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On finite groups with a 2-Sylow subgroup isomorphic to that of the symmetric group of degree $4n$

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§ 0. Introduction.

Let G be a finite group with a 2-Sylow subgroup isomorphic to that of the symmetric group of degree $4n$. The purpose of the present paper is to make some remarks on the fusion of involutions of G, which are useful for the investigations of certain finite simple groups, especially the alternating group of degree $4n+2$ or $4n+3$ and the orthogonal commutator groups $\Omega_{2n+2}(\varepsilon, q)$ $(q^{n+1}\equiv-\varepsilon \bmod 4$ and $q\equiv\pm 3 \bmod 8)$[1].

The main results are Theorem A and Theorem B in § 7. We note that the Thompson subgroup of a 2-Sylow subgroup of G plays the important role in the discussions in § 2∼§ 6. These can be regarded as a generalization of a part of [**6**]. Moreover, as an application of Theorem A, the author has obtained a characterization of the alternating groups of degrees $4n+2$ and $4n+3$ in terms of the centralizer of an involution $(1, 2)(3, 4)\cdots(4n-1, 4n)$. This will be published in a subsequent paper. Also H. Yamaki [**9**] has treated such characterizations of $\mathfrak{A}_m$ $(m=12, 13, 14$ and $15)$, though, for $m=12$ and 13, Theorem A can not be applied and an additional condition is necessary on account of the existence of the finite simple group $Sp_6(2)$.

Notations and Terminology.

$J(X)$	The Thompson subgroup of a group X (cf. [8])[2]
$Z(X)$	the center of a group X
X'	the commutator subgroup of X
$X \wr Y$	a wreath product of a group X by a permutation group Y
$x\sim y$ in X	x is conjugate to y in a group X
y^x	$x^{-1}yx$
$x: y\rightarrow z$	$y^x=z$
$[x, y]$	$x^{-1}y^{-1}xy$

1) For the notations of orthogonal groups, see [1] and [10]. Note that if $q^{n+1}\equiv-\varepsilon \bmod 4$, $\Omega_{2n+2}(\varepsilon, q)$ has the trivial center.

2) Recently, the slightly different definition of $J(X)$ from that of [8] is used, but for groups treated in the present paper, both definitions are the same.

$\langle \cdots \mid \cdots \rangle$	a group generated by $\cdots$ subject to the relations $\cdots$.
$\mathfrak{S}_n$	the symmetric group of degree n
$\mathfrak{A}_n$	the alternating group of degree n
Z_n	a cyclic group of order n.

Let X be a group isomorphic to $\mathfrak{S}_l$. X is generated by $l-1$ elements $x_1, x_2, \cdots, x_{l-1}$ subject to the relations;

$$x_1^2 = \cdots = x_{l-1}^2 = (x_i x_{i+1})^3 = (x_j x_k)^2 = 1 \quad (1 \leqq i, j, k \leqq l-1 \text{ and } |j-k| > 1)^{3)}.$$

We call an ordered set of such generators of X *a set of canonical generators of* X.

§ 1. The symmetric groups and the orthogonal groups.

(1.1) Let G be a finite group satisfying the following conditions:

(i) *G has a subgroup N, which is isomorphic to a wreath product of a dihedral group of order 8 by the symmetric group of degree n, and*

(ii) *a 2-Sylow subgroup of N is that of G.*

Then N has a set of generators λ_k, π'_k, π_k and σ_i $(1 \leqq k \leqq n$ and $1 \leqq i \leqq n-1)$ subject to the following relations:

$$(*) \quad \begin{gathered} \lambda_k^2 = \pi'^2_k = (\lambda_k \pi'_k)^4 = 1 \qquad \pi_k = (\lambda_k \pi'_k)^2, \\ [\langle \lambda_k, \pi'_k \rangle, \langle \lambda_h, \pi'_h \rangle] = 1 \qquad (k \neq h), \\ \sigma_1^2 = \cdots = \sigma_{n-1}^2 = (\sigma_i \sigma_{i+1})^3 = (\sigma_j \sigma_k)^2 = 1 \quad (1 \leqq i, j, k \leqq n-1, |j-k| > 1), \\ \lambda_i^{\sigma_i} = \lambda_{i+1}, \ \pi_i'^{\sigma_i} = \pi'_{i+1} \text{ and } [\sigma_i, \lambda_k] = [\sigma_i, \pi'_k] = 1 \quad (k \neq i, i+1). \end{gathered}$$

Put

$$\begin{gathered} J = J_1 \times J_2 \times \cdots \times J_n \qquad J_k = \langle \lambda_k, \pi'_k \rangle, \\ S = S_1 \times S_2 \times \cdots \times S_n \qquad S_k = \langle \pi_k, \pi'_k \rangle, \\ M = M_1 \times M_2 \times \cdots M_n \qquad M_k = \langle \pi_k, \lambda_k \rangle, \\ P = \langle \sigma_1, \sigma_2, \cdots, \sigma_{n-1} \rangle, \\ \alpha_n = \pi_1 \pi_2 \cdots \pi_n, \end{gathered}$$

and

$$H = C_G(\alpha_n).$$

Then J is normal in N. N is a semidirect product of P and J, and is a subgroup of H. J is a direct product of n copies J_k $(1 \leqq k \leqq n)$ of a dihedral group of order 8. S and M are elementary abelian subgroups of order 2^{2n}. P is isomorphic to the symmetric group of degree n.

In this section, we shall give some examples which may be useful for the

3) Cf. [**2**; p. 287].

understanding of the discussions in §2~§7.

(1.2) *Examples.*

(i) *The Symmetric Groups:* $G=\mathfrak{S}_{4n}$. Let π_k, π'_k, λ_k and σ_i be involutions in $\mathfrak{S}_{4n}$ as follows:

$$\pi_k=(4k-3,\ 4k-2)(4k-1,\ 4k),$$
$$\pi'_k=(4k-3,\ 4k-1)(4k-2,\ 4k),$$
$$\lambda_k=(4k-3,\ 4k-2),$$

and

$$\sigma_i=(4i-3,\ 4i+1)(4i-2,\ 4i+2)(4i-1,\ 4i+3)(4i,\ 4i+4).$$

Then these involutions satisfy the conditions (*).

(ii) *The Alternating Group:* $G=\mathfrak{A}_{4n+r}$ ($r=2$ or 3). Put $\lambda_k=(4k-3, 4k-2)(4n+1, 4n+2)$ and let π_k, π'_k and σ_i be the same as (i). Then these involutions satisfy the conditions (*).

(iii) *The Orthogonal Group:* $G=O_{2n}(\varepsilon', q)$ where $q^n\equiv\varepsilon' \bmod 4$ and $q\equiv\pm 3 \bmod 8$. Let $\sum_{i=1}^{2n} x_i^2$ be the underlying quadratic form of the orthogonal group $O_{2n}(\varepsilon', q)$. By I_k we denote the $k\times k$ unit matrix. Put

$$\pi_k=\begin{pmatrix} I_{2(k-1)} & & \\ & -I_2 & \\ & & I_{2(n-k)}\end{pmatrix}$$

$$\pi'_k=\begin{pmatrix} I_{2(k-1)} & & \\ & U & \\ & & I_{2(n-k)}\end{pmatrix} \qquad U=\begin{pmatrix} & 1\\ 1 & \end{pmatrix}$$

$$\lambda_k=\begin{pmatrix} I_{2(k-1)} & & \\ & V & \\ & & I_{2(n-k)}\end{pmatrix} \qquad V=\begin{pmatrix} -1 & \\ & 1\end{pmatrix}$$

$$\sigma_i=I_2\times P_i,$$

where P_i denotes the $n\times n$ permutation matrix corresponding to the permutation $(i, i+1)$ and $I_2\times P_i$ denotes the Kronecker product of marices.

(iv) *The Orthogonal Commutator Groups:* $G=\Omega_{2n+2}(\varepsilon, q)$, where $q^{n+1}\equiv-\varepsilon \bmod 4$ and $q\equiv\pm 3 \bmod 8$. Let a be a nonsquare element of the finite field of q elements and $\sum_{i=1}^{2n} x_i^2+x_{2n+1}^2+ax_{2n+2}^2$ be the underlying quadratic form of the group $\Omega_{2n+2}(\varepsilon, q)$. There is an injective isomorphism of $O_{2n}(\varepsilon', q)$ with the quadratic form $\sum_{i=1}^{2n} x_i^2$ into the group $\Omega_{2n+2}(\varepsilon, q)$ (cf. [**10**, p. 419]). In the present case, let π_k, π'_k, λ_k and σ_i be the image by this isomorphism of the corresponding elements in $O_{2n}(\varepsilon', q)$.

(v) *The Wreath Products:* $G = Z_2 \wr \mathfrak{S}_{2n}$. Let X_n be an elementary abelian group of order 2^{2n} with *a* set $\{x_1, x_2, \cdots, x_{2n}\}$ of generators and Y_n be a group isomorphic to $\mathfrak{S}_{2n}$ with $\{y_1, z_1, y_2, \cdots, z_{n-1}, y_n\}$ as a set of canonical generators of Y_n. Define the action on X_n of Y_n as follows;

$$x_{2i-1}^{y_i} = x_{2i},\ [x_j, y_i] = 1 \qquad (1 \leqq i \leqq n,\ j \neq 2i-1,\ 2i)$$
$$x_{2i}^{z_i} = x_{2i+1},\ [x_j, z_i] = 1 \qquad (1 \leqq i \leqq n-1,\ j \neq 2i,\ 2i+1).$$

Construct a semidirect product $G = X_n \cdot Y_n$. Then G is isomorphic to a wreath product $Z_2 \wr S_{2n}$.

Put

$$\lambda_i = x_{2i-1},$$
$$\pi_i' = y_i,$$
$$\pi_i = x_{2i-1}x_{2i},$$
$$\sigma_i = (y_i y_{i+1})^{z_i}.$$

Then these involutions satisfy the conditions (*).

REMARK. In § 5, we shall use the following fact: *the representatives of conjugacy classes of involutions of* $X_n \cdot Y_n$ *are* $\pi_1' \cdots \pi_k' \pi_{k+1} \cdots \pi_{k+l}$ $(0 < k+l \leqq n)$ *and* $\pi_1' \cdots \pi_k' \pi_{k+1} \cdots \pi_{k+l} \lambda_n$ $(0 \leqq k+l \leqq n-1)$ (cf. W. Specht [7]). This can be proved directly without difficulties.

