ON FINITE EXPANSION OF A HOLE IN A THIN INFINITE PLATE*

BY

P. G. HODGE, JR.† AND R. SANKARANARAYANAN

Polytechnic Institute of Brooklyn

Summary. A complete solution for all values of applied pressure is obtained for the expansion of a hole of zero radius in an initially uniform infinite sheet. The analysis is compared with those of previous investigators and found to be simpler and/or more complete. It is shown that the results are applicable to the expansion of a finite hole in a uniform sheet and to the expansion of a hole in a tapered sheet.

1. Introduction. The problem of finite expansion of a circular hole in a thin infinite plate has been discussed repeatedly in the literature. Taylor [3][1] treated the problem by using Tresca's yield condition and the flow rule associated with Mises' yield condition, but he considered a rigid perfectly plastic material. Hill's [4] extension of Taylor's work was also restricted to a non-strain hardening material. Prager [1] solved the problem for a rigid strain hardening material which satisfies Tresca's initial yield condition and the associated flow rule, but his solution is restricted to a certain finite value of the pressure. Alexander and Ford [5] have treated the same problem, using Mises' yield condition and the associated Prandtl-Reuss relations.

The present paper deals with the same problem treated by Prager, but the solution is obtained for all values of the internal pressure. As in Hill's treatment [4] the mathematical analysis in the plastic regions is simplified by reducing the problem to the solution of ordinary differential equations by means of a substitution of independent variables.

Specifically, a hole is expanded from zero radius in a thin infinite sheet of work-hardening material which satisfies Tresca's yield condition and the associated flow rule. An incompressible material is envisaged, and since finite plastic deformations are considered, elastic deformations are neglected.

In the case of plane stress problems Tresca's yield condition can be represented by a hexagon in the σ_1, σ_2 plane (Fig. 1). If the "stress point" with coordinates σ_1, σ_2 remains in the interior of the hexagon, the material is rigid. For a point on the boundary, plastic flow may commence in accord with the plastic potential flow law [6]. According to this law, on a side of the hexagon the strain rate vector must be the unique normal, while at a corner it may assume any position between the limiting normals.

For a perfectly plastic material the hexagon in Fig. 1 is fixed, and the stress point is not permitted to leave it. For the isotropic strain hardening material considered in this paper, the yield curve maintains its shape, center and orientation, but merely changes its size.

It has been shown [7, 8] that under certain restrictive conditions the plastic flow laws can be explicitly integrated. These conditions, defined as a "regular progression"

---

*Received March 29, 1957. The results presented in this paper were obtained in the course of research sponsored by the Office of Naval Research.

†Now at Illinois Institute of Technology.

[1]Numbers in square brackets refer to the references collected at the end of the paper.

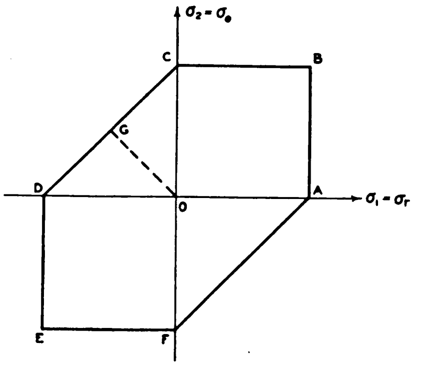

FIG. 1. Tresca hexagon for plane stress.

of the stress point, state that it must never lose contact with a side of the hexagon. In other words, the stress point may stay on a side, stay in a corner, or move from a side to a corner; it may not move from one side to another, from a corner to a side, or back into the interior.

Prager [1] considered the enlargement of a circular hole of radius a_0 to $1.48a_0$ by the application of a gradually increasing uniform pressure p to the edge of the hole. The stress point of any particle starts at the origin for $p = 0$ and moves along the straight line OG as p increases (Fig. 2). It first becomes plastic at G and moves along GD to D. Next, the stress point remains a finite time at D, engaging in plastic flow. Finally, it moves back along DC to the accompaniment of further plastic flow. However, Prager did not consider the final stage of plastic flow, when "non-regular progression" of the stress point takes place from the corner D, back to the side CD of the yield hexagon. Therefore, as he pointed out, his solution is only valid for $a/a_0 < 1.48$. In particular, Prager's solution is not applicable to the expansion of a hole from zero radius.

Since the stress point progresses non-regularly in the final stage of plastic deformation, the explicit integrated forms of the flow law cannot be used. A similar condition was encountered in an earlier paper [2] concerned with the bending of a simply supported annular plate. There it was found that the complete solution was built up of zones and that appropriate initial conditions were always available from plastic zones already formed. The analysis of the present paper reveals that a similar condition holds in the finite expansion of a hole in an infinite sheet.

