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On finite groups with their character tables having at most $p + 1$ zeros in each column

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Abstract
In the present paper, the authors classify finite solvable groups whose character tables have at most $p + 1$ zeros in each column, where p is the minimal prime divisor of their orders.
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1 Introduction

The distribution of zeros in the character table of a group is a classical problem with an extensive literature that goes back to the well-known theorem of Burnside stating that any non-linear irreducible character of a finite group vanishes at some element of the group. Hence it is interesting to investigate the properties of a finite group with 'many' zeros or without zeros at some special places in its character table. There are many results about the number of zeros in the rows of a character table (see [2, 3, 6, 17] and [7], etc.).

On the other hand, we find that it is also interesting to study the distribution of zeros in terms of a column. Isaacs *et al.* [5] investigated the property of an element x of odd order such that every irreducible character does not take zero on x, and proved that such element x was contained in the Fitting subgroup. Hence the number of zeros in the columns of a character table will have effect on the group structure. Xu *et al.* [13] classified finite groups of odd order with their character tables having at most p zeros on each column, where p is the smallest prime divisor of the group order. In this paper, we classify finite solvable groups with their character tables having at most $p + 1$ zeros in each column, where p is the minimal prime divisor of their orders.

Definition 1.1 A group is called $V(k)$-group if its character table has at most k zeros in each column.

Definition 1.2 Let $\theta(x)$ be a character, and A be a subset of G. For $x \in A$, if $\theta(x) = 0$, we say θ vanishes at x. If $\theta(x) = 0$ for any $x \in A$, we say that θ vanishes on A.

All further unexplained symbols and notation are standard and can be found, for instance, in [4]. In particular, Z_p^n denotes an elementary abelian group of order p^n.

In this paper, we study solvable $V(p + 1)$-groups, where p is a minimal prime divisor of $|G|$. Hence there are two cases that need to be considered: $p = 2$ and $p \neq 2$.

For $p = 2$, we have the following theorem.

Theorem A *Let G be solvable such that* $2||G|$. *Then G is a* $V(2+1)$*-group if and only if one of the following holds:*

(1) G has exactly one non-linear irreducible character;
(2) G has exactly two non-linear irreducible characters;
(3) G has exactly three non-linear irreducible characters;
(4) G has a normal series $1 < G'' < G' < G$, *where* $|G/G'| = 2$, $|G'/G''| = 3$, $G/G'' \cong S_3$, *and* $G'' \cong D_8$ *or* Q_8*;*
(5) $G \cong G' \rtimes Z_2$, *where* $G' = Z_2^3 \rtimes Z_7$ *is a Frobenius group;*
(6) $G \cong G' \rtimes Z_2$, *where* $G' = Z_2^4 \rtimes Z_5$ *is a Frobenius group, and the action of G on the set* $\mathrm{Irr}_1(G')$ *is non-trivial;*
(7) $G \cong G' \rtimes Z_3$, *where* $G' \cong Q_8$ *or* D_8*;*
(8) $G \cong G' \rtimes Z_3$, *where* $G' \cong Z_2^3 \rtimes Z_7$ *is a Frobenius group;*
(9) G is abelian.

For $p \neq 2$, *then G is an odd order by the minimality of p, and G must be solvable. And we get:*

Theorem B *Let G be a finite group and p be the minimal odd prime divisor of its order. Then G is a* $V(p+1)$*-group if and only if one of the following holds:*

(i) G is abelian;
(ii) G is an extra-special p-group;
(iii) $G = G' \rtimes L$ *is a Frobenius group with an elementary abelian kernel* G' *and a cyclic complement L, and* $|G'| - 1 \leq (p+1)|L|$.

2 Preliminaries

In this section, we give some results which will be applied to our further investigations.

Lemma 2.1 ([12, Theorem 1]) *Let G be a solvable finite group. Then the character table of G has at most one zero in each column if and only if one of the following holds:*

(1) G is abelian;
(2) G has exactly one non-linear irreducible character, and one of the following holds:
(2.1) G is an extra-special 2*-group;*
(2.2) $G = N \rtimes H$ *is a Frobenius group with a kernel N and a complement H, where H is an abelian group and N is an elementary abelian p-group such that* $|H| = |N| - 1$.

Lemma 2.2 ([1, Theorem B]) *Let x be an element in a p-group P with* $|x^P| = p^b$. *Then there exist at least* $b(p-1)$ *non-linear irreducible characters vanishing at x.*

Lemma 2.3 ([8, Proposition 3.2.2]) *Let G be a solvable group. Assume that G has at most two irreducible characters of degree m* $(\neq 1)$. *If* $|G/G'| = 2$, *then one of the following occurs:*

(1) $G \cong S_3$*;*
(2) $G \cong Z_5 \rtimes Z_2$ *is a Frobenius group with a kernel* Z_5 *and a complement* Z_2*;*
(3) $G \cong S_4$.

If $|G/G'| \neq 2$, *G is* π*-nilpotent and a normal* π*-complement of G is non-abelian, where* $\pi = \pi(G/G')$, *then one of the following occurs:*

(i) $G = K \rtimes H$ *is a Frobenius group with a kernel K of order* 25 *and a complement H, where K is an elementary abelian group and H is a dihedral group* D_{12}*;*

(ii) *$G = K \rtimes A$ is a Frobenius group with an elementary abelian kernel K of order* 81 *and a complement A, where $A/Z(A) \cong Z_5 \rtimes Z_4$ and $|Z(A)| = 2$;*
(iii) *$G \cong AB$, where A is a cyclic subgroup of order* 4 *and B is an extra-special group of order* 27*;*
(iv) *G is the normalizer of a Sylow* 2*-group of the simple group* $\mathrm{Sz}(2^{2m+1})$*.*

Lemma 2.4 ([14]) *Let G be a finite group and $1 \neq N \lhd G$. Then G is a Frobenius group with a kernel N if and only if θ^G is irreducible for any $\theta \in \mathrm{Irr}(N)$ and $\theta \neq 1_N$.*