(1.3) In the above examples, we can verify the following statements without difficulty. The verifications are left to the reader.

(i) A 2-Sylow subgroup of N is that of G,

(ii) J is generated by all abelian subgroups of N of order 2^{2n} and so, it is the Thompson subgroup of a 2-Sylow subgroup of G,

(iii) α_n is an involution in the center of a 2-Sylow subgroup of G,

(iv) every element of $N_H(S)$ induces a permutation on the set $\{\pi_1', \pi_1'\pi_1, \pi_2', \pi_2'\pi_2, \cdots, \pi_n', \pi_n'\pi_n\}$ which consists of members in a basis of S, and so does one of $N_H(M)$ on the set $\{\lambda_1, \lambda_1\pi_1, \lambda_2, \lambda_2\pi_2, \cdots, \lambda_n, \lambda_n\pi_n\}$, and

(v) the structure of the normalizers of S and M are given in the following table;

	$N_H(S)/C_H(S)$	$N_H(M)/C_H(M)$	$N_G(S)/C_G(S)$	$N_G(M)/C_G(M)$
$\mathfrak{S}_{4n}$	$Z_2 \wr \mathfrak{S}_n$	$\mathfrak{S}_{2n}$	$\mathfrak{S}_3 \wr \mathfrak{S}_n$	$\mathfrak{S}_{2n}$
$O_{2n}(\varepsilon', q)$	$\mathfrak{S}_{2n}$	$\mathfrak{S}_{2n}$	$\mathfrak{S}_{2n}$	$\mathfrak{S}_{2n}$
A_{4n+r}	$Z_2 \wr \mathfrak{S}_n$	$\mathfrak{S}_{2n}$	$\mathfrak{S}_3 \wr \mathfrak{S}_n$	$\mathfrak{S}_{2n+1}$
$\Omega_{2n+2}(\varepsilon, q)$	$\mathfrak{S}_{2n}$	$\mathfrak{S}_{2n}$	$\mathfrak{S}_{2n+1}$	$\mathfrak{S}_{2n+1}$

§ 2. Elementary abelian subgroups of G.

(2.1) Throughout the rest of the present paper, *G denotes a finite group satisfying the conditions* (i) *and* (ii) *in* (1.1). Also all notations introduced in (1.1) will be preserved in the same meanings as there.

We note that J is the Thompson subgroup of a 2-Sylow subgroup of G, all elementary abelian subgroups of order 2^{2n} of J are normal in J, and, S and M are normal in N.

J, S and M play the important roles in the discussions in § 2∼§ 6.

(2.2) LEMMA. *Let D be a group isomorphic to a direct product of n copies D_i $(1 \leqq i \leqq n)$ of a dihedral group of order 2^{m+1} $(m \geqq 2)$. Put $Z(D_i)=\langle z_i \rangle$. Define* $\mathrm{Aut}_0(D)=\{\sigma \in \mathrm{Aut}(D) \mid z_i^\sigma = z_i \ (1 \leqq i \leqq n)\}$, *where* $\mathrm{Aut}(D)$ *denotes the automorphism group of D. Then we have* (i) *every element of* $\mathrm{Aut}(D)$ *induces a permutation on the set* $\{z_1, z_2, \cdots, z_n\}$ *and* (ii) $\mathrm{Aut}_0(D)$ *is a 2-group.*

PROOF. Let a_i and b_i be generators of D_i subject to the relations: $a_i^2=b_i^2=(a_ib_i)^{2^m}=1$ $(1 \leqq i \leqq n)$. Put $c_i=a_ib_i$. From a theorem of Remak-Schmidt [**5**, p. 130], it follows that, for $\sigma \in \mathrm{Aut}(D)$, there exists an element τ of $\mathfrak{S}_n$ such that D_i^σ and $D_{\tau(i)}$ are centrally isomorphic. This implies that $(a_ic_i^{s_i})^\sigma = a_{\tau(i)}u_i$ and $(b_ic_i^{t_i})^\sigma = b_{\tau(i)}u_i'$, where $s_i \equiv t_i \bmod 2$ and $u_i, u_i' \in Z(D)$. Then we get $z_i^\sigma = z_{\tau(i)}$ by taking the product of both equalities and doing its 2^{m-1}-powers. This proves (i). By counting all the possible choices of s_i, t_i, u_i and u_i', we see that $\mathrm{Aut}_0(D)$ is a 2-group.

(2.3) LEMMA. $N_G(J)=NC_G(J)$.

PROOF. Put $N_0=\{\sigma \in N_G(J) \mid \pi_i^\sigma = \pi_i \ (1 \leqq i \leqq n)\}$. Then we have $N_0 \supseteqq JC_G(J)$. From (2.2), it follows that $N_G(J)=PN_0$, $P \cap N_0 = 1$ and $N_0/JC_G(J)$ is a 2-group. By the assumption (1.1: (ii)), we must have $N_0=JC_G(J)$. Hence we get $N_G(J)=NC_G(J)$.

(2.4) LEMMA. $N_G(S) \cap N_G(M) \supseteqq N_G(J)$.

PROOF. This is obvious, because S and M are normal in N and $N_G(J) = NC_G(J)$ by (2.3).

(2.5) LEMMA. *S and M are weakly closed in a 2-Sylow subgroup of G with respect to G.*

PROOF. Let D be a 2-Sylow subgroup of N. Suppose that $S^x \subset D$ for some $x \in G$. Then we have $S^x \lhd J$. Hence we get $N_G(S) \supset J, J^{x^{-1}}$ and we can find an element y of $N_G(S)$ such that $J^y = J^{x^{-1}}$. Since $N_G(S) \supseteqq N_G(J)$ by (2.4), we get $N_G(S) \ni yx$ and so $S=S^{yx}=S^x$. Thus we have proved that S is weakly closed in D with respect to G. Similarly we can prove that M is weakly closed in D.

(2.6) LEMMA. *If any two elements of S (resp. M) are conjugate in G, they are conjugate in $N_G(S)$ (resp. $N_G(M)$). If X is a 2-subgroup of G containing S*

(resp. M), X normalizes S (resp. M).

PROOF. This is an immediate consequence of (2.5).

§ 3. General remarks on the fusion of involutions of G.

(3.1) DEFINITION. We define some elements of G as follows;

$$\alpha_k=\pi_1\pi_2\cdots\pi_k \qquad (1\leqq k\leqq n)$$

$$\left.\begin{aligned}\pi_{k,l}&=\pi_1'\pi_2'\cdots\pi_k'\pi_{k+1}\cdots\pi_{k+l}\\ \lambda_{k,l}&=\lambda_1\lambda_2\cdots\lambda_k\pi_{k+1}\cdots\pi_{k+l}\end{aligned}\right\} \qquad (0<k+l\leqq n)$$

$$\tau_{k,l}=\pi_1'\pi_2'\cdots\pi_k'\pi_{k+1}\cdots\pi_{k+l}\lambda_n \qquad (0\leqq k+l\leqq n-1).$$

We note that $\pi_{k,l}$'s (resp. $\lambda_{k,l}$'s) are representatives of the orbits of elements in S (resp. M) under the action on S (resp. M) of N.

Throughout the present paper, we shall assume $n\geqq 2$. The special case $n=2$ was treated in [**6**].

(3.2) LEMMA. *Any two elements of $\alpha_1, \alpha_2, \cdots, \alpha_n$ are not conjugate in G.*

PROOF. By the definition of N and (2.3), any two of α_k's are not conjugate in $N_G(J)$. On the other hand, if two elements of $Z(J)$ are conjugate in G, they are conjugate in $N_G(J)$ since J is weakly closed in a 2-Sylow subgroup of G. From this, our lemma follows.

(3.3) For convenience, we shall introduce the following definition. If an involution x of G is conjugate to an involution of $Z(J)$, we say that x is *of positive length*. Then it follows from the structure of N that x is conjugate to one of $\alpha_1, \alpha_2, \cdots, \alpha_n$. If $x\sim\alpha_k$ in G, we say that x is *of length k*. Note that, in $Z(J)$, there is exactly one element of length n, namely α_n. Further we introduce some notations frequently used in subsequent lemmas.

Assume that $\pi_{k,l}$ is of positive length. Put

$$\bar{U}_{k,l}=C_J(\pi_{k,l})=S_1\times\cdots\times S_k\times J_{k+1}\times\cdots\times J_n.$$

Then we have $Z(\bar{U}_{k,l})=S_1\times S_2\times\cdots\times S_k\times\langle\pi_{k+1},\cdots,\pi_n\rangle$ and $\bar{U}'_{k,l}=\langle\pi_{k+1},\cdots,\pi_n\rangle$. Denote by $P_{k,l}$ a 2-Sylow subgroup of $C_G(\pi_{k,l})$ with $\bar{U}_{k,l}\subset P_{k,l}\subset C_G(\pi_{k,l})$. Since $\pi_{k,l}$ is of positive length, $P_{k,l}$ contains a subgroup conjugate to J, which is the Thompson subgroup $J(P_{k,l})$ of $P_{k,l}$. Since $\bar{U}_{k,l}$ is generated by elementary abelian subgroups of order 2^{2n}, we have $\bar{U}_{k,l}\subset J(P_{k,l})$. Put $U_{k,l}=\langle J, J(P_{k,l})\rangle$. Then we have

(i) $Z(J(P_{k,l}))\ni\pi_{k,l}, \pi_{k+1},\cdots,\pi_n$,

(ii) $Z(U_{k,l})\ni\pi_{k+1},\cdots,\pi_n$, and

(iii) $U_{k,l}$ normalizes $\bar{U}_{k,l}$, $Z(\bar{U}_{k,l})$, $\bar{U}'_{k,l}$ and all elementary abelian subgroups of $\bar{U}_{k,l}$ of order 2^{2n}.