For conceptual simplicity, it is first assumed that the initial radius of the hole is zero. Since the plate is infinite, this implies that there is no characteristic length in the plane of the sheet. It follows that the geometric solution at all times must be similar to itself. Therefore, rather than the solution depending upon time and space independently, it depends only upon a conveniently chosen space-time ratio. This technique was first used by Hill [4].

The solution for a hole expanded from a finite radius is obtained from that for expanding a hole from zero radius simply by discarding the part of the solution which is not required. This is so because it is immaterial whether the pressure at any radius is applied by an external agency or through the displacement of an inner annulus of the sheet. The stress and velocity in an element depend only on what happens beyond the radius.

The basic theory is presented in the next section. Section 3 gives the details of the solution. The paper will conclude with a discussion of some limitations and extensions of the analysis, as well as its relation to other investigations cited.

2. Basic theory. The state of a thin infinite sheet of initially uniform thickness is fully defined by the principal stresses σ_r and σ_ϑ, the radial displacement u, and the thickness h. The stresses must satisfy the equation of radial equilibrium

$$\frac{\partial(h\sigma_r)}{\partial r} + \frac{h(\sigma_r - \sigma_\vartheta)}{r} = 0. \tag{1}$$

Where the material is rigid, it may be regarded as the limiting case of an elastic material as Young's modulus becomes infinite. Therefore, in a rigid region the stresses must also satisfy the compatibility equation for an incompressible material

$$r\left(\frac{\partial\sigma_\vartheta}{\partial r} - \frac{1}{2}\frac{\partial\sigma_r}{\partial r}\right) + \frac{3}{2}(\sigma_\vartheta - \sigma_r) = 0. \tag{2}$$

A stress point in the interior of the hexagon must satisfy Eqs. (1) and (2). The flow law in a plastic region is expressed in terms of the plastic strain rates

$$\epsilon_r = \partial v/\partial r, \qquad \epsilon_\vartheta = v/r, \qquad \epsilon_z = (1/h)(\partial h/\partial t), \tag{3}$$

where v is the radial velocity. Since the material is incompressible,

$$\partial v/\partial r + v/r + \partial(\log h)/\partial t = 0. \tag{4}$$

The plastic flow law assumes different forms for each side or corner of the yield hexagon (Fig. 1). For the type of loading considered here, the relevant plastic regimes are CD and D. On CD, the strain rate vector must be directed normal to the side, hence

$$\epsilon_r + \epsilon_\vartheta = -\epsilon_z = 0. \tag{5}$$

Further, the stresses must satisfy

$$\sigma_r \leq 0, \qquad \sigma_\vartheta \geq 0 \tag{6}$$

if the stress point is to remain on CD.

In the corner D, the stress point must always lie on the σ_r axis; hence

$$\sigma_\vartheta = 0. \tag{7}$$

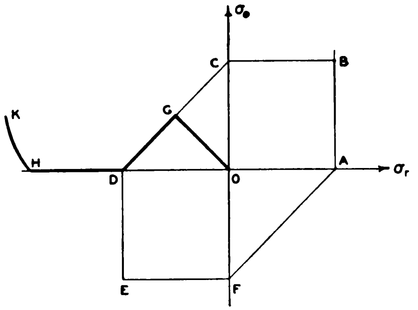

FIG. 2. The stress profile.

The strain rate vector must lie between the values corresponding to the sides CD and DE; hence

$$0 < \epsilon_\vartheta < -\epsilon_r . \tag{8}$$

It remains to consider the rate of hardening. As in Prager's treatment [1] this will be assumed to be proportional to the energy dissipation. Thus

$$\partial\sigma/\partial t = \alpha D, \tag{9}$$

where α is a constant, D the dissipation function, and

$$\sigma = \max [| \sigma_r |, \quad | \sigma_\vartheta |, \quad | \sigma_r - \sigma_\vartheta |]. \tag{10}$$

