

On Finite Groups in which Every Solvable Non-cyclic Proper Subgroup is either Self-normalizing or Normal¹

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Abstract

In this paper, we call a finite group G an SN -group if every solvable non-cyclic proper subgroup of G is either self-normalizing or normal in G . It is shown that any SN -group is solvable. Also, the structure of nilpotent SN -groups is obtained.

Mathematics Subject Classification: 20D10

Keywords: non-cyclic subgroup; self-normalizing; normal

1 Introduction

Let G be a finite group and H a subgroup of G . By $N_G(H)$ we denote the normalizer of H in G . It is clear that the following inequality holds for any subgroup H of G

$$H \leq N_G(H) \leq G.$$

Recall that H is said to be self-normalizing in G if $H = N_G(H)$ and H be normal in G if $N_G(H) = G$.

In [3], S. Li et al proved the following result:

Theorem 1.1 [3] *Let G be a finite group. Suppose that every non-cyclic subgroup of G is self-normalizing, then G is supersolvable.*

¹This work is founded by NSFS (No. ZR2011AM005).

As a generalization of above theorem, the main goal of this paper is to investigate the structure of a finite group in which every solvable non-cyclic proper subgroup is either self-normalizing or normal. For convenience, such a finite group is called an *SN*-group. Our main result is as follows, the proof of which is given in Section 2.

Theorem 1.2 *Let G be an SN-group, then G is solvable.*

Theorem 1.2 implies that G is an *SN*-group if and only if G is a group in which every non-cyclic proper subgroup is either self-normalizing or normal.

Remark 1.1 A finite group in which every solvable non-abelian proper subgroup is either self-normalizing or normal may be non-solvable. For example, every non-abelian proper subgroup of the alternating group A_5 is self-normalizing but A_5 is non-solvable.

Remark 1.2 An *SN*-group may be non-supersolvable. For example, let $G \cong A_4 \times Z_3$, where A_4 is the alternating group of degree 4. It is easy to see that G is an *SN*-group but G is non-supersolvable.

In [1] and [5], Z. Božikov, Z. Janko and D.S. Passman classified finite p -groups in which every non-cyclic subgroup is normal. For convenience, we call such a p -group a *PBJ*-group, in particular, we call G a *PBJ**-group if G is a *PBJ*-group but G is not a Dedekind group.

For nilpotent *SN*-groups, we have the following result, the proof of which is given in Section 3.

Theorem 1.3 *Let G be a finite group and $\pi(G)$ be the set of prime divisors of $|G|$. Then G is a nilpotent SN-group if and only if one of the following statements holds:*

- (1) *If $|\pi(G)| = 1$, then G is a PBJ-group.*
- (2) *If $|\pi(G)| > 1$, then G is either a Dedekind group or a direct product of a PBJ*- p -group and a cyclic p' -group for some prime p .*

For convenience, by $A \rtimes B$ we denote the semidirect product of the normal subgroup A and the subgroup B .

2 Proof of Theorem 1.2

Proof. (1) Claim: If every solvable non-cyclic subgroup of G is self-normalizing, then G is solvable.

Suppose not. Let G be a counterexample of minimal order.

(i) Claim: G is a non-abelian simple group.

Otherwise, assume that N is a nontrivial normal subgroup of G . By the minimality of G , we have that both N and G/N are solvable, which implies that G is solvable, a contradiction. Thus, G is a non-abelian simple group.

Let P be a Sylow 2-subgroup of G . Assume that $|P| = 2^n$, where $n \geq 1$.

(ii) Claim: P is non-cyclic.

Otherwise, assume that P is cyclic. By [6, Theorem 10.1.9], G is 2-nilpotent. Then there exists a nontrivial normal subgroup L of G such that $G = L \rtimes P$, which contradicts G is a non-abelian simple group. Thus, P is non-cyclic.

(iii) Claim: P is non-abelian.

Otherwise, assume that P is abelian. Since P is non-cyclic, by the hypothesis, $P = N_G(P)$. By [6, Theorem 10.1.8], we have that G is 2-nilpotent, which contradicts G is a non-abelian simple group. Thus, P is non-abelian. It follows that $n \geq 3$.

(a) Suppose that $n = 3$, by the hypothesis, every subgroup of P of order 4 is cyclic. Then P is isomorphic to the quaternion group Q_8 .