In fact, since J normalizes all elementary abelian subgroups of J of order 2^{2n}

and $J\cap J(P_{k,l})\supseteqq \bar{U}_{k,l}$, $U_{k,l}$ normalizes all such subgroups of $\bar{U}_{k,l}$. Since $\bar{U}_{k,l}$ is generated by elementary abelian subgroups of order 2^{2n}, we get $U_{k,l}\triangleright \bar{U}_{k,l}$ and so $U_{k,l}\triangleright Z(\bar{U}_{k,l})$, $\bar{U}'_{k,l}$ because $Z(\bar{U}_{k,l})$ and $\bar{U}'_{k,l}$ are characteristic subgroups of $\bar{U}_{k,l}$. This proves (iii). (i) follows from the fact that $Z(J(P_{k,l}))=J(P_{k,l})'$ and $\bar{U}'_{k,l}\subset J(P_{k,l})'$. Then (ii) is obvious. Similarly, under the assumption that $\lambda_{k,l}$ is of positive length, we define the followings:

$\bar{V}_{k,l}=C_J(\lambda_{k,l})$,

$L_{k,l}=$ a 2-Sylow subgroup of $C_G(\lambda_{k,l})$ with $\bar{V}_{k,l}\subseteqq L_{k,l}\subseteqq C_G(\lambda_{k,l})$,

$V_{k,l}=\langle J, J(L_{k,l})\rangle$.

Then we have

(i)′ $Z(J(L_{k,l}))\ni\lambda_{k,l},\ \pi_{k+1},\cdots,\pi_n$,

(ii)′ $Z(V_{k,l})\ni\pi_{k+1},\cdots,\pi_n$, and

(iii)′ $V_{k,l}$ normalizes $\bar{V}_{k,l}$, $Z(\bar{V}_{k,l})$ and all elementary abelian subgroups of $\bar{V}_{k,l}$ of order 2^{2n}.

Finally, under the assumption that $\tau_{k,l}$ is of positive length, we construct the followings:

$\bar{W}_{k,l}=C_J(\tau_{k,l})$,

$T_{k,l}=$ a 2-Sylow subgroup of $C_G(\tau_{k,l})$ with $\bar{W}_{k,l}\subseteqq T_{k,l}\subseteqq C_G(\tau_{k,l})$,

$W_{k,l}=\langle J, J(T_{k,l})\rangle$.

Then we have

(i)″ $Z(J(T_{k,l}))\ni\lambda_{k,l},\ \pi_{k+1},\cdots,\pi_{n-1}$,

(ii)″ $Z(W_{k,l})\ni\pi_{k+1},\cdots,\pi_{n-1}$, and

(iii)″ $W_{k,l}$ normalizes $\bar{W}_{k,l}$, $Z(\bar{W}_{k,l})$, $\bar{W}'_{k,l}$ and all elementary abelian subgroups of $\bar{W}_{k,l}$ of order 2^{2n}.

(3.4) LEMMA. (i) *$\pi_{k,l}\sim\alpha_n$ in $G\Rightarrow l=0$ or $k+l=n$*, (ii) *$\lambda_{k,l}\sim\alpha_n$ in $G\Rightarrow l=0$ or $k+l=n$ and* (iii) *$\tau_{k,l}\sim\alpha_n$ in $G\Rightarrow l=0$ or $k+l=n-1$.*

PROOF. Suppose that $\pi_{k,l}\sim\alpha_n$ in G. Then we can construct $P_{k,l}$ as in (3.3). By (3.3; (i)), we have $Z(J(P_{k,l}))\ni\pi_{k,l},\ \pi_{k+1},\cdots,\pi_n$, Assume by way of contradiction that $l\geqq 1$ and $n>k+l$. Then, since $\pi_{k,l}\sim\pi_{k,l}\pi_{k+1}\pi_n$ in G and $\pi_{k,l},\ \pi_{k,l}\pi_{k+1}\pi_n\in Z(J(P_{k,l}))$, $Z(J(P_{k,l}))$ has two elements of length n, which is impossible because $Z(J)$ has only one element of length n. This proves (i). Similarly, by using $L_{k,l}$ and $T_{k,l}$ in (3.3), we obtain (ii) and (iii).

(3.5) LEMMA. (i) *$\alpha_1\sim\pi_{1,0}$ in $G\Leftrightarrow\alpha_n\sim\pi_{1,n-1}$ in G and* (ii) *$\alpha_1\sim\lambda_{1,0}$ in $G\Leftrightarrow\alpha_n\sim\lambda_{1,n-1}$ in G.*

PROOF. Suppose that $\alpha_1\sim\pi_{1,0}$ in G. We can construct $P_{1,0}$ as in (3.3). Then we have $Z(J(P_{1,0}))=\langle\pi'_1,\pi_2,\cdots,\pi_n\rangle$. Since there are exactly n elements of length 1 in $Z(J(P_{1,0}))$ which must be $\pi'_1,\pi_2,\cdots,\pi_n$, we get $\alpha_n\sim\pi'_1\pi_2\cdots\pi_n=\pi_{1,n-1}$ in G. Conversely, if $\alpha_n\sim\pi_{1,n-1}$ in G, we have $Z(J(P_{1,n-1}))=\langle\pi'_1,\pi_2,\cdots,\pi_n\rangle$ where $P_{1,n-1}$ is a group constructed for $\pi_{1,n-1}$ as in (3.3). Then we get $\pi'_1=\pi_{1,n-1}(\pi_2\cdots\pi_n)\sim\alpha_1$ in G because $\pi_{1,n-1}$ is of length n and π_k's $(2\leqq k\leqq n)$

are of length 1. This proves (i). Similarly, we can prove (ii) by using $L_{1,0}$ and $L_{1,n-1}$ constructed for $\lambda_{1,0}$ and $\lambda_{1,n-1}$ as in (3.3).

(3.6) LEMMA. *We may assume $\alpha_n \not\sim \pi_1'$ and $\alpha_n \not\sim \lambda_1$ in G without loss of generality. (Therefore we shall assume $\alpha_n \not\sim \pi_1'$ and $\alpha_n \not\sim \lambda_1$ in G throughout the rest of this paper.)*

PROOF. This follows from (3.2) and (3.5), by interchanging π_k's (resp. λ_k's) by $\alpha_n\pi_k$'s (resp. $\alpha_n\lambda_k$'s) if necessary.

(3.7) LEMMA. (i) *If π_1' is of positive length, we have $\pi_1' \sim \pi_1$ and $\pi_{1,l} \sim \alpha_{l+1}$.* (ii) *If λ_1 is of positive length, we have $\lambda_1 \sim \pi_1$ and $\lambda_{1,l} \sim \alpha_{l+1}$.*

PROOF. Suppose that $\pi_1' \sim \alpha_k$ in G. We have $Z(J(P_{1,0})) = \langle \pi_1', \pi_2, \cdots, \pi_n \rangle$ by (3.3; (i)). By (3.6), we have $n > k$. If $k > 1$, by taking suitable $n-k$ elements of π_s's $(2 \leqq s \leqq n)$, for example $\pi_{k+1}, \cdots, \pi_n$, $(\pi_{k+1} \cdots \pi_n)\pi_1'$ would be of length n. This is impossible since $\pi_1'\pi_{k+1} \cdots \pi_n \sim (\pi_1'\pi_{k+1} \cdots \pi_n)\pi_k\pi_{k+1}$ in N and $Z(J(P_{1,0}))$ has only one element of length n. Thus we have shown that, if π_1' is of positive length, π_1' must be of length 1 and so $\pi_1' \sim \pi_1$ in G. Since $Z(J(P_{1,0})) = \langle \pi_1', \pi_2, \cdots, \pi_n \rangle$ and π_1' is of length 1, $\pi_1'\pi_2 \cdots \pi_{l+1}$ must be of length $l+1$. This proves (i). Similarly we can prove (ii).

(3.8) LEMMA. (i) *$\pi_1' \not\sim \pi_1$ in $G \Rightarrow N_G(S) = N_H(S)$, where $H = C_G(\alpha_n)$.* (ii) *$\lambda_1 \not\sim \pi_1$ in $G \Rightarrow N_G(M) = N_H(M)$.*

PROOF. We shall prove (i). Similarly we can work in the case (ii). It is sufficient to see that α_n is not conjugate in G to any element of S other than α_n, and so, by (3.4; (i)) it suffices to see $\alpha_n \not\sim \pi_{k,0}$ and $\alpha_n \not\sim \pi_{k,n-k}$ in G $(1 \leqq k \leqq n)$. We shall show this by induction on k. Since $\pi_1' \not\sim \pi_1$ in G by our assumption, it follows from (3.5; (i)) that $\alpha_n \not\sim \pi_{1,n-1}$ in G. This implies that our assertion is true for $k=1$. Suppose by the inductive hypothesis that, if $1 \leqq h < k$, we have $\pi_{h,0} \not\sim \alpha_n$ and $\pi_{h,n-h} \not\sim \alpha_n$ in G. Firstly, we shall show that $\pi_{k,n-k} \not\sim \alpha_n$ in G. Assume by way of contradiction that $\pi_{k,n-k} \sim \alpha_n$ in G. Then, since $Z(J(P_{k,n-k})) \ni \pi_{k,n-k}, \pi_{k+1}, \cdots, \pi_n$ and $\pi_{k,n-k} \sim \alpha_n$ in G, we have $\pi_{k,0} \sim \alpha_k$ in G. We know by (3.3; (iii)) that $U_{k,n-k}$ normalizes $Z(\bar{U}_{k,n-k}) = \langle \pi_1', \pi_2', \cdots, \pi_k', \pi_1, \pi_2, \cdots, \pi_n \rangle$. From the inductive hypothesis, (3.4; (i)) and $\pi_{k,0} \sim \alpha_k$ in G, it follows that the totality of elements in $Z(\bar{U}_{k,n-k})$ of length n is as follows:

$$\alpha_n \quad \text{and} \quad \pi_{k,n-k}x,$$

where x ranges over all elements of $\langle \pi_1, \cdots, \pi_k \rangle$. Denote by X the group generated by them. Then we have $X = \langle \pi_{k,0}, \pi_{k+1} \cdots \pi_n, \pi_1, \pi_2, \cdots, \pi_k \rangle$ and $X \lhd U_{k,n-k}$. The totality of elements in X of length 1 is