Lemma 2.5 ([10]) *A finite solvable group G has exactly one non-linear irreducible character if and only if one of the following holds:*
(1) *G is an extra-special* 2*-group;*
(2) *$G = N \rtimes H$ is a Frobenius group with an elementary abelian kernel N and an abelian complement H such that $|H| = |N| - 1$.*

Lemma 2.6 ([15, Theorem 2]) *A finite solvable group G has exactly two non-linear irreducible characters if and only if one of the following holds:*
(1) *G is an extra-special* 3*-group;*
(2) *$G = N \rtimes H$ is a Frobenius group with an elementary abelian kernel N and an abelian complement H such that $2|H| = |N| - 1$;*
(3) *$G = (Z_3 \times Z_3) \rtimes Q_8$ is a Frobenius group with a Frobenius complement Q_8;*
(4) *G is an* 2*-group of class* 3*, and G has a normal series $1 \lhd Z(G) \lhd G' \lhd G$ satisfying the conditions: $G/Z(G)$ is an extra special* 2*-group, $|G'| = 4$ and $|Z(G)| = 2$;*
(5) *G is an* 2*-group of class* 2*, and G has a normal series $1 \lhd G' \lhd Z(G) \lhd G$ satisfying the conditions: $G/Z(G)$ is an elementary abelian group, $|G'| = 2$ and $|Z(G)| = 4$;*
(6) *$G/Z(G) \cong N \rtimes H$ is a Frobenius group with an elementary abelian kernel N and an abelian complement H such that $|H| = |N| - 1$. Furthermore, $Z(G) \cong Z_2$ and $Z(G) \cap G' = 1$.*

Lemma 2.7 ([9, Corollary 1]) *Let G be a non-abelian group of odd order and p be the minimal prime divisor of $|G|$. Then G has at most $2p$ irreducible characters of degree m where $1 \neq m \in \mathrm{cd}(G)$ if and only if one of the following occurs:*
(1) *G is an extra-special p-group;*
(2) *$G = K \rtimes H$ is a Frobenius group with an elementary abelian kernel K and a cyclic complement H. Moreover, $|K| - 1 \leq (2p - 2)|H|$.*

Lemma 2.8 ([16, Theorem 2.3]) *Let G be non-nilpotent and $|\mathrm{Irr}_1(G)| = q$, where q is a prime number. Then $|\mathrm{cd}(G)| = 2$ if and only if one of the following assertions holds:*
(1) *$F(G)$ is abelian, $|G : F(G)| = |F(G) : G'| = p$, p is a prime divisor of $|G'|$, $|G'| - 1$ is a prime number;*
(2) *$G = G' \rtimes A$, G' is an abelian p-group and A is cyclic, $|A| = q(|G'| - 1)$ (q is a prime number), $G/Z(G)$ is a Frobenius group with a Frobenius kernel isomorphic to G' and a cyclic Frobenius complement of order $|G'| - 1$;*
(3) *$G = N_1 \times N_2$, where N_1 is a Frobenius group with an elementary abelian kernel N and a cyclic complement H such that $|N| - 1 = |H|$. Moreover, $|N_2| = q$, where q is a prime number;*

(4) G is a Frobenius group with an elementary abelian kernel N and a cyclic complement H such that $q|H| = |N| - 1$, where q is a prime number.

3 Proof of Theorem A

To make the proof clear, we give the following lemma:

Lemma 3.1 *Let G be a $V(k)$-group, $\varphi \in \mathrm{Irr}(G') \setminus \{1\}$ and T_φ be the inertia group of φ. There exists a subgroup $G' \le U_\varphi \le T_\varphi$ such that $|U_\varphi/G'| \le k$. Moreover, for each $\phi \in \mathrm{Irr}(U_\varphi)$ such that $\phi_{G'} = \varphi$, there exists a unique $\chi \in \mathrm{Irr}(G)$, lying over ϕ and vanishing on $G \setminus U_\varphi$. Hence there are at least $|U_\varphi/G'|$ irreducible characters of G, lying over φ and vanishing on $G \setminus U_\varphi$.*

Proof By [11, Lemma 2.2], there exists $G' \le U_\varphi \le T_\varphi$ such that φ is extendible to $\phi \in \mathrm{Irr}(U_\varphi)$ and ϕ and φ are fully ramified with respect to T_φ/U_φ.

Note $T_\varphi \lhd G$ for $G' \le T_\varphi$. Applying Clifford theory, we see that there exists a unique $\chi \in \mathrm{Irr}(G)$ such that $[\chi_{U_\varphi}, \phi] \neq 0$ and χ vanishes on $G \setminus U_\varphi$.

Suppose ϕ vanishes at x. So is $\lambda\phi$ for each $\lambda \in \mathrm{Irr}(U_\varphi/G')$. Note that φ is also extendible to $(\lambda\phi)$. Since G is a $V(k)$-group, $|U_\varphi/G'| \le k$. □

In practice, we always choose U_φ as big as possible.

Proof of Theorem A Obviously, we need only to show the necessity.

Since an abelian group is obviously a $V(2+1)$-group, so we may assume that G is a non-abelian solvable group. We divide the proof into 3 cases by the order of G/G'.

Case 1. If $|G/G'| > 3$, we claim that G is one of the groups in (1), (2) and (3).

Let φ be a non-linear irreducible character of G'. If φ is extendible to an irreducible character $\chi \in \mathrm{Irr}(G)$, then the irreducible characters of the form $\lambda\chi$ for $\lambda \in \mathrm{Irr}(G/G')$ are $|G/G'|$ distinct irreducible characters of G by [4, Theorem 6.17], which vanish at the elements as φ. Hence $|G/G'| \le 3$, a contradiction. For each non-principle character $\varphi \in \mathrm{Irr}(G')$, by Lemma 3.1, there exists U_φ such that $|U_\varphi/G'| \le 3$. Now we consider the action of G on the set of $\mathrm{Irr}(G') \setminus \{1_{G'}\}$ by $g : \theta \to \theta^g$ for $\theta \in \mathrm{Irr}(G')$ and $g \in G$.

