g_2(x) = e^{-\alpha x}, \quad g_3(x) = x e^{-\alpha x}, \quad g_4(x) = x e^{-2\alpha x},$$

for some fixed $\alpha > 0$. Moreover,

$$Z_t^1 \equiv Z_0^1, \quad Z_t^3 = Z_0^3 e^{-\alpha t}, \quad Z_t^4 = Z_0^4 e^{-2\alpha t} \quad (Z_0^4 \geq 0)$$

and Z^2 satisfies

$$dZ_t^2 = \left(Z_t^3 + Z_t^4 - \alpha Z_t^2\right) dt + \sqrt{\alpha Z_t^4}\, dW_t. \tag{6.5}$$

Here W is a real-valued Brownian motion.

We now shall find a 2-dimensional local HJM model that is of the form (6.4) whenever $r_0 = \sum_{j=1}^4 z_j g_j$ with $z_4 \geq 0$. In view of (6.5), a candidate for σ is given on the half space $\mathcal{U} := \{\ell > 0\}$ by

$$\sigma(h) = \sqrt{\alpha \ell(h)}\, g_2,$$

where ℓ is some continuous linear functional on $C(\mathbb{R}_{\geq 0}, \mathbb{R})$ with $\ell(g_1) = \ell(g_2) = \ell(g_3) = 0$ and $\ell(g_4) = 1$. Straightforward calculations show, for $h \in \mathcal{U} \cap D(A^\infty)$,

$$\mu(h) = Ah + \ell(h) g_2 - \ell(h) g_2^2$$

$$[\mu, \sigma](h) = -\alpha \sqrt{\alpha \ell(h)}\, g_2 - \frac{\ell(\mu(h))}{2\sqrt{\alpha \ell(h)}} g_2.$$

(the clue is that $\ell \circ \sigma \equiv 0$). Hence indeed the Lie algebra generated by σ and μ has dimension 2 on $\mathcal{U} \cap D(A^\infty) \setminus \Sigma$.

References

[1] T. Björk and B. J. Christensen, *Interest rate dynamics and consistent forward rate curves*, Math. Finance **9** (1999), 323–348.

[2] T. Björk and C. Landén, *On the construction of finite dimensional realizations for nonlinear forward rate models*, forthcoming in Finance Stochast.

[3] T. Björk and L. Svensson, *On the existence of finite dimensional realizations for nonlinear forward rate models*, Math. Finance **11** (2001), 205–243.

[4] J. Cox, J. Ingersoll, and S. Ross, *A theory of the term structure of interest rates*, Econometrica **53** (1985), 385–408.

[5] G. Da Prato and J. Zabczyk, *Stochastic equations in infinite dimensions*, Cambridge University Press, 1992.

[6] K. J. Engel and R. Nagel, *One-parameter semigroups for linear evolution equations*, Springer-Verlag, New York, 2000.

[7] D. Filipović, *Exponential-polynomial families and the term structure of interest rates*, Bernoulli **6** (2000), no. 6, 1–27.

[8] ———, *Invariant manifolds for weak solutions to stochastic equations*, Probab. Theory Relat. Fields **118** (2000), no. 3, 323–341.

[9] ———, *Consistency problems for Heath–Jarrow–Morton interest rate models*, Springer-Verlag, Berlin, 2001.

[10] D. Filipović and J. Teichmann, *Existence of invariant manifolds for stochastic equations in infinite dimensions*, forthcoming in J. Funct. Anal.

[11] ———, *Regularity of finite-dimensional realizations for evolution equations*, forthcoming in J. Funct. Anal.

[12] ———, *On the geometry of the term structure of interest rates*, invited contribution to the Proceedings of the Royal Society, submitted, 2002.

[13] M. Heidari and L. Wu, *Are interest rate derivatives spanned by the term structure of interest rates?*, Working paper, Fordham University, 2001.

[14] J. Hull and A. White, *Pricing interest-rate derivative securities*, Rev. Finan. Stud. **3** (1990), 573–592.

[15] M. Musiela, *Stochastic PDEs and term structure models*, Journées Internationales de Finance, IGR-AFFI, La Baule, 1993.

[16] L. E. O. Svensson, *Estimating and interpreting forward interest rates: Sweden 1992-1994*, IMF Working Paper No. 114, September 1994.

[17] O. Vasicek, *An equilibrium characterization of the term structure*, J. Finan. Econom. **5** (1977), 177–188.

[18] J. Zabczyk, *Stochastic invariance and consistency of financial models*, Ren. Mat. Acc. Lincei **11** (2000), no. 9, 67–80.

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