$$\pi_1, \pi_2, \cdots, \pi_k \qquad \text{if } k < n-1$$

and

$$\pi_1, \pi_2, \cdots, \pi_n \qquad \text{if } k \geqq n-1.$$

Since $X \lhd U_{k,n-k}$, we have $U_{k,n-k} \rhd \langle \pi_1, \pi_2, \cdots, \pi_k \rangle$ or $\langle \pi_1, \pi_2, \cdots, \pi_n \rangle$ according to whether $k < n-1$ or $k \geqq n-1$. In the second case, we have $[U_{k,n-k}, \pi_1\pi_2 \cdots \pi_n] = 1$. In the former case, we have $[U_{k,n-k}, \pi_1 \cdots \pi_k] = 1$ and so $[U_{k,n-k}, \alpha_n] = 1$ because $Z(U_{k,n-k}) \ni \pi_{k+1}, \cdots, \pi_n$ by (3.3; (ii)). Thus, in any case, we get $Z(U_{k,n-k}) \ni \alpha_n$. Then we have $\alpha_n \in Z(J(P_{k,n-k}))$, which is impossible since $\alpha_n, \pi_{k,n-k} \in Z(J(P_{k,n-k}))$ and they are of length n. Hence we have proved that $\alpha_n \not\sim \pi_{k,n-k}$ in G. Secondly assume that $\alpha_n \sim \pi_{k,0}$ in G. We have $Z(\bar{U}_{k,0}) = \langle \pi_1', \pi_2', \cdots, \pi_k', \pi_1, \pi_2, \cdots, \pi_n \rangle$ and the totality of elements in $Z(\bar{U}_{k,0})$ of length n is α_n and $\pi_{k,0}x$, where x ranges over all elements of $\langle \pi_1, \pi_2, \cdots, \pi_k \rangle$. If we denote by Y the group generated by them, we have $Y = \langle \pi_{k,0}, \pi_{k+1} \cdots \pi_n, \pi_1, \pi_2, \cdots, \pi_k \rangle$ and $U_{k,0} \triangleright Y$ by (3.3; (iii)). By the same argument as above, we get $Z(U_{k,0}) \ni \alpha_n$ and so $\alpha_n \in Z(J(P_{k,0}))$, which is impossible because $\alpha_n, \pi_{k,0} \in Z(J(P_{k,0}))$ and they are of length n. Hence we have proved that $\alpha_n \not\sim \pi_{k,0}$ in G. This completes the proof of our lemma.

§ 4. The case $N_G(S) > N_H(S)$.

(4.1) *In this section, we shall assume* $N_G(S) > N_H(S)$. Then, by (3.8), we have $\pi_1' \sim \pi_1$ in G. Further, we note that, if we work with M and λ_k's $(1 \leqq k \leqq n)$ in place of S and π_k's $(1 \leqq k \leqq n)$ respectively, we can obtain the corresponding results for M under the assumption $N_G(M) > N_H(M)$.

(4.2) LEMMA. *We have two possibilities Case I or Case II for the fusion in G of elements of S according to whether* $\alpha_2 \sim \pi_1'\pi_2'$ *or* $\alpha_1 \sim \pi_1'\pi_2'$. *More precisely, we have*

Case I (i) $\pi_{k,l} \sim \alpha_{k+l}$ *in G, and*

(ii) *there exist n elements* β_s $(1 \leqq s \leqq n)$ *of* $N_G(S)$ *of odd order such that* $\beta_s : \pi_s \to \pi_s' \to \pi_s\pi_s'$ *and* $[\beta_s, \pi_t] = [\beta_s, \pi_t'] = 1$ *for* $s \neq t$, *or*

Case II (i)′ $\pi_{2k-1,l} \sim \pi_{2k,l} \sim \alpha_{k+l}$ *in G and*

(ii)′ *there exist n elements* β_s $(1 \leqq s \leqq n)$ *of* $N_G(S)$ *of odd order such that* $\beta_s : \pi_s \to \pi_s' \to \pi_s\pi_s'$ *and* $[\beta_s, \pi_t] = [\beta_s, \pi_s\pi_t'] = 1$ *for* $s \neq t$.

PROOF. Since we have $\pi_1' \sim \pi_1$ in G, we can construct $\bar{U}_{1,0}$, $P_{1,0}$ and $U_{1,0}$ for an element $\pi_1' = \pi_{1,0}$ as in (3.3). For simplicity, we write $\bar{U}_{1,0} = \bar{U}$, $P_{1,0} = P$ and $U_{1,0} = U$. Then, by (i) and (iii) of (3.3), we know that $Z(\bar{U}) = \langle \pi_1, \pi_1' \rangle \times \langle \pi_2, \cdots, \pi_n \rangle$ and U normalizes $Z(\bar{U})$. Since $Z(U) \supseteqq \langle \pi_2, \cdots, \pi_n \rangle$, and, π_1, π_1' and $\pi_1\pi_1'$ are only elements of length 1 of $Z(\bar{U}) - \langle \pi_2, \cdots, \pi_n \rangle$, we get $U \triangleright \langle \pi_1', \pi_1 \rangle$. Further, since $Z(U) \supseteqq \langle \pi_2, \cdots, \pi_n \rangle$, $Z(J(P)) = \langle \pi_1', \pi_2, \cdots, \pi_n \rangle$ and $J(P)$ is conjugate in U to J, we have $\pi_1' \sim \pi_1$ in U. Therefore we have $U/C_U(\langle \pi_1', \pi_1 \rangle) \cong \mathfrak{S}_3$. This implies that there is an element β of U of odd order such that $\beta : \pi_1 \to \pi_1' \to \pi_1\pi_1'$. By (3.3; (iii)) we know that β normalizes all elementary subgroups of $\bar{U}$ of order 2^{2n}, in particular $S = S_1 \times \cdots \times S_n$ and $S_1 \times M_2 \times \cdots \times M_{k-1} \times S_k \times M_{k+1} \times \cdots \times M_n$. Hence β normalizes their intersection $\langle Z(\bar{U}), \pi_k' \rangle$. Since β normalizes

$Z(\bar{U})$ by (3.3; (iii)) and is of odd order, β must centralize an element of $\langle Z(\bar{U}), \pi'_k\rangle - Z(\bar{U})$, and so one of π'_k, $\pi'_k\pi_1$, $\pi'_k\pi'_1$ and $\pi'_k\pi_1\pi'_1$ because β centralizes $\langle\pi_2, \pi_3, \cdots, \pi_n\rangle$ and $\pi_1 \to \pi'_1 \to \pi_1\pi'_1$. Suppose that $[\beta, \pi'_k\pi'_1]=1$. Then we get $\pi'^{\beta}_k = \pi'_k\pi_1$, which is impossible because $\pi'_k \sim \pi_1$ and $\pi'_k\pi_1 \sim \pi'_1\pi_2 \sim \pi_1\pi_2$ by (3.7; (i)). Hence we get $[\beta, \pi'_k\pi'_1] \neq 1$. Similarly we have $[\beta, \pi'_k\pi_1\pi'_1] \neq 1$. Hence we get $[\beta, \pi'_k]=1$ or $[\beta, \pi'_k\pi_1]=1$. Firstly suppose that $[\beta, \pi'_k]=1$. Then we have $\beta: \pi'_k\pi_1 \to \pi'_k\pi'_1 \to \pi'_k\pi_1\pi'_1$. Since $\pi'_k\pi'_1 \sim \pi'_1\pi'_2$ in N and $\pi'_k\pi_1 \sim \alpha_2$ by (3.7; (i)), we get $\pi'_1\pi'_2 \sim \alpha_2$. Secondly suppose that $[\beta, \pi'_k\pi_1]=1$. Then we have $\pi'^{\beta}_k = \pi'_k\pi_1\pi'_1$. Hence we get $\beta: \pi'_k \to \pi'_k\pi_1\pi'_1 \to \pi'_k\pi'_1$. Since $\pi'_k \sim \pi'_1 \sim \pi_1$ by the assumption $N_G(S) > N_H(S)$ and (3.8), we get $\pi'_1\pi'_2 \sim \alpha_1$. From these facts it follows that we have $[\beta, \pi'_k]=1$ or $[\beta, \pi'_k\pi_1]=1$ according to whether $\alpha_2 \sim \pi'_1\pi'_2$ or $\alpha_1 \sim \pi'_1\pi'_2$. This implies that, if $\pi'_1\pi'_2 \sim \alpha_2$ in G, we must have $[\beta, \pi'_l]=1$ for *any* l $(2 \leqq l \leqq n)$, and if $\pi'_1\pi'_2 \sim \alpha_1$, we must have $[\beta, \pi'_l\pi_1]=1$ for *any* l $(2 \leqq l \leqq n)$.

Case I. Suppose that $\alpha_2 \sim \pi'_1\pi'_2$. If, for every l $(1 \leqq l \leqq n)$, we start with π'_l in place of π'_1 in the above discussions, we can find an element β_l of $N_G(S)$ of odd order such that $\beta_l: \pi_l \to \pi'_l \to \pi_l\pi'_l$ and $[\beta_l, \pi_k]=[\beta_l, \pi'_k]=1$ for $k \neq l$. Then we have $\beta_1^2\beta_2^2 \cdots \beta_k^2 : \pi_{k,t} \to \alpha_{k+t}$. Thus we get the first case in our lemma.

Case II. Suppose that $\alpha_1 \sim \pi'_1\pi'_2$ in G. If we start with π'_l in place of π'_1 in the above discussions, we can find an element β_l of $N_G(S)$ of odd order such that $\beta_l: \pi_l \to \pi'_l \to \pi_l\pi'_l$ and $[\beta_l, \pi_k]=[\beta_l, \pi'_k\pi_l]=1$ for $k \neq l$. If s is even $(1 \leqq s \leqq n)$, we have $\beta_1: \pi_{s,t} \to \pi'_2 \cdots \pi'_s\pi_{s+1} \cdots \pi_{s+t}$ since $\pi_{s,t} \sim (\pi_1\pi'_1) \cdots (\pi_1\pi'_s)\pi_{s+1} \cdots \pi_{s+t}$, β_1; $\pi_1\pi'_1 \to \pi_1$ and $[\beta_1, \pi'_k\pi_1]=1$ $(2 \leqq k \leqq s)$. If s is odd $(1 \leqq s \leqq n)$, we have $\beta_1^2: \pi_{s,t} \to \pi_1\pi'_2\pi'_3 \cdots \pi'_s\pi_{s+1} \cdots \pi_{s+t} \sim \pi_{s-1,t+1}$ since $\beta_1^2: \pi'_1 \to \pi_1$ and $\pi'_k \to \pi'_k\pi'_1$ $(2 \leqq k \leqq s)$. From these it follows that we have $\pi_{s,t} \sim \alpha_{s/2+t}$ or $\alpha_{s+1/2+t}$ according to whether s is even or odd. This yields the second case in our lemma.