Subcase 1.1. The action of G on $\{\mathrm{Irr}(G')\} \setminus \{1_{G'}\}$ has at least four orbits.

Let φ_i $(i = 1,2,3,4)$ be non-principal irreducible characters of G' from four different orbits. By Lemma 3.1, there exists $\chi_i \in \mathrm{Irr}(G)$ such that $[\chi_{iG'}, \varphi_i] \neq 0$. Then χ_i vanishes on $G \setminus \bigcup_{i=1}^4 U_{\varphi_i}$, $i = 1,2,3,4$, which implies $G = \bigcup_{i=1}^4 U_{\varphi_i}$. Since $|U_\varphi/G'| \le 3$, $|G/G'| \le 9$. If $|G/G'| = 9$, then $|U_{\varphi_i}/G'| = 3$, where $i = 1,2,3,4$. By Lemma 3.1, for each i, there are exactly three irreducible characters χ_{i1}, χ_{i2} and χ_{i3} of G, lying over φ_i and vanishing on $G \setminus U_{\varphi_i}$. Since U_{φ_i} is proper in G, $(G \setminus U_{\varphi_i}) \cap (G \setminus U_{\varphi_j}) \neq \varnothing$ for $i \neq j$. Let $x \in (G \setminus U_{\varphi_i}) \cap (G \setminus U_{\varphi_j})$. Then there are six irreducible characters of G vanishing at x, which is a contradiction. Similarly, we also get a contradiction if $|G/G'| = 4,5,6,7$ or 8.

Subcase 1.2. The action of G on $\{\mathrm{Irr}(G')\} \setminus \{1_{G'}\}$ has exactly three orbits.

Let $\varphi_i \in \mathrm{Irr}(G')$ $(i = 1,2,3)$ be non-principal irreducible characters from different orbits. If φ_i satisfies $|U_{\varphi_1}/G'| = 2$ or 3, then, by Lemma 3.1, there are at least four irreducible characters of G, lying over φ_1, φ_2 or φ_3 and vanishing on $G \setminus \bigcup_{i=1}^3 U_{\varphi_i}$, which implies $G = \bigcup_{i=1}^3 U_{\varphi_i}$ for G is a $V(2+1)$-group. Since $|G/G'| \ge 4$, there are at least two of these irreducible characters φ_i, satisfying $|U_{\varphi_i}/G'| \ge 2$. By Lemma 3.1, there are at least four irreducible characters of G, vanishing at an element, a contradiction. Hence $|U_{\varphi_i}/G'| = 1$ for $i = 1,2,3$, which implies that G has exactly 3 non-linear irreducible characters.

Subcase 1.3. The action of G on $\{\mathrm{Irr}(G')\} \setminus \{1_{G'}\}$ has exactly two orbits.

We can choose two irreducible characters of G', say φ_1 and φ_2, from two different orbits respectively. If one of φ_1 and φ_2, for example, φ_1 satisfies $|U_{\varphi_1}/G'| = 3$, then there are at least four irreducible characters of G lying over φ_1 and φ_2 and vanishing on $G \setminus \bigcup_{i=1}^{2} U_{\varphi_i}$. Since G is a $V(2+1)$-group, it follows that $G = \bigcup_{i=1}^{2} U_{\varphi_i}$. Hence, $G = U_{\varphi_1}$ or U_{φ_2}, which is clearly impossible since $|G : G'| > 3$. If φ_1 and φ_2 satisfy $|U_{\varphi_i}/G'| = 2$, where $i = 1, 2$, then there are at least four non-linear irreducible characters of G lying over φ_1 and φ_2 and vanishing on $G \setminus \bigcup_{i=1}^{2} U_{\varphi_i}$, which implies $G = \bigcup_{i=1}^{2} U_{\varphi_i}$. Hence, $|G : G'| \leq 3$, a contradiction. Therefore, there is at most one of the two irreducible characters φ_1 and φ_2 satisfying $|U_{\varphi_i}/G'| = 2$, which implies that G has at most 3 non-linear irreducible characters. Hence, G can only be isomorphic to one of the groups listed in (1), (2) or (3) by Lemma 2.5.

Subcase 1.4. The action of G on $\{\mathrm{Irr}(G')\} \setminus \{1_{G'}\}$ has exactly one orbit.

Since $|U_\varphi/G'| \leq 3$, G has at most three non-linear irreducible characters. By Lemma 2.5, it is easy to see that G is isomorphic to one of the groups listed in (1), (2) or (3).

Case 2. $|G/G'| = 2$. Then G is one of the groups listed in conclusion (1)-(6).

Let χ be a non-principle irreducible character of G and θ an irreducible constituent of $\chi_{G'}$. In this case, we have that $\chi_{G'} = \theta$ or $\chi = \theta^G$. If $\chi = \theta^G$, then χ vanishes on $G \setminus G'$, which implies that G has at most three non-linear irreducible characters induced by those of G'.

Subcase 2.1. G has exactly one irreducible character, say β, induced from an irreducible characters of G', say λ.

We have seen that $\lambda(1) = 1$. There are exactly three linear characters in $\mathrm{Irr}(G')$. Using Lemma 3.1, there exists at most one zero in each column of the character table of G'. By Lemma 2.1, G' is abelian or has exactly one non-linear irreducible character. If G' is a non-abelian group, let φ be the unique non-linear irreducible character of G' and $\chi_{G'} = \varphi$, where $\chi \in \mathrm{Irr}(G)$. Then G has exactly three non-linear irreducible characters χ, $\xi\chi$ and λ^G, where $\xi \in \mathrm{Irr}(G/G') \setminus \{1_G\}$. Clearly, $\lambda^G(1) = 2$. If $\chi(1) = 2$, then $|G| = 2 + 12 = 14$, and $|G'| = 7$, contrary to G is non-abelian. Therefore, $\chi(1) = \xi\chi(1) \neq 2$. By Lemma 2.3, $G \cong S_4$, and G is in (3). If G' is abelian, then $|G'| = 3$. Hence $|G| = 6$ and $G \cong S_3$, and G is in (1).