On the side CD, as well as the corner D, Eq. (9) can be written

$$\partial\sigma/\partial t = -\alpha\sigma\epsilon_r ,$$

where

$$\sigma = \sigma_\vartheta - \sigma_r . \tag{11}$$

At $r = \infty$, the radial stress must vanish, and at $r = a$, the current hole radius, it must be in equilibrium with the applied load. Thus

$$\text{at} \quad r = a, \qquad h\sigma_r = -p; \tag{12}$$

$$\text{at} \quad r = \infty, \qquad \sigma_r = 0. \tag{13}$$

Finally, at the beginning of the expansion the thickness must be equal to its initial constant value

$$\text{at} \quad t = 0, \qquad h = h_0 .$$

3. Solution. When a hole is expanded from zero radius, the material at a sufficiently large distance from the hole is rigid because it is stressed below the yield limit. Therefore, $h = h_0$ and $u = 0$. At $r = \infty$, the stresses must vanish, while at the elastic-plastic boundary they must satisfy

$$\sigma_r - \sigma_\vartheta = -\sigma_0 . \tag{14}$$

Therefore, Eqs. (1) and (2) determine the stresses to be

$$\sigma_r = -\sigma_\vartheta = \frac{-\sigma_0}{2} \frac{\rho^2}{r^2} , \tag{15}$$

where σ_0 denotes the yield stress of the virgin material and ρ is the radius of the elastic plastic interface. The solution given by Eqs. (15) is valid for $r > \rho$.

For r somewhat less than ρ, the stress point will be in regime CD. For this portion Eq. (5) shows that h is still equal to h_0 . Since there is no thickening of the material, there can be no straining and hence it is still rigid, even though plastic. The stresses are now given by Eqs. (1) and (14), with Eq. (15) as a boundary condition at $r = \rho$. Thus

$$\begin{aligned} \sigma_r &= -\sigma_0\left(\frac{1}{2} + \log \frac{\rho}{r}\right) ; \\ \sigma_\vartheta &= \sigma_0\left(\frac{1}{2} - \log \frac{\rho}{r}\right). \end{aligned} \tag{16}$$

It is clear from the second Eq. (16) that the hoop stress vanishes for

$$r = \rho e^{-1/2}. \tag{17}$$

Therefore, it follows from inequalities (6) that Eqs. (16) are valid for $\mu < r < \rho$, where ρ and μ are related by

$$\rho = \mu e^{1/2}. \tag{18}$$

The portion of the sheet just inside the circle $r = \mu$ corresponds to the regime D. It is evident that μ will increase with the pressure, and hence it may be conveniently taken as the time parameter. The ratio of the original radius r_0 of a particle to the radius of this boundary is then taken as the space-time ratio mentioned in the Introduction. Thus the entire solution will be a function of

$$x = r_0/\mu. \tag{19a}$$

The equilibrium equation, the incompressibility relation, the law of strain hardening, and the flow rule may now be expressed in terms of this parameter x. The resulting equations will be ordinary differential equations, and the solutions will be obtained relatively easily. It is convenient to define the following dimensionless quantities

$$\xi = r/\mu, \qquad s_r = \sigma_r/\sigma_0\,, \qquad s_\vartheta = \sigma_\vartheta/\sigma_0\,, \qquad s = \sigma/\sigma_0\,, \qquad \eta = h/h_0\,, \tag{19b}$$

all as function of x only. If φ is any function of x alone, then the following relations are seen to hold:

$$\begin{aligned} \partial\varphi/\partial r_0 &= (1/\mu)\varphi', \\ \partial\varphi/\partial\mu &= -(1/\mu)x\varphi', \\ \partial r/\partial r_0 &= \frac{\mu\partial(r/\mu)}{\partial r_0} = \mu\,\frac{\partial\xi}{\partial r_0} = \xi', \\ \partial\varphi/\partial r &= \frac{\partial\varphi/\partial r_0}{\partial r/\partial r_0} = \varphi'/\mu\xi', \end{aligned} \tag{20}$$

where primes denote differentiation with respect to x. Therefore, the velocity and strain rates are

$$v = \partial r/\partial\mu = \xi - x\xi', \tag{21}$$

$$\epsilon_r = \partial u/\partial r = -x\xi''/\mu\xi', \tag{22}$$

$$\epsilon_\vartheta = v/r = (\xi - x\xi')/\mu\xi, \tag{23}$$

$$\epsilon_z = \frac{1}{h}\frac{\partial h}{\partial\mu} = -(x\eta')/\mu\eta. \tag{24}$$