(4.3) REMARK. (i) If we choose S as in §1, the first case in (4.2) occurs when $G = \mathfrak{S}_{4n}$, $\mathfrak{A}_{4n+2}$ or $\mathfrak{A}_{4n+3}$, and the second case in (4.2) does when $G = \Omega_{2n+2}(\varepsilon, q)$. (ii) If we take M in §1 as "S" in this section, then only the second case occurs in both "orthogonal" and "symmetric" cases.

(4.4) LEMMA. *Every element of $N_H(S)$ induces a permutation on the set $\{\pi'_1, \pi'_1\pi_1, \cdots, \pi'_n, \pi'_n\pi_n\}$, which consists of members of a basis of S.*

PROOF. Firstly suppose that we have case I for the fusion in G of elements of S. By (4.2), it is sufficient to see that $\pi'_k \not\sim \pi_l$ in $N_H(S)$ $(1 \leqq k, l \leqq n)$. If $\pi'^x_k = \pi_l$ for some $x \in N_H(S)$, we would have $(\pi'_k\alpha_n)^x = \pi_l\alpha_n$, which is impossible because $\pi'_k\alpha_n \sim \alpha_n$ and $\pi_l\alpha_n \sim \alpha_{n-1}$ in G. Secondly, suppose that we have case II. By (4.2), it is sufficient to see that $\pi'_k \not\sim \pi_l$ and $\pi'_k \not\sim \pi'_l\pi'_m$ in $N_H(S)$ $(1 \leqq k, l, m \leqq n)$. In the same way as case I, "$\pi'_k \sim \pi_l$ in $N_H(S)$" is impossible. If $\pi'^x_k = \pi'_l\pi'_m$ for some $x \in N_H(S)$, we would have $(\alpha_n\pi'_k)^x = \pi'_l\pi'_m\alpha_n$, which is impossible because $\alpha_n\pi'_k \sim \alpha_n$ and $\pi'_l\pi'_m\alpha_n \sim \pi_{2,n-2} \sim \alpha_{n-1}$ in G. This completes the proof of our lemma.

(4.5) LEMMA.

(i) $$N_H(S)/C_H(S) \cong \begin{cases} Z_2 \wr \mathfrak{S}_n & \text{for case I} \\ \mathfrak{S}_{2n} & \text{for case II}, \end{cases}$$

and

(ii) $$N_G(S)/C_G(S) \cong \begin{cases} \mathfrak{S}_3 \wr \mathfrak{S}_n & \text{for case I} \\ \mathfrak{S}_{2n+1} & \text{for case II}. \end{cases}$$

PROOF. Case I. Firstly we shall determine the structure of $N_H(S)/C_H(S)$. We note that, if we have case I, every element of $N_H(S)$ induces a permutation on the set $\{\pi_1, \pi_2, \cdots, \pi_n\}$ of n elements by (4.2). Put $\Pi = \{\pi_1', \pi_1'\pi_1, \cdots, \pi_n', \pi_n'\pi_n\}$ and $\Pi_k = \{\pi_k', \pi_k'\pi_k\}$ $(1 \leqq k \leqq n)$. Suppose that $\Pi_k^x \cap \Pi_l \neq \phi$, where $x \in N_H(S)$ and ϕ denotes the empty set. Then we have $\pi_l' = \pi_k'^x$ or $(\pi_k'\pi_k)^x$ if $\pi_l' \in \Pi_k^x \cap \Pi_l$, and $\pi_l'\pi_l = \pi_k'^x$ or $(\pi_k'\pi_k)^x$ if $\pi_l'\pi_l \in \Pi_k^x \cap \Pi_l$. For example, if $\pi_l' = \pi_k'^x$, we must have $\pi_l = \pi_k^x$. In fact, if $\pi_k^x = \pi_h$ $(h \neq l)$, we would have $(\pi_k'\pi_k)^x = \pi_l'\pi_h$ and so $(\alpha_n\pi_k'\pi_k)^x = \pi_l'\pi_h\alpha_n$, which is impossible because $\alpha_n\pi_k'\pi_k \sim \alpha_n$ and $\pi_l'\pi_h\alpha_n \sim \alpha_{n-1}$ if $h \neq l$. Thus we get $\Pi_h^x = \Pi_l$. Also in any other cases, we get $\Pi_k^x = \Pi_l$ if $\Pi_k^x \cap \Pi_l \neq \phi$. This implies that $N_H(S)/C_H(S)$ is an imprimitive permutation group on the set Π with Π_k's $(1 \leqq k \leqq n)$ as a class of sets of imprimitivity. On the other hand, N is a subgroup of $N_H(S)$ and $N \cap C_H(S) = S$. Further, from the structure of N, it follows that $NC_H(S)/C_H(S)$ is the maximal imprimitive group on the set Π with Π_k's $(1 \leqq k \leqq n)$ as a class of sets of imprimitivity. Hence we have $N_H(S) = NC_H(S)$. This implies that $N_H(S)/C_H(S) \cong Z_2 \wr \mathfrak{S}_n$. Denote by $\bar{x}$ the image of an element x by the canonical homomorphism of $N_G(S)$ onto $N_G(S)/C_G(S)$. Let β_k $(1 \leqq k \leqq n)$ be n elements defined in (4.2). Then from the action on S of β_k, λ_k and $\sigma \in P$, it follows that $\bar{\beta}_k^{\bar{\lambda}_k} = \bar{\beta}_k^{-1}$, $[\bar{\lambda}_k, \bar{\beta}_l] = [\bar{\beta}_k, \bar{\beta}_l] = 1$ $(k \neq l)$, and $\bar{\beta}_k^{\bar{\sigma}} = \bar{\beta}_{\sigma(k)}$. Remark that, in the right hand side of the last equality, σ is identified with an element of $\mathfrak{S}_n$ (cf. (1.1)). This implies that $N_G(S)/C_G(S)$ contains a subgroup isomorphic to $\mathfrak{S}_3 \wr \mathfrak{S}_n$. On the other hand, since S has 3^n elements conjugate in $N_G(S)$ to α_n by case I in (4.2) and (2.6), we have $[N_G(S) : N_H(S)] = 3^n$. This yields that we must have $N_G(S)/C_G(S) \cong \mathfrak{S}_3 \wr \mathfrak{S}_n$.

Case II. Let β_k $(1 \leqq k \leqq n)$ be n elements defined in (4.2: case II). Put $\delta_k = \beta_k^{-1}\beta_{k+1}\beta_k\lambda_{k+1}$ $(1 \leqq k \leqq n-1)$. Then from the action on S of λ_k $(1 \leqq k \leqq n)$ and δ_k $(1 \leqq k \leqq n-1)$, it follows that $N_H(S) \ni \delta_k$ and the set $\{\bar{\lambda}_1, \bar{\delta}_1, \bar{\lambda}_2, \cdots, \bar{\delta}_{n-1}, \bar{\lambda}_n\}$ is a set of canonical generators of $\mathfrak{S}_{2n}$ (for this terminology, see the introduction). Then, by (4.4), we must have $N_H(S)/C_H(S) \cong \mathfrak{S}_{2n}$. Further, from the action on S of $\beta_1\lambda_1$, it follows that the set $\{\bar{\beta}_1\bar{\lambda}_1, \bar{\lambda}_1, \bar{\delta}_1, \cdots, \bar{\lambda}_{n-1}, \bar{\delta}_{n-1}, \bar{\lambda}_n\}$ is a set of canonical generators of $\mathfrak{S}_{2n+1}$. Since S has $2n+1$ elements conjugate in $N_G(S)$ to α_n by (4.2: case II) and (2.6), we have $[N_G(S) : N_H(S)] = 2n+1$. This yields that $N_G(S)/C_G(S) \cong \mathfrak{S}_{2n+1}$. This completes the proof of (4.5).

(4.6) In the rest of the present paper, we shall consider the following conditions for S and M:

(Π) *every element of* $N_H(S)$ *induces a permutation on the set* $\{\pi_1', \pi_1'\pi_1, \cdots, \pi_n', \pi_n'\pi_n\}$,

(Λ) *every element of* $N_H(M)$ *induces a permutation on the set* $\{\lambda_1, \lambda_1\pi_1, \cdots, \lambda_n, \lambda_n\pi_n\}$.

If $N_G(S) > N_H(S)$ (resp. $N_G(M) > N_H(M)$), S (resp. M) satisfies the conditions (Π) (resp. (Λ)) by (4.4). For all examples in § 1, S and M satisfy the conditions (Π) and (Λ) respectively. Furthermore we note that

(Λ) *implies* $\lambda_1 \not\sim \lambda_1\pi_2$ *in* G, and

(Π) *implies* $\pi_1' \not\sim \pi_1'\pi_2$ *in* G.

In fact, if $\lambda_1 \sim \lambda_1\pi_2$ in G, (2.6) and (Λ) yield that $N_G(M) > N_H(M)$. Hence by (4.2), we have $\lambda_1 \sim \alpha_1$ and $\lambda_1\pi_2 \sim \alpha_2$ which is impossible if $\lambda_1 \sim \lambda_1\pi_2$, because $\alpha_1 \not\sim \alpha_2$ in G. Quite similarly the second statement follows.

(4.7) LEMMA. *Assume that* $N_G(S) > N_H(S)$ *and the condition* (Λ). *Then we have one of the followings:*

Case I′ $[\beta_k, \lambda_l] = 1$ *for any pair* $\{k, l\}$ $(k \neq l)$, *or*

Case II′ $[\beta_k, \lambda_l\pi_k] = 1$ *for any pair* $\{k, l\}$ $(k \neq l)$,

according to whether $\pi_1'\lambda_2 \sim \pi_1\lambda_2$ *or* $\pi_1'\lambda_2 \sim \lambda_1$.