Subcase 2.2. G has exactly two irreducible characters induced from irreducible characters of G'. Let β, α be the two irreducible characters such that $\beta = \lambda^G$ and $\alpha = \mu^G$, where $\beta, \alpha \in \mathrm{Irr}(G)$ and $\lambda, \mu \in \mathrm{Irr}(G')$.

Subsubcase 2.2.1. If λ and μ are linear, then G is a group as in the conclusion (2).

In the case G' has five linear characters ($|G'/G''| = 5$), and all non-linear irreducible characters of G' are extendible to G. By the same arguments as above, one has that there is at most one zero in each column of the character table of G'. By Lemma 2.1, we have that G' either is abelian or has exactly one non-linear irreducible character.

If G' is abelian, then G has exactly two non-linear irreducible characters. Hence, G is isomorphic to the group listed in (2) by Lemma 2.5.

If G' is a non-abelian group, let φ be the unique non-linear irreducible character of G' and $\chi_{G'} = \varphi$, where $\chi \in \mathrm{Irr}(G)$. Then G has exactly four non-linear irreducible characters χ, $\xi\chi$, λ^G and μ^G, where $1_G \neq \xi \in \mathrm{Irr}(G/G')$. Since $|G/G'| = 2$, it follows that $\lambda^G(1) = \mu^G(1) = 2$. If $\chi(1) = \xi\chi(1) = \lambda^G(1) = \mu^G(1) = 2$, then $|G| = 2 + 16 = 18$. Hence, $|G'| = 9$, which implies that G' is abelian, a contradiction. Therefore, $\chi(1) = \xi\chi(1) \neq 2$, $\lambda^G(1) = \mu^G(1) = 2$. So G satisfies the assumptions of Lemma 2.3. But it is easy to check by Lemma 2.3 that no such group exists.

Subsubcase 2.2.2. If either λ or μ is non-linear, then G is one of the groups in (2) and (4).

Without loss of generality, let $\mu(1) > 1$ and $\lambda(1) = 1$. In this case, one has that $|G'/G''| = 3$ and $|G/G'|$=2, and so $G/G'' \cong S_3$. Now, we assert that G' has at most two non-linear irreducible characters except μ and μ^g for each $g \in G \setminus G'$. Otherwise, we can choose three such irreducible characters of G', say ϕ_1, ϕ_2 and ϕ_3. We first assume that one of the three irreducible characters is extended by an irreducible character of G''. Let $\phi_{1G''} = \delta$ where $\delta \in \mathrm{Irr}(G'')$. Since ϕ_i is extendible to G, we have that ϕ_i does not vanish for any element while ϕ_j vanishes if $j \neq i$, which implies that μ, μ^g and ϕ_1 are the three extensions of β, but ϕ_2 and ϕ_3 can not be extended by an irreducible character of G''. Let $\xi_1, \xi_2 \in \mathrm{Irr}(G'')$ such that $\phi_2 = \xi_1^{G'}$ and $\phi_3 = \xi_2^{G'}$. Then both ϕ_2 and ϕ_3 vanish on $G' \setminus G''$, a contradiction. In this case, the above assertion holds. Next, we assume that ϕ_1, ϕ_2 and ϕ_3 are not extended by an irreducible character of G''. Then each of them is induced by an irreducible character of G'', which implies that the irreducible characters ϕ_1, ϕ_2 and ϕ_3 vanish on $G' \setminus G''$, a contradiction.

Now, we have that G' has at most four non-linear irreducible characters. If G' has exactly two non-linear irreducible characters, then G has exactly two non-linear irreducible characters λ^G and μ^G, which concludes (2) in this case. Suppose G' has exactly three non-linear irreducible characters, say μ, μ^g and ϕ_1. By the same arguments as above, we have that either the irreducible characters μ, μ^g and ϕ_1 can be extended by an irreducible character of G'' or induced by an irreducible characters of G''. If μ, μ^g and ϕ_1 are three extensions of an irreducible character of G'', then $\mu(1) = \mu^g(1) = \phi_1(1)$, which is impossible by Lemma 2.8. In the latter case, one has that G' is a *Frobenius* group with a kernel G'' by Lemma 2.4. Furthermore, since $|G'/G''| = 3$ and G' has exactly three non-linear irreducible characters, we get that G'' is an abelian group. Hence $|G''| = 10$, a contradiction to $G/G'' \cong S_3$. Now, we have that G' has four irreducible characters, denoted by μ, μ^g, ϕ_1 and ϕ_2. By the same argument as before, one has that μ, μ^g and ϕ_1 are three extensions of an irreducible character of G'' and $\phi_2 = \mu^{G'}$, where $\mu \in \mathrm{Irr}(G'')$. Since each linear character of G'' is not extendible to G', $\mu(1) = 1$, which implies that G'' has exactly four linear characters and one non-linear irreducible character. By Lemma 2.5, we have that $G'' \cong D_8$ or Q_8, and so G has normal series $1 < G'' < G' < G$, where $|G/G'| = 2$, $|G'/G''| = 3$, $G/G'' \cong S_3$, $G'' \cong D_8$ or Q_8. Thus G is a group as in the conclusion (4).

Subcase 2.3. G has exactly three irreducible characters induced from irreducible characters of G'. Let $\beta = \lambda^G$, $\alpha = \mu^G$ and $\gamma = \eta^G$ be the three irreducible characters such that $\beta = \lambda^G$, $\alpha = \mu^G$ and $\gamma = \eta^G$, where $\beta, \alpha, \gamma \in \mathrm{Irr}(G)$ and $\lambda, \mu, \eta \in \mathrm{Irr}(G')$.