At the boundary of the plastic region $x = 1$, the displacement and velocity must vanish, the thickness must equal its initial value, and the stress point must be at D. It follows that, at $x = 1$,

$$\xi = 1, \qquad \xi' = 1, \qquad \eta = 1, \qquad s = 1. \tag{25}$$

The strain hardening law and the incompressibility relation, Eqs. (10) and (5), respectively, may now be written as

$$-xs' = \alpha x\xi''/\xi',$$

$$-x\xi''/\xi + 1 - x\xi'/\xi - x\eta'/\eta = 0. \tag{26}$$

Integrals of Eqs. (26) which satisfy Eqs. (25) are then found in the form

$$s^{1/\alpha}\xi' = 1, \tag{27}$$

$$\xi\xi'\eta = x. \tag{28}$$

Finally, the equilibrium equation in the corner D becomes

$$(\xi\eta s_r')/\xi' = 0. \tag{29}$$

The solution of Eqs. (27), (28) and (29) satisfying the boundary conditions (25) are

$$s = -s_r = (x)^{-\alpha/(1+\alpha)}, \qquad s_\vartheta = 0,$$

$$\xi = \frac{1}{2+\alpha}\,[1 + (1+\alpha)x^{(2+\alpha)/(1+\alpha)}], \tag{30}$$

$$\eta = \frac{(2+\alpha)x^{\alpha/(1+\alpha)}}{1 + (1+\alpha)x^{(2+\alpha)/(1+\alpha)}}.$$

This solution will be valid so long as Inequality (8) is satisfied. It is readily verified that this is true for all $1/2 < \xi < 1$. Substitution of Eqs. (19) into Eqs. (30) yields the results previously obtained by Prager [1].

For $\xi < 1/2$, the solutions given by Eqs. (32) to (35) predict a strain rate vector which points too far up, suggesting that the portion of the plate just inside the circle $\xi = 1/2$, corresponds to plastic regime CD.

For the solution in the region $\xi \leq 1/2$, Eqs. (27) and (28) are still valid since Eqs. (10) and (5) hold in regimes D and CD. In addition Eq. (5) implies that

$$\eta = \eta_1 = 2\left[\frac{\alpha}{2(1+\alpha)}\right]^{\alpha/(2+\alpha)}. \tag{31}$$

Here, as in the following, integration constants are determined to give continuity with Eqs. (30) at $x = 1/2$.

Because η is a constant, Eq. (28) is easily solved, the result being

$$\xi = \left[\frac{x^2}{\eta_1} + \frac{1}{4(1+\alpha)}\right]^{1/2}. \tag{32}$$

With ξ known, Eq. (27) furnishes

$$s = (\eta_1/x)^\alpha\left[\frac{x^2}{\eta_1} + \frac{1}{4(1+\alpha)}\right]^{\alpha/2}. \tag{33}$$

Hence, from the equilibrium equation

$$s_r = \int_{x_1}^{x} (x/\eta_1)^{1-\alpha}\left[\frac{x^2}{\eta_1} + \frac{1}{4(1+\alpha)}\right]^{(\alpha-2)/2} dx - \left[\frac{\alpha}{2(1+\alpha)}\right]^{-\alpha/(2+\alpha)}. \tag{34}$$

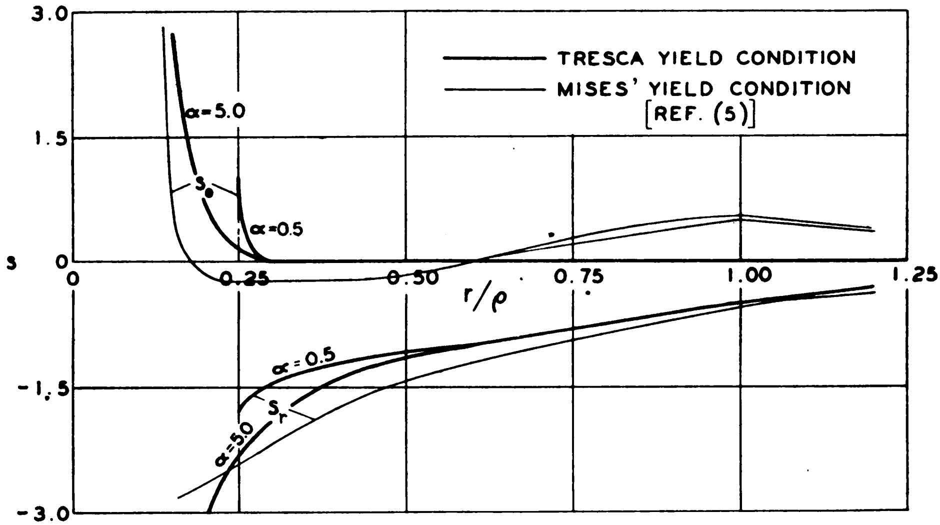

FIG. 3. Distribution of radial and circumferential stresses.

Finally, since $s_\vartheta = s + s_r$, the value of s_ϑ is obtained by adding Eqs. (33) and (34).