PROOF. By (4.2), we know that $\beta_k: \pi_k \to \pi_k' \to \pi_k'\pi_k$ and $[\beta_k, \pi_l] = 1$ $(k \neq l)$ in both cases of (4.2). By the proof of (4.2), β_k normalizes all elementary abelian subgroups of $C_J(\pi_k')$ of order 2^{2n}, in particular $\langle \pi_k', \pi_k \rangle \times M_l \times \prod_{i\neq k,l} S_i$ and $\langle \pi_k', \pi_k \rangle \times M_l \times \prod_{i\neq k,l} M_i$. Hence β_k normalizes their intersection $Y_k = Z(J) \times \langle \pi_k', \lambda_l \rangle$. Then β_k must centralize an element of $Y_k - Z(J) \times \langle \pi_k' \rangle$ because β_k normalizes $Z(J) \times \langle \pi_k' \rangle$ and is of odd order. Therefore β_k centralizes one of λ_l, $\lambda_l\pi_k'$, $\pi_k'\pi_k\lambda_l$ and $\pi_k\lambda_l$ since $[\beta_k, \pi_l] = 1$ $(k \neq l)$. Suppose that $[\beta_k, \lambda_l\pi_k'] = 1$. Then, from $\lambda_l\pi_k' = (\lambda_l\pi_k')^{\beta_k} = \lambda_l^{\beta_k}\pi_k\pi_k'$, we get $\lambda_l^{\beta_k} = \lambda_l\pi_k$, which is impossible as remarked in (4.6) because $\lambda_l \sim \lambda_1$ and $\lambda_l\pi_k \sim \lambda_1\pi_2$ in G. Secondly suppose that $[\beta_k, \pi_k'\pi_k\lambda_l] = 1$. Then we get $\lambda_l^{\beta_k^2} = \lambda_l\pi_k$, which is impossible by the same reason as above. Thus we have $[\beta_k, \lambda_l] = 1$ or $[\beta_k, \lambda_l\pi_k] = 1$. If $[\beta_k, \lambda_l] = 1$, we must have $\lambda_l\pi_k' = (\lambda_l\pi_k)^{\beta_k}$, and so $\pi_1'\lambda_2 \sim \pi_1\lambda_2$ because $\lambda_2\pi_1' \sim \lambda_l\pi_k'$ and $\lambda_2\pi_1 \sim \lambda_l\pi_k$ in N. If $[\beta_k, \lambda_l\pi_k] = 1$, we must have $\lambda_l = (\lambda_l\pi_k')^{\beta_k}$, and so $\lambda_1 \sim \pi_1'\lambda_2$ in G. Therefore, if $\pi_1'\lambda_2 \sim \pi_1\lambda_2$ in G, we must have $[\beta_k, \lambda_l] = 1$ for any pair $\{k, l\}$ $(k \neq l)$, and if $\pi_1'\lambda_2 \sim \lambda_1$, we must have $[\beta_k, \lambda_l\pi_k] = 1$ for any pair $\{k, l\}$ $(k \neq l)$. The proof is complete.

(4.8) LEMMA. *Assume that* $N_G(M) > N_H(M)$ *and* (Π). *Then we have one of the followings:*

Case I″ $[\gamma_k, \pi_l'] = 1$ *for any pair* $\{k, l\}$ $(k \neq l)$, *or*

Case II″ $[\gamma_k, \pi_l'\pi_k] = 1$ *for any pair* $\{k, l\}$ $(k \neq l)$

according to whether $\pi_1'\lambda_2 \sim \pi_1'\pi_2$ *or* $\pi_1'\lambda_2 \sim \pi_1'$. *Here* γ_k's $(1 \leqq k \leqq n)$ *are the ele-*

ments constructed for M in place of S in (4.2) (cf. (4.1)).

(4.9) LEMMA. *Assume that $N_G(S) > N_H(S)$ and $N_G(M) > N_H(M)$. Then we have $[\beta_k, \lambda_l] = 1$ and $[\gamma_k, \pi_l'] = 1$ $(k \neq l)$.*

PROOF. By (4.4) S and M satisfy the assumptions of (4.7) and (4.8) respectively. Furthermore we know that $\pi_1' \sim \lambda_1 \sim \alpha_1$ and $\pi_1'\pi_2 \sim \pi_1\lambda_2 \sim \alpha_2$ in G by (4.2). Therefore by (4.7) and (4.8), it is sufficient to see that $\pi_1'\lambda_2 \sim \alpha_2$. Put

$$F = \langle \pi_1', \pi_1 \rangle \times \langle \lambda_2, \pi_2 \rangle \quad \text{and} \quad X = N_G(F)/C_G(F).$$

We shall determine the structure of X. Firstly we note that, from (4.7) and (4.8), we have $N_G(F) \ni \beta_1, \gamma_2$ for any cases of the lemmas. Take a 2-Sylow subgroup D of $N_G(F)$ containing J. (Note that $J \triangleright F$.) Then we have $D \triangleright J$ and so $D \subset N_G(J) \cap N_G(F)$. Since $N_G(J) = N \cdot C_G(J)$, it follows from the structure of N that $D \cdot C_G(F) = \langle \lambda_1, \pi_2' \rangle \cdot C_G(F)$. This implies that the four group $\langle \bar{\lambda}_1, \bar{\pi}_2' \rangle$ is a 2-Sylow subgroup of X. From the action of λ_1 and $\lambda_1\pi_2'$ on F, we see that $\bar{\lambda}_1$ and $\bar{\lambda}_1\bar{\pi}_2'$ are not conjugate in X. Therefore X has a normal 2-complement, and so $|X| = 4 \cdot 3^a$ $(0 \leqq a \leqq 2)$ by the structure of $GL(4, 2)$ because X can be regarded as a subgroup of $GL(4, 2) \cong A_8$. Since $N_G(F) - C_G(F) \ni \beta_1, \gamma_2$, we get $N_G(F) = \langle \lambda_1, \pi_2', \beta_1, \gamma_2 \rangle \cdot C_G(F)$. This yields that $[N_G(F) \cap C_G(\alpha_2) : C_G(F)] = 4$ and so $[N_G(F) : N_G(F) \cap C_G(\alpha_2)] = 9$. Namely, α_2 has nine conjugates in $N_G(F)$. Since $\pi_1, \pi_1', \pi_1'\pi_1, \lambda_2, \pi_2$ and $\lambda_2\pi_2$ are of length 1 by (4.2), we must have $\pi_1'\lambda_2 \sim \alpha_2$ in $N_G(F)$. This completes the proof of our lemma.

(4.10) LEMMA. *Assume that $N_G(S) > N_H(S)$ and (Λ). Without loss of generality, we may assume that $[\beta_k, \lambda_l] = 1$ $(k \neq l)$.*

PROOF. If $N_G(M) > N_H(M)$, our lemma follows from (4.9). Assume that $N_G(M) = N_H(M)$ and we have case II′ in (4.7), namely $[\beta_k, \lambda_l\pi_k] = 1$ for any pair $\{k, l\}$ $(k \neq l)$. Then we have $[\beta_k, \lambda_l\alpha_n] = 1$, because $[\beta_k, \pi_h] = 1$ $(k \neq h)$. We can replace λ_l's by $\lambda_l\alpha_n$'s $(1 \leqq l \leqq n)$ from the structure of N. (Note that, since $N_G(M) = N_H(M)$ and so $\lambda_l\alpha_n \not\sim \alpha_n$, this replacement does not conflict with that of (3.6) and does not destroy the condition (Λ).) Thus we may assume that $[\beta_k, \lambda_l] = 1$ by the suitable choice of notations.

(4.11) LEMMA. *Assume that $N_G(M) > N_H(M)$ and (Π). Then without loss of generality, we may assume that $[\gamma_k, \pi_l'] = 1$ $(k \neq l)$.*

(4.12) Summarizing the results of this section, we obtain the following theorem.

THEOREM. (1) *Assume that $N_G(S) > N_H(S)$ and M satisfies the condition (Λ). Then we have one of the followings:*

Case I (i) *there exist n elements β_s $(1 \leqq s \leqq n)$ of $N_G(S)$ such that*

(i-1) *β_s is of odd order,*

(i-2) $\beta_s : \pi_s \to \pi_s' \to \pi_s'\pi_s$,

(i-3) $[\beta_s, \pi_t] = [\beta_s, \pi_t'] = [\beta_s, \lambda_t] = 1$ $(s \neq t)$, *and*

(ii) $N_G(S)/C_G(S) \cong \mathfrak{S}_3 \wr \mathfrak{S}_n$ *and* $N_H(S)/C_H(S) \cong Z_2 \wr \mathfrak{S}_n$,

or

Case II (i) *there exist n element* β_s $(1 \leqq s \leqq n)$ *of* $N_G(S)$ *such that*

(i-1) β_s *is of odd order,*

(i-2) $\beta_s: \pi_s \to \pi'_s \to \pi'_s \pi_s$,

(i-3) $[\beta_s, \pi_t] = [\beta_s, \pi_s \pi'_t] = [\beta_s, \lambda_t] = 1$ $(s \neq t)$,

and

(ii) $N_G(S)/C_G(S) \cong \mathfrak{S}_{2n+1}$ *and* $N_H(S)/C_H(S) \cong \mathfrak{S}_{2n}$.

(2) *Assume that* $N_G(M) > N_H(M)$ *and S satisfies the condition* (Π). *Then we have one of the followings:*

Case I (i) *there exist n elements* γ_s *of* $N_G(M)$ *such that*

(i-1) γ_s *is of odd order,*

(i-2) $\gamma_s: \pi_s \to \lambda_s \to \lambda_s \pi_s$,

(i-3) $[\gamma_s, \pi_t] = [\gamma_s, \lambda_t] = [\gamma_s, \pi'_t] = 1$ $(s \neq t)$,

(ii) $N_G(M)/C_G(M) \cong \mathfrak{S}_3 \wr \mathfrak{S}_n$ *and* $N_H(M)/C_H(M) \cong Z_2 \wr \mathfrak{S}_n$,

or

Case II (i) *there exist n elements* γ_s $(1 \leqq s \leqq n)$ *of* $N_G(M)$ *such that*

(i-1) γ_s *is of odd order,*

(i-2) $\gamma_s: \pi_s \to \lambda_s \to \lambda_s \pi_s$,

(i-3) $[\gamma_s, \pi_t] = [\gamma_s, \lambda_t \pi_s] = [\gamma_s, \pi'_t] = 1$ $(s \neq t)$,

and

(ii) $N_G(M)/C_G(M) \cong \mathfrak{S}_{2n+1}$ *and* $N_H(M)/C_H(M) \cong \mathfrak{S}_{2n}$.