Subsubcase 2.3.1. If λ, μ and η are linear characters, then G is one of the groups in (3) and (5).

In this case, one has that G' has seven linear characters ($|G'/G''| = 7$) and all non-linear irreducible characters of G' are extendible to G. By the same arguments as before, we have that there are at most one zero in each column of the character table of G'. By Lemma 2.1, we have that G' either is abelian or has exactly one non-linear irreducible character.

If G' is abelian, then G has exactly three non-linear irreducible characters. In such case, (3) follows by Lemma 2.5.

If G' is non-abelian, then $G' \cong K \rtimes H$, where K is an elementary abelian group of order 8 and $H \cong Z_7$. Therefore, $G \cong (K \rtimes H) \rtimes Z_2$, and so (5) holds.

Subsubcase 2.3.2. If one of λ, μ or η is non-linear, then G is one of groups in (3) and (6).

Without loss of generality, let $\mu(1) > 1$ and $\eta(1) = \lambda(1) = 1$. So $|G'/G''| = 5$. Take $\xi \in \mathrm{Irr}(G'')$. If ξ is extendible to $\delta \in \mathrm{Irr}(G')$, then the irreducible characters of the form $\rho\delta$

for any $\rho \in \mathrm{Irr}(G'/G'')$ are all of the irreducible constituents of $\xi^{G'}$. Hence, one has that there are at least three irreducible characters of the set $\{\rho\delta|\lambda \in \mathrm{Irr}(G'/G'')\}$ extendible to G, and the extensions have common zeros, a contradiction. Therefore, $\xi^{G'} \in \mathrm{Irr}(G')$ for every $\xi \in \mathrm{Irr}(G'')$. Since $\xi^{G'}$ vanish on $G' \setminus G''$, we have that G' has at most three non-linear irreducible characters. Otherwise, we can take four non-linear irreducible characters μ, μ^g, φ and ψ of G', where $g \in G' \setminus G''$. Clearly, φ and ψ are extendible to G. Moreover, we have four irreducible characters of G, two lying over φ and other two lying over ψ, vanishing on $G' \setminus G''$, a contradiction. By Lemma 2.4, G' is a *Frobenius* group with an abelian kernel G''. We know that $|G'/G''| = 5$ and G'' is an elementary abelian subgroup of order 16 or a cyclic subgroup of order 11. If $|G''| = 11$, then G has exactly three irreducible characters λ^G, μ^G and η^G. Hence (3) follows. If $|G''| = 16$, then $G \cong G' \rtimes Z_2$, where $G' = K \rtimes G''$ is a *Frobenius* group with an elementary abelian kernel G'' of order 16 and a cyclic complement K of order 5; and the action of G on $\mathrm{Irr}_1(G')$ is non-trivial. In such case, (6) follows.

Subcase 2.3.3. If two of λ, μ and η are non-linear, then we claim that no such group exists.

Without loss of generality, let $\mu(1) > 1$, $\eta(1) > 1$ and $\lambda(1) = 1$. In this case, $|G'/G''| = 3$. Let $\xi \in \mathrm{Irr}(G'')$. Then either ξ is extendible to G' or $\xi^{G'} \in \mathrm{Irr}(G')$. Moreover, we can get that any two irreducible characters of G' which are not μ, μ^g, η and η^g for some $g \in G \setminus G'$ have no common zeros. According to the properties of the distribution of zeros of G, we have that the non-linear irreducible characters of G' can only be one of the following two cases:

(a) G' has 7 irreducible characters, μ, μ^g, η, η^g, δ, θ and χ, where $\xi_1, \xi_2 \in \mathrm{Irr}(G'')$ and $\xi_1 \neq \xi_2 \in \mathrm{Irr}(G'')$ such that $\delta_{G''} = \xi_1$ and $\theta_{G''} = \xi_2$. On the other hand, μ, μ^g, η, η^g are the four extensions of ξ_1 and ξ_2 while $\chi = \rho^{G'}$ where $\rho \in \mathrm{Irr}(G'')$ and $\rho(1) = 1$.

(b) G' has 5 or 6 irreducible characters μ, μ^g, η, η^g, δ and θ (maybe there exists no such θ), where $\xi, \zeta, \varepsilon \in \mathrm{Irr}(G'')$ such that $\delta_{G''} = \mu_{G''} = \mu^g_{G''} = \xi$, $\eta = \zeta^{G'}$ and $\eta^g = \varepsilon^{G'}$. If there exists such an irreducible character θ of G', then $\theta = \nu^{G'}$, where $\nu \in \mathrm{Irr}(G'')$.

In the former case (a), one has that G'' has only two non-linear irreducible characters with no common zeros, which is impossible by Lemma 2.1.

In the latter case (b), if $\zeta(1) = 1$, then $\varepsilon(1) = 1$, which implies that $|G''/G'''| = 7$ or 10. Since ζ is a conjugate to ε in G, it follows that the quotient group G/G'' isomorphic to a subgroup of Z_4 or Z_6, a contradiction to $G/G'' \cong S_3$. Hence, $\nu(1) = 1$, and so $|G''/G'''| = 4$. By Clifford theorem, it is easy to see that η^G vanishes on $G'' \setminus G'''$. Let ω be an irreducible character of $\mathrm{Irr}(G''/G''')$. Then $\omega\xi \in \mathrm{Irr}(G'')$. If $\omega\xi \neq \xi$, then $\omega\xi$ is a conjugate to μ or η, and so ξ vanishes on some elements in $G'' \setminus G'''$, which implies that μ^G, η^G and the two extensions of δ to G have common zeros in $G'' \setminus G'''$, a contradiction.

Case 3. $|G/G'| = 3$. Then G is one of the groups in (1), (3) and (7).