It is easily verified that Inequalities (6) are always valid for $0 \leq \xi < 1/2$. Therefore, the solutions represented by Eqs. (33) and (34) give the complete stress distribution in the plastic region. It is further verified that s increases as the value of ξ is decreased from 0.5, so that the strain hardening continues.

4. Conclusions. The complete solution has been obtained for the finite expansion of a circular hole from zero radius in a thin infinite plate of initially uniform thickness. The material of the plate is rigid-plastic, satisfies Tresca's yield condition and the associated flow rule, and hardens isotropically.

As already stated, the solution for a hole expanded from a finite radius is obtained from the present analysis simply by discarding the part of the solution around the edge of the hole which is not required. This is a consequence of the fact that the outer annular

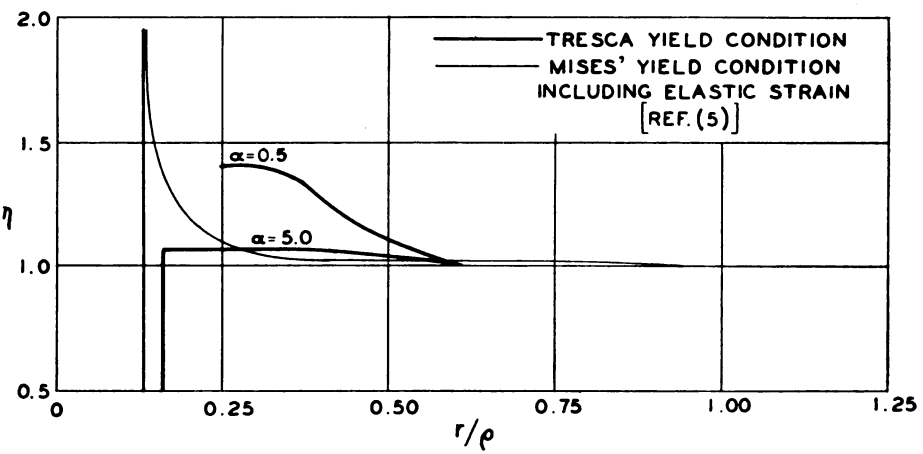

FIG. 4. Thickness variation in deformed sheet.

fibres of the plate receive radial pressure transmitted only from the inner fibres, and the inner fibres may be equally well replaced by some externally applied pressure.

As pointed out by Alexander and Ford [5], the governing equations in the plastic region of a plate of initially uniform thickness are identical with those of a plate which varies in thickness proportionately with the radius. Therefore, the plastic region analysis of the paper applies equally well to such a plate.

The great simplicity achieved by using Tresca's yield condition and the associated flow rule will be obvious by a comparison of the present analysis with that of Alexander and Ford [5], who treated the same problem by using Mises' yield condition and the associated Prandtl-Reuss relations. Therefore, it is of some interest to compare the results of the two theories. Figures (3) and (4) show the stress and thickness distributions, plotted as functions of r/ρ, for two values $\alpha = 5.0$ and $\alpha = 0.5$, of the hardening.

Figures (3) and (4) show that there is fairly good agreement between the two distributions of stress, though there is some discrepancy between the two distributions of strain. Experiment alone, of course, can decide which analysis predicts the real behavior of the material more closely. However, even if experimental evidence were to show the Mises-type solution to be more accurate, the present treatment might still be valuable as an approximate solution due to its much greater simplicity.

References

1. W. Prager, *On the use of singular yield conditions and associated flow rules*, J. Appl. Mech. **20**, 317-320 (1953)
2. P. G. Hodge, Jr., *Plastic bending of an annular plate*, J. Math. & Phys. **36**, 130–137 (1957)
3. G. I. Taylor, *The formation and enlargement of a circular hole in a thin plastic sheet*, Quart. J. Mech. Appl. Math. **1**, 103-124 (1948)
4. R. Hill, *Plastic distortion of non-uniform sheets*, Phil. Mag. Ser. 7, vol. **xi**, 971-983 (1949)
5. J. M. Alexander and H. Ford, *On expanding a hole from zero radius in a thin infinite plate*, Proc. Roy. Soc. **A226**, 543-561 (1954)
6. R. v. Mises, *Mechanik der plastischen Formaenderung von Kristallen*, ZAMM **8**, 161-185 (1928)
7. W. Prager, *The theory of plasticity—a survey of recent achievements*, James Clayton Lecture, Inst. Mech. Engrs., London, 1955
8. P. G. Hodge, Jr., *Minimum principles of piecewise linear isotropic plasticity*, J. Ratl. Mech. and Anal. **5**, 917-938 (1956)