(3) *If* $N_G(S) > N_H(S)$ *and* $N_G(M) > N_H(M)$, *S and M satisfy* (Π) *and* (Λ) *respectively, and so* (1) *and* (2) *hold.*

§ 5. The fusion under the additional assumption to M.

(5.1) In the rest of the present paper, besides the fundamental assumption to G in (1.1), we shall assume that

(i) $N_H(M)/C_H(M) \cong \mathfrak{S}_{2n}$

and

(ii) M satisfies the condition (Λ) in (4.6).

We remark that, if $N_H(M) < N_G(M)$, (ii) is an immediate consequence of (4.4) applied to M in place of S and we must have case II for the fusion in G of M, and $N_G(M)/C_G(M) \cong \mathfrak{S}_{2n+1}$ by (4.5). If we choose M as in § 1, all examples in § 1 satisfy the conditions (i) and (ii).

Since M is self-centralizing normal subgroup of a 2-Sylow subgroup of H, we have $C_H(M)=M\times F$ and $|F|=\text{odd}$. Put $\overline{W}=N_H(M)/F$ and, for a subset X of $W=N_H(M)$, denote by $\bar{X}$ the image of X by the canonical homomorphism from W onto $\overline{W}$.

LEMMA. *There exists a complement $\bar{K}$ of $\overline{W}$ over $\bar{M}$ and $n-1$ involutions $\bar{\sigma}'_i$ $(1\leqq i\leqq n-1)$ of $\bar{K}$ such that $\{\bar{\pi}'_1, \bar{\sigma}'_1, \cdots, \bar{\sigma}'_{n-1}, \bar{\pi}'_n\}$ is a set of canonical generators of $\bar{K}$.*

PROOF. By a theorem of Gaschütz [3], there is a complement $\bar{K}$ of $\overline{W}$ over $\bar{M}$. Then the above assumptions (i) and (ii) to M yield that there are $2n-1$ involutions $\{\bar{y}_1, \bar{z}_1, \bar{y}_2, \cdots, \bar{z}_{n-1}, \bar{y}_n\}$ of $\bar{K}$ such that

$$\bar{\lambda}_i^{\bar{y}_i}=\bar{\lambda}_i\bar{\pi}_i,\ [\bar{\lambda}_j, \bar{y}_i]=[\bar{\lambda}_j\bar{\pi}_j, \bar{y}_i]=1 \qquad (j\neq i)$$

$$(\bar{\lambda}_i\bar{\pi}_i)^{\bar{z}_i}=\bar{\lambda}_{i+1},\ [\bar{\lambda}_j, \bar{z}_i]=[\bar{\lambda}_k\bar{\pi}_k, \bar{z}_i]=1 \qquad (j\neq i+1,\ k\neq i)\,.$$

From the action of $\bar{\pi}'_i$ on $\bar{M}$, we see that $\bar{y}_i\equiv\bar{\pi}'_i \bmod \bar{M}$. Now we claim that $\bar{y}_i=\bar{\pi}'_i$ for any i $(1\leqq i\leqq n)$ or $\bar{y}_i=\bar{\pi}'_i\bar{\alpha}_n$ for any i $(1\leqq i\leqq n)$. In fact as is easily seen from (1.2; (v)), $\bar{N}_1=\langle \bar{y}_i, \bar{\pi}_i, \bar{\lambda}_i, (\bar{y}_j\bar{y}_{j+1})^{\bar{z}_j} | 1\leqq i\leqq n,\ 1\leqq j\leqq n-1\rangle$ is conjugate in $\overline{W}$ to $\bar{N}$ and the cardinality of the orbit containing $\bar{y}_i$ under the action on $\langle \bar{y}_i, \bar{\pi}_i | 1\leqq i\leqq n\rangle$ of $\bar{N}_1$ is $2n$. Considering the orbit under the action on S of N (cf. (2.1)) and using the fact that $\bar{y}_i\equiv\bar{\pi}'_i \bmod \bar{M}$, it follows that $\bar{y}_i=\bar{\pi}'_i$ or $\bar{\pi}'_i\bar{\alpha}_n$ $(1\leqq i\leqq n)$. Since $\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_n$ are conjugate in $\overline{W}$, we must have $\bar{y}_i=\bar{\pi}'_i$ for any i $(1\leqq i\leqq n)$ or $\bar{y}_i=\bar{\pi}'_i\bar{\alpha}_n$ $(1\leqq i\leqq n)$. If we have the former case, our lemma holds, while if we have the latter case, the subgroup $\langle \bar{y}_1\bar{\alpha}_n, \bar{z}_1\bar{\alpha}_n, \cdots, \bar{z}_{n-1}\bar{\alpha}_n, \bar{y}_n\bar{\alpha}_n\rangle$ has the required properties.

(5.2) LEMMA. *The representatives of conjugacy classes of involutions of $N_H(M)$ are $\pi_{k,l}$ $(0<k+l\leqq n)$ and $\tau_{k,l}$ $(0\leqq k+l\leqq n-1)$, where $\tau_{k,l}$'s are elements defined in* (3.1).

PROOF. We note that two involutions x, y of W are conjugate in W if and only if $\bar{x}$ and $\bar{y}$ are conjugate in $\overline{W}$ because F is of odd order. Then our lemma follows from Lemma in (5.1) and Remark in (1.2; (v)).

(5.3) LEMMA. *If G has no normal subgroup of index* 2, *every involution of G must be conjugate in G to an element of S.*

PROOF. It is sufficient to see that every involution of $N_H(M)$ fuses to an element of S, because $N_H(M)$ contains a 2-Sylow subgroup of G. From the structure of $N_H(M)$, it follows that there is a subgroup K_0 of $N_H(M)$ of index 2 such that K_0 contains S but does not contain $\tau_{k,l}$'s. By (5.2), every involution of K_0 must be conjugate in $N_H(M)$ to an element of S. Further, since G has no normal subgroup of index 2, a lemma of J. G. Thompson yields that $\tau_{k,l}$ is conjugate to an element of K_0, and so one of S. This completes the proof of our lemma.

(5.4) LEMMA. *Assume that $N_G(M)>N_H(M)$ and S satisfies the condition*

(II) in (4.6). *Then we have* $\tau_{k,l}\sim\pi_{k,l+1}$ *in* G.

PROOF. Let γ_n be as in (4.11). Then we have $\tau_{k,l}^{\gamma_n^{-1}}=\pi_{k,l}\pi_n$. Since $\pi_{k,l}\pi_n\sim\pi_{k,l+1}$ in N, we get $\tau_{k,l}\sim\pi_{k,l+1}$ in G.

§ 6. The degenerate case $N_G(S)=N_H(S)$.

(6.1) In this section, *we shall assume the conditions* (i) *and* (ii) *in* (5.1) *for* M.

(6.2) LEMMA. *Assume that* $N_H(M)=N_G(M)$ *and* $N_H(S)=N_G(S)$. *Then we have* $G=HO(G)$, *where* $O(G)$ *denotes the largest normal subgroup of* G *of odd order. In particular,* G *is not simple.*

PROOF. We shall show that α_n is not conjugate in G to any element of H other than α_n. By (5.2), we know that the representatives of conjugacy classes of involutions in H are $\pi_{k,l}$ $(0<k+l\leqq n)$ and $\tau_{k,l}$ $(0\leqq k+l\leqq n-1)$. Then, by the assumption $N_G(S)=N_H(S)$, we have $\pi_{k,l}\not\sim\alpha_n$. Hence, by (3.4: (iii)) it is sufficient to see that $\tau_{k,0}\not\sim\alpha_n$ and $\tau_{k,n-1-k}\not\sim\alpha_n$ in G. We shall prove this by induction on k. By (3.6) and the assumption $N_H(M)=N_G(M)$, we have $\tau_{0,0}\not\sim\alpha_n$ and $\tau_{0,n-1}\not\sim\alpha_n$ in G. This implies that our assertion is true for $k=0$. Assume by the inductive hypothesis that, if $0\leqq h<k$, $\tau_{h,0}\not\sim\alpha_n$ and $\tau_{h,n-1-h}\not\sim\alpha_n$ in G. Suppose by way of contradiction that $\tau_{k,n-1-k}\sim\alpha_n$ in G. Then we can construct $\overline{W}_{k,n-1-k}$, $T_{k,n-1-k}$ and $W_{k,n-1-k}$ for an element $\tau_{k,n-1-k}$ as in (3.3). Put $\overline{W}_{k,n-1-k}=\overline{W}$, $T_{k,n-1-k}=T$ and $W_{k,n-1-k}=W$. Then we have $Z(\overline{W})=S_1\times S_2\times\cdots\times S_k\times\langle\pi_{k+1},\cdots,\pi_{n-1}\rangle\times\langle\pi_n,\lambda_n\rangle$. From the assumption of our lemma, inductive hypothesis and (3.4; (iii)), it follows that the totality of elements in $Z(\overline{W})$ of length n is α_n and $\tau_{k,n-1-k}x$, where x is an arbitrary element in $\langle\pi_1,\pi_2,\cdots,\pi_k\rangle\times\langle\pi_n\rangle$. (Remark that, if $\tau_{k,n-1-k}\sim\alpha_n$ in G, we have $\tau_{k,0}\not\sim\alpha_n$ in G. Otherwise, $Z(J(W_{k,0}))$ would have two elements $\tau_{k,n-1-k}$ and $\tau_{k,0}$ of length n.) Denote by X the group generated by α_n and $\tau_{k,n-1-k}x$'s. Then we have $X=\langle\tau_{k,n-1-k},\pi_1,\pi_2,\cdots,\pi_k,\pi_{k+1}\cdots\pi_{n-1},\pi_n\rangle$. Since $W\triangleright Z(\overline{W})$ by (3.3: (iii)$''$), we get $W\triangleright X$. The totality of elements in X of length 1 is $\{\pi_1,\pi_2,\cdots,\pi_k,\pi_n\}$ or $\{\pi_1,\pi_2,\cdots,\pi_n\}$ according to whether $k<n-2$ or $k\geqq n-2$. In the second case, $W\triangleright\langle\pi_1,\pi_2,\cdots,\pi_n\rangle$ and so $[W,\alpha_n]=1$. In the former case, we have $W\triangleright\langle\pi_1,\pi_2,\cdots,\pi_k,\pi_n\rangle$ and so $[W,\alpha_n]=1$, because $[W,\pi_{k+1}\cdots\pi_{n-1}]=1$ by (3.3: (ii)$''$). Then $Z(J(T))$ has two elements $\tau_{k,n-1-k}$ and α_n of length n, which is impossible. Thus we have proved that $\alpha_n\not\sim\tau_{k,n-1-k}$ in G. Secondly suppose that $\alpha_n\sim\tau_{k,0}$ in G. We have $Z(\overline{W}_{k,0})=S_1\times\cdots\times S_k\times\langle\pi_{k+1},\cdots,\pi_{n-1}\rangle\times\langle\pi_n,\lambda_n\rangle$ and the totality of elements in $Z(\overline{W}_{k,0})$ of length n is α_n and $\tau_{k,0}x$, where x is an arbitrary element in $\langle\pi_1,\cdots,\pi_k\rangle\times\langle\pi_n\rangle$. If we denote by Y the group generated by them, we have $Y=\langle\tau_{k,0},\pi_1,\cdots,\pi_k,\pi_{k+1}\cdots\pi_{n-1},\pi_n\rangle$. By the same argument as above, we get $Z(W_{k,0})\ni\alpha_n$ and so $\alpha_n\in Z(J(T_{k,0}))$, which is im-