Let χ be a non-principle irreducible character of G and θ be an irreducible constituent of $\chi_{G'}$. In this case, we have that $\chi_{G'} = \theta$ or $\chi = \theta^G$. If $\chi = \theta^G$, then χ vanishes on $G \setminus G'$, which implies that G has at most three irreducible characters induced from irreducible characters of G'.

Subcase 3.1. If G has exactly one irreducible character induced from irreducible characters of G', then G is isomorphic to the group listed in (7) or A_4, the alternating group of degree 4.

By the same arguments as in Subcase 2.1, one has that there exists at most one zero in each column of the character table of G'. By Lemma 2.1, G' is abelian or has exactly one non-linear irreducible character.

If G' is a non-abelian group, G' has exactly one irreducible non-linear character. Then $|G'/G''| = 4$ and $|G/G'| = 3$. By Lemma 2.5, $G' \cong D_8$ or Q_8. Moreover, we can get that $G \cong D_8 \rtimes Z_3$ or $Q_8 \rtimes Z_3$. Hence, (7) holds.

If G' is an abelian group, then $|G'| = 4$, and so $|G| = 12$. Hence, $G \cong A_4$, (1) holds.

Subcase 3.2. G has exactly two irreducible characters induced from irreducible characters of G'. Let $\beta = \lambda^G$ and $\alpha = \mu^G$ be the two irreducible characters, where $\beta, \alpha \in \mathrm{Irr}(G)$ and $\lambda, \mu \in \mathrm{Irr}(G')$.

Subsubcase 3.2.1. If λ and μ are linear characters, then G is one of the groups in (2) and (8).

In this case, we know that G' has seven linear characters since $|G'/G''| = 7$ and all the non-linear irreducible characters of G' are extendible to G. By the same arguments as above, one has that there is at most one zero in each column of the character table of G'. By Lemma 2.1, we get that G' either is abelian or has exactly one non-linear irreducible character.

If G' is an abelian group, then G has exactly two non-linear irreducible characters. In the case (2) follows by Lemma 2.5.

If G' is a non-abelian group, then G' is a *Frobenius* group with an elementary abelian kernel G'' of order 8 and a complement H of order 7. Thus $G \cong G' \rtimes Z_3$, and so (8) holds.

Subsubcase 3.2.2. If one of λ and μ is non-linear, then there is no such group G.

Without loss of generality, let $\mu(1) > 1$ and $\lambda(1) = 1$. In this case, we have that $|G'/G''| = 4$ and $|G/G'|=3$. Thus $G/G'' \cong A_4$. We assert that G' has exactly three non-linear irreducible characters μ, μ^g and μ^{g^2}, where $g \in G \setminus G'$. Assume the contrary; let δ be an irreducible character of G' such that $\delta \neq \mu, \mu^g$ and μ^{g^2}. According to the properties of the distribution of zeros of G', it is easy to know that μ vanishes on $G' \setminus G''$ and δ has zeros in $G' \setminus G''$. Hence, all of the three irreducible characters lying over δ and μ^G have common zeros, a contradiction. Therefore, G has exactly two non-linear irreducible characters μ^G and λ^G. Moreover, $\mu^G(1) \neq \lambda^G(1)$. But no such groups exist by Lemma 2.6, a contradiction.

Subcase 3.3. G has exactly three irreducible characters induced from irreducible characters of G'. Let $\beta = \lambda^G$, $\alpha = \mu^G$ and $\gamma = \eta^G$ be the three irreducible characters, where $\beta, \alpha, \gamma \in \mathrm{Irr}(G)$ and $\lambda, \mu, \eta \in \mathrm{Irr}(G')$.

Subsubcase 3.3.1. λ, μ and η are linear characters.

In this case, $|G/G'| = 3$ and $|G'/G''| = 10$, which is obviously impossible.

Subsubcase 3.3.2. One of λ, μ and η is non-linear.

Without loss of generality, let $\mu(1) > 1$ and $\eta(1) = \lambda(1) = 1$. In this case, one has that $|G'/G''| = 7$. By the same arguments as in Subsubcase 2.3.2, for every $\xi \in \mathrm{Irr}_1(G')$, it follows that there necessarily exists an irreducible character $\vartheta \in \mathrm{Irr}(G'')$ such that $\xi = \vartheta^{G'}$. Hence, G' has exactly three non-linear irreducible characters μ, μ^g and μ^{g^2}, where $g \in G \setminus G'$. Then G' is a *Frobenius* group with an abelian kernel G'' of order 22. But $|G'/G''| = 7$, which implies that G' is abelian, a contradiction.

Subsubcase 3.3.3. Two of λ, μ and η are non-linear.

Without loss of generality, let $\mu(1) > 1$, $\eta(1) > 1$ and $\lambda(1) = 1$. In this case, we have that $|G'/G''| = 4$. By the same arguments as in Subsubcase 3.2.2, G' has exactly six non-linear irreducible characters μ, μ^g, μ^{g^2}, η, η^g and η^{g^2}, where $g \in G \setminus G'$. It follows that G has exactly three non-linear irreducible characters, and so (3) holds. This completes the proof of Theorem A. □

4 Proof of Theorem B

Let G be a $V(p+1)$-groups, where p is the minimal odd prime divisor of its order. Obviously, G is a solvable group since G is a group of odd order. In this section, we give the proof of Theorem B.

Proof of Theorem B If G is an abelian group, obviously G is a $V(p+1)$-group, and so we may assume that G is a non-abelian group. We divide the proof into two cases up to the order $|G/G'|$.

Case 1. $|G/G'| > p$.

Let φ be a non-linear irreducible character of G'. If φ is extendible to an irreducible character $\chi \in \mathrm{Irr}(G)$, then there are $|G/G'|$ distinct irreducible characters of the form $\lambda\chi$ of $G, \lambda \in \mathrm{Irr}(G/G')$. Obviously, $\lambda\chi$ vanishes at which φ vanishes. Hence $|G/G'| = p+1$. Since p is the minimal prime divisor $|G|$, $p = 2$, a contradiction. Therefore, only the principal character of G' is extendible to an irreducible character of G.