First, suppose that P is a maximal subgroup of G . By [7], $G \cong PSL_2(p)$, where p is a Fermat prime or a Mersenne prime such that $p \geq 17$. Since the Sylow 2-subgroup of $PSL_2(p)$ is a dihedral group, it implies that P is Q_8 -free, a contradiction.

Second, suppose that P is not a maximal subgroup of G . Let M be a maximal subgroup of G such that $P < M$. By the minimality of G , M is solvable. Then there exists a maximal subgroup H of M such that $H \trianglelefteq M$. By the hypothesis, H is cyclic. It implies that $P \not\leq H$ and then $M = HP$. Let K be a Hall-subgroup of H of odd order, then $M = K \rtimes P$. For each cyclic subgroup $\langle a \rangle$ of P of order 4, let $T = K \rtimes \langle a \rangle$. Since $|M : T| = |K \rtimes P : K \rtimes \langle a \rangle| = |P : \langle a \rangle| = 2$, we have that $T \trianglelefteq M$. By the hypothesis, T is cyclic, which implies that $\langle a \rangle \leq C_G(K)$. Then we can get that $P \leq C_G(K)$. It follows that $P \trianglelefteq M$, a contradiction.

(b) Suppose that $n \geq 4$. Since P is non-cyclic, by [4], we can easily get that P has at least one non-cyclic subgroup Q of order 8. Then $Q < N_P(Q) \leq N_G(Q)$, a contradiction.

It implies that the counterexample does not exist and then G is solvable.

(2) Final conclusion.

Let G be a counterexample of minimal order. We have that G is a minimal non-solvable group and then $G/\Phi(G)$ is a minimal non-abelian simple group. For every solvable non-cyclic proper subgroup $H/\Phi(G)$ of $G/\Phi(G)$, one has that H is a solvable non-cyclic subgroup of G . Since $G/\Phi(G)$ is a non-abelian simple group, H is not normal in G . By the hypothesis, $H = N_G(H)$. Then $H/\Phi(G) = N_G(H)/\Phi(G) = N_{G/\Phi(G)}(H/\Phi(G))$, which implies that every solvable non-cyclic subgroup of $G/\Phi(G)$ is self-normalizing. By Claim (1), $G/\Phi(G)$ is solvable, a contradiction. It follows that the counterexample does not exist.

and so G is solvable. \square

3 Proof of Theorem 1.3

Proof. The sufficiency part is evident, we only need to prove the necessity part. Since G is a nilpotent SN -group, we can easily get that every non-cyclic subgroup of G is normal.

(1) If $|\pi(G)| = 1$, then G is a PBJ -group.

(2) Suppose that $|\pi(G)| > 1$ and $G = P_1 \times P_2 \times \dots \times P_s$, where $P_i \in \text{Syl}_{p_i}(G)$, p_1, p_2, \dots, p_s are distinct prime divisors of $|G|$, $s \geq 2$.

(i) If every Sylow subgroup of G is cyclic, then G is a cyclic group.

(ii) Suppose that G has at least one non-cyclic Sylow subgroup, we can assume that P_1 is non-cyclic. Then P_1 is a non-cyclic PBJ -group. So P_1 is either a PBJ^* -group or a non-cyclic Dedekind group.

First, suppose that P_1 is a PBJ^* -group. For every $i \geq 2$, we claim that P_i is cyclic. Otherwise, assume that P_j is non-cyclic for some $j \geq 2$. For every subgroup H of P_1 , we have that HP_j is non-cyclic. Since H is a characteristically subgroup of HP_j and $HP_j \trianglelefteq G$, one has $H \trianglelefteq G$. Hence $H \trianglelefteq P_1$, which implies that P_1 is a Dedekind group, a contradiction. So P_i is cyclic for every $i \geq 2$. It follows that G is a direct product of a PBJ^* - p -group and a cyclic p' -group for some prime p .

Second, suppose that P_1 is a non-cyclic Dedekind group. For every $i \geq 2$ and every subgroup K of P_i , P_1K is non-cyclic. Since K is a characteristically subgroup of P_1K and $P_1K \trianglelefteq G$, one has $K \trianglelefteq G$. Hence $K \trianglelefteq P_i$, which implies that P_i is a Dedekind group for every $i \geq 2$. It follows that G is a non-cyclic Dedekind group. \square

Acknowledgements

The author is grateful to Professor Y. Wang for her encouragement and motivation. The author is also grateful to Dr. J. Shi for his valuable comments and suggestions.

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Received: May, 2012