possible because α_n, $\tau_{k,0} \in Z(J(T_{k,0}))$ and they are of length n. Thus we have proved that α_n is not conjugate in G to any element of H other than α_n. Then our lemma follows from Glauberman's theorem [4] and Frattini argument.

(6.3) LEMMA. *Assume that H has a normal subgroup of index 2 and S satisfies the condition (Π) in* (4.6). *Then if $N_G(S)=N_H(S)$, G has a normal subgroup of index* 2.

PROOF. If $N_G(M)=N_H(M)$, our lemma follows from (6.2). Assume that $N_G(M)>N_H(M)$. Put $D_1=MP\langle\pi_1'\pi_2', \pi_1'\pi_3', \cdots, \pi_1'\pi_n'\rangle$ and then we have $N=D_1\langle\pi_1'\rangle$. Then N contains a 2-Sylow subgroup of G by (1.1; (ii)) and $[N:D_1]=2$. From (5.2) and (5.4) it follows that every involution of D_1 is conjugate in G to an element $S\cap D_1$. If G has no normal subgroup of index 2, a lemma of Thompson yields that π_1' must fuse to an element of D_1 and so one of $S\cap D_1$. Since π_1' is not conjugate in $N_H(S)$ to any element of $S\cap D_1$ by the assumption of our lemma, (2.6) yields that $N_H(S)<N_G(S)$. This is a contradiction.

(6.4) THEOREM. *Assume that M satisfies the conditions* (i) *and* (ii) *in* (5.1), *and, S and H satisfy the same assumptions as* (6.3). *If G has no normal subgroup of index* 2, *the followings hold;*

(i) *$N_H(S)<N_G(S)$ and $N_H(M)<N_G(M)$,*

(ii) *G has exactly n classes of involutions with the representatives $\alpha_1, \alpha_2, \cdots, \alpha_n$, and*

(iii) *G has two possibilities for the fusion of involutions.*

PROOF. By (6.3), we have $N_H(S)<N_G(S)$. Then (4.2) yields that each element of S must be conjugate in G to one of $\alpha_1, \alpha_2, \cdots, \alpha_n$. From (5.3) it follows that λ_n must fuse in G to one of α_k's $(1\leqq k\leqq n)$ and so $\lambda_n\sim\alpha_1$ in G by (3.7). By (3.5) and (2.6), we have $N_H(M)<N_G(M)$. Then (5.3), (4.2) and (4.5) yield that G has exactly n classes of involutions and two possibilities for the fusion of involutions.

§ 7. Applications.

(7.1) *The Alternating Case.* Let α_n be an involution of $\mathfrak{A}_{4n+r}$ ($r=2$ or 3) which has a cycle decomposition

$$(1, 2)(3, 4) \cdots (4n-1, 4n),$$

and λ_k, π_k', π_k, H, S and M be as in (1.2: (ii)). Let G be a finite group satisfying the following conditions:

(i) *G has no normal subgroup of index* 2, *and*

(ii) *G contains an involution $\tilde{\alpha}_n$ in the center of a* 2*-Sylow subgroup of G whose centralizer $\tilde{H}$ is isomorphic to H.*

For simplicity, we identify elements and subgroups of H with the corresponding ones of $\tilde{H}$. Then we have the following

THEOREM A. *G has exactly n classes of involutions with the representatives* $\alpha_1, \alpha_2, \cdots, \alpha_n$. *More precisely, there exist elements* β_s *and* γ_s $(1 \leqq s \leqq n)$ *of odd order with the following properties;*

(i) $\beta_s \in N_G(S)$ *and* $\gamma_s \in N_G(M)$,

(ii) $\beta_s : \pi_s \to \pi'_s \to \pi_s \pi'_s$ *and* $[\beta_s, \pi_t] = [\beta_s, \pi'_t] = [\beta_s, \lambda_t] = 1$ $(s \neq t)$, *and*

(iii) $\gamma_s : \pi_s \to \lambda_s \to \lambda_s \pi_s$, $[\gamma_s, \pi_t] = [\gamma_s, \pi_s \lambda_t] = 1$ $(s \neq t)$ *and* $[\gamma_s, \pi'_t] = 1$ $(1 \leqq s,\ t \leqq n$ *and* $s \neq t)$.

In particular, we have

(iv) $\pi_{s,t} \sim \alpha_{s+t}$,

(v) $\lambda_{2s-1,t} \sim \lambda_{2s,t} \sim \alpha_{s+t}$, *and*

(vi) $\tau_{s,t} \sim \alpha_{s+t+1}$,

where $\pi_{s,t}$, $\lambda_{s,t}$ *and* $\tau_{s,t}$ *are involutions defined in* (3.1).

PROOF. G satisfies all assumptions of Theorem (6.4) (cf. (1.3)). Hence G has exactly n classes of involutions. Further we have $N_H(S) < N_G(S)$. Since $N_H(S)/C_H(S) \cong Z_2 \wr \mathfrak{S}_n$ (cf. (1.3)), we must have case I for the fusion in G of S by (4.5). Then our theorem follows from (4.2), (4.10), (4.11) and (5.4).

(7.2) *The Orthogonal Case.* Let $\Omega_{2n+2}(\varepsilon, q)$ $(q^{n+1} \equiv -\varepsilon \bmod 4$ and $q \equiv \pm 3 \bmod 8)$ be the orthogonal commutator group with the underlying quadratic form $\sum_{i=1}^{2n} x_i^2 + x_{2n+1}^2 + a x_{2n+2}^2$, where a is a nonsquare element of the finite field of q elements. Put $\alpha_n = \begin{pmatrix} -I_{2n} & \\ & I_2 \end{pmatrix}$, where $I_k =$ the $k \times k$ unit matrix. Then α_n is an involution in the center of 2-Sylow subgroup of $\Omega_{2n+2}(\varepsilon, q)$. By H we denote the centralizer in $\Omega_{2n+2}(\varepsilon, q)$ of α_n. Let λ_k, π_k, π'_k, S and M be as in (1.2: (iv)) and (1.1).

Let G be a finite group satisfying the following conditions;

(i) *G has no normal subgroup of index* 2, *and*

(ii) *G contains an involution* $\tilde{\alpha}_n$ *in the center of a 2-Sylow subgroup of G whose centralizer* $\tilde{H}$ *is isomorphic to H.*

We identify elements and subgroups of H with the corresponding elements of $\tilde{H}$. Then we have the following

THEOREM B. *G has exactly n classes of involutions with representatives* $\alpha_1, \alpha_2, \cdots, \alpha_n$. *More precisely, there exist elements* β_s *and* γ_s $(1 \leqq s \leqq n)$ *of odd order such that*

(i) $\beta_s \in N_G(S)$ *and* $\gamma_s \in N_G(M)$,

(ii) $\beta_s : \pi_s \to \pi'_s \to \pi_s \pi'_s$, $[\beta_s, \pi_t] = [\beta_s, \pi_s \pi'_t] = 1$ *and* $[\beta_s, \lambda_t] = 1$ $(1 \leqq s,\ t \leqq n,\ s \neq t)$, *and*

(iii) $\gamma_s : \pi_s \to \lambda_s \to \pi_s \lambda_s$, $[\gamma_s, \pi_t] = [\gamma_s, \pi_s \lambda_t] = 1$ *and* $[\gamma_s, \pi'_t] = 1$ $(1 \leqq s,\ t \leqq n,\ s \neq t)$.

In particular, we have

(iv) $\pi_{2s-1,t} \sim \pi_{2s,t} \sim \alpha_{s+t}$,

(v) $\lambda_{2s-1,t} \sim \lambda_{2s,t} \sim \alpha_{s+t}$, *and*

(vi) $\tau_{2s-1,t} \sim \tau_{2s,t} \sim \alpha_{s+t+1}$.

PROOF. G satisfies all assumptions of Theorem (6.4) (cf. (1.3)). Hence G has n classes of involutions with the representatives $\alpha_1, \cdots, \alpha_n$. Further we have $N_H(S) < N_G(S)$ and $N_H(M) < N_G(M)$. Since $N_H(S)/C_H(S) \cong N_H(M)/C_H(M) \cong \mathfrak{S}_{2n}$ (cf. (1.3)), we must have case II for the fusion in G of S and M by (4.5). Then our theorem follows from (4.2), (4.11) and (5.4).

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