For a non-principal character $\varphi \in G'$, by Lemma 3.1, U_φ exists and $|U_\varphi/G'| = 1$ or p by the minimality of p. Now, consider the action of G on $\mathrm{Irr}(G') \setminus \{1_{G'}\}$ by $g : \theta \to \theta^g$ for $\theta \in \mathrm{Irr}(G')$ and $g \in G$.

Subcase 1.1. This action has at least $p+2$ orbits.

In this case, we can take $p+2$ irreducible characters of G', say $\varphi_1, \varphi_2, \dots, \varphi_{p+2}$, from $p+2$ different orbits respectively. So $\varphi_1, \varphi_2, \dots, \varphi_{p+2}$ can give rise to at least $p+2$ distinct irreducible characters of G and all of them vanish on $G \setminus \bigcup_{i=1}^{p+2} U_{\varphi_i}$, which implies $G = \bigcup_{i=1}^{p+2} U_{\varphi_i}$. Since $|U_\varphi/G'| = 1$ or p, $|G/G'| \le p(p+1)$. Hence, $|G/G'| = p^2$ or r, where r is a prime and $r > p$. If $|G/G'| = p^2$, then G has at least $p+1$ subgroups U_{φ_i} satisfying the equality $|U_{\varphi_i}/G'| = p$, where $i = 1, 2, \dots, p+1$. It follows that there are exactly p irreducible characters of G lying over φ_i and vanishing on $G \setminus U_{\varphi_i}$ for each i. Since $G = \bigcup_{i=1}^{p+2} U_{\varphi_i}$, we have that the intersection $(G \setminus U_{\varphi_i}) \cap (G \setminus U_{\varphi_j})$ is not empty for $i \neq j$. Let x be an element of $(G \setminus U_{\varphi_i}) \cap (G \setminus U_{\varphi_j})$. Then there are $2p$ irreducible characters of G lying over φ_i, φ_j and vanishing at x, which leads to a contradiction. Now, we consider the case that $|G/G'| = r$. In this case, $\varphi_i^G \in \mathrm{Irr}(G)$, where $i = 1, 2, \dots, p+2$, and so the $p+2$ irreducible characters $\varphi_1^G, \varphi_2^G, \dots, \varphi_{p+2}^G$ vanish on $G \setminus G'$, an obvious contradiction. Therefore, there are at most $p+1$ orbits of the action of G on $\{\mathrm{Irr}(G')\} \setminus \{1_{G'}\}$.

Subcase 1.2. This action has k orbits, where $3 \le k \le p+1$.

We can choose k irreducible characters of G', say $\varphi_1, \varphi_2, \dots, \varphi_k$, lying in each of k different orbits respectively. If one of the k irreducible characters, for example, φ_1 satisfies $|U_{\varphi_1}/G'| = p$, then there are at least $p+2$ irreducible characters of G lying over $\varphi_1, \varphi_2, \dots, \varphi_k$ and vanishing on $G \setminus \bigcup_{i=1}^{k} U_{\varphi_i}$, which forces $G = \bigcup_{i=1}^{k} U_{\varphi_i}$. Hence $p < |G/G'| < kp$, and then $|G/G'| = q$. But p is the minimal divisor of $|G|$, a contradiction to $|U_{\varphi_1}/G'| = p$. Therefore, we have that $|U_{\varphi_i}/G'| = 1$ where $i = 1, 2, \dots, k$, and hence G has at most $p+1$ non-linear irreducible characters. By Lemma 2.7, one has that either G is an extra special p-group or a *Frobenius* group with an elementary abelian kernel.

Let $\varphi \in \mathrm{Irr}(G')$ and $\theta \in \mathrm{Irr}(T_\varphi|\varphi)$. Since $U_\varphi = G'$, we have that $\theta(1) = \sqrt{|T_\varphi/G'|} \cdot \varphi(1)$. It follows that $\chi(1) = |G/T_\varphi| \cdot \theta(1) = |G/T_\varphi| \cdot \sqrt{|T_\varphi/G'|} \cdot \varphi(1)$, where $\chi \in \mathrm{Irr}(G|\varphi)$. Hence, for each $\chi \in \mathrm{Irr}_1(G)$, $\chi(1)$ is divided by every prime divisor of $|G/G'|$. By [4, Theorem 12.2], G

has a normal q-complement, and so G has a normal $\pi(G/G')$-complement. Let N_q denote the normal q-complement of G, where q is a prime divisor of $|G/G'|$.

Subsubcase 1.2.1. If $\pi(G/G') = \pi(G)$, then either G is an abelian group or an extra special p group. In the case that the desired conclusion (i) or (ii) holds.

Subsubcase 1.2.2. Suppose that $\pi(G/G') \neq \pi(G)$ and G is not nilpotent. In the following, we will show that G/N_q is abelian for every prime divisor q of $|G/G'|$.

We first prove that if $p||G/G'|$, then G/N_p is an abelian group. Suppose G/N_p is non-abelian. By Lemma 2.7, G/N_p is an extra special group. So G/N_p has $p-1$ non-linear irreducible characters, which implies that $|\operatorname{Irr}_1(G|N_p)| \leq 2$. Hence, one has that the action of G/N_p on the set $\{\operatorname{Irr}(N_p)\} \setminus \{1_{N_p}\}$ has at most 2 orbits. If N_p is non-abelian, then the action of G/N_p on the set $\{\operatorname{Lin}(N_p)\} \setminus \{1_{N_p}\}$ has exactly one orbit. Thus, $(|N_p/N_p'|-1)|p^{2n+1}$, which implies that $|N_p/N_p'|$ is an even number, a contradiction. Hence, N_p is an abelian group. Next, we assert that N_p is the minimal normal subgroup of G. Assume the contrary. Let $1 \neq M \trianglelefteq G$ and $M < N_p$. Obviously, the action of G on the set $\{\operatorname{Irr}(M)\} \setminus \{1_M\}$ has exactly one orbit. By the same reasoning as above, one has that $|M/M'|$ is an even number, a contradiction; and so the assertion holds. Assume that N_p is an elementary abelian q-group $(q \neq p)$. Till now, we get that G is a *Frobenius* group with a kernel N_p and an extra-special complement of order p^n. However, it is easy to check that no such group G exists, which leads to a contradiction.

Next, we prove that G/N_q is an abelian group for $q \neq p$. If G/N_q is non-abelian, then G/N_q has at least $q-1$ non-linear irreducible characters by Lemma 2.2. But G has at most $p+1$ non-linear irreducible characters, which implies that $N_q = 1$, a contradiction.

Set $K = \bigcap N_q$. Then K is a normal $\pi(G/G')$-complement of G and $G' \subseteq K$. Since G is solvable, we have that G' has a *Hall*-$(\pi(G) - \pi(G/G'))$ subgroup H. Obviously, H is a *Hall*-$(\pi(G) - \pi(G/G'))$ subgroup of G. In this case, one has that $K = H \leq G'$, and hence $K = G'$. Set $1_{G'} \neq \lambda \in \operatorname{Irr}(G')$ and $\mu \in \operatorname{Irr}(T_\lambda|\lambda)$. Since $(|G'|, |G/G'|) = 1$, one has that λ is extendible to μ. Furthermore, λ is fully-ramified with respect to T_λ, which implies that $T_\lambda = G'$. By Lemma 2.4, G is a *Frobenius* group with a kernel G' and a cyclic complement L. Since G has at most $p+1$ non-linear irreducible characters, one has that $|G'| \leq (p+1)|L| + 1$, where $p \leq q$. Hence, (iii) holds.

Subcase 1.3. This action has at most two orbits.

If the action of G on $\{\operatorname{Irr}(G') \setminus 1_{G'}\}$ has exactly one orbit, then $|G'/G''|$ is an even number, a contradiction. Now, we have that the action of G on the set $\{\operatorname{Irr}(G') \setminus 1_{G'}\}$ has exactly two orbits. By the same reasoning as above, one has that G' is an abelian and a minimal normal subgroup of G. We can take two irreducible characters of G', say φ_1 and φ_2, from the two different orbits respectively. If both φ_1 and φ_2 satisfy $|U_{\varphi_i}/G'| = p$, then there are $2p$ irreducible characters of G lying over φ_1 and φ_2 and vanishing on $G \setminus \bigcup_{i=1}^{2} U_{\varphi_i}$, which implies $G = \bigcup_{i=1}^{2} U_{\varphi_i}$. But this is impossible since $U_{\varphi_i} \neq G$. If one of φ_1 and φ_2, for example φ_1 satisfies $|U_{\varphi_1}/G'| = p$, then G has exactly $p+1$ non-linear irreducible characters. Moreover, we have $G = HK$ is a *Frobenius* group with an elementary abelian kernel K and a cyclic complement H. Since $H \cong G/K$ is abelian, one has that $G' \leq K$. On the other hand, we know that $|\operatorname{Irr}_1(G|G')| = p + 1$, which implies that $G' = K$. Thus $|U_{\varphi_1}/G'| = 1$, a contradiction. Therefore, we have that $|U_{\varphi_i}/G'| = 1$, where $i = 1, 2$. In the case that G has exactly two non-linear irreducible characters. By Lemma 2.6, one has that G is an extra special p-group, as desired.

Case 2. $|G/G'| = p$.

Let χ be a non-principle irreducible character of G and θ an irreducible constituent of $\chi_{G'}$. In this case, we have that $\chi_{G'} = \theta$ or $\chi = \theta^G$. If $\chi = \theta^G$, then χ vanishes on $G \setminus G'$, which implies that G has at most $p + 1$ irreducible characters induced from irreducible characters of G'.

Since the non-principle linear irreducible characters of G' are not extendible to G, one has that $|G'/G''| = kp + 1$, where $0 < k \leq p + 1$. One the other hand, since p is the minimal prime divisor of $|G|$, we get that $|G'/G''| = kp + 1$. Let β be a non-principle irreducible character of G' and φ an irreducible character of G'' such that $[\beta_{G''}, \varphi] \neq 0$. Then $\beta_{G''} = \varphi$ or $\beta = \varphi^{G''}$.

Subcase 2.1. If G' is an abelian group, then G is isomorphic to the group listed in (iii).

Since G' is abelian, it follows by Lemma 2.1 that G is a *Frobenius* group with a kernel G' and a cyclic complement L of order p, where G' is a cyclic group of order $kp + 1$ and $k < p$, as desired.

Subcase 2.2. If G' is not abelian, then there is no such group. In order to prove this, we write the proof into two subsubcases.

Subsubcase 2.2.1. Suppose there exists an irreducible character $\delta \in \mathrm{Irr}(G')$ such that δ is extendible to G.

We first prove that $\delta_{G''}$ is reducible and δ is the unique character of G' which is extendible to G. Let $\varphi \in \mathrm{Irr}(G'')$ such that $[\delta_{G''}, \varphi] \neq 0$. Suppose $\delta_{G''}$ is irreducible, then $\delta_{G''} = \varphi$. Let $L = \{\lambda\delta | \lambda \in \mathrm{Irr}(G'/G'')\}$ be the set of all extensions of φ. If one of the elements in $L \setminus \{\delta\}$, say $\lambda\delta$, is extendible to G, then G has $2p$ non-linear irreducible characters lying over δ and $\lambda\delta$, and the $2p$ non-linear irreducible characters vanish at the elements where φ vanishes, contrary to G is a $V(p + 1)$ group. Therefore, all the elements of $L \setminus \{\delta\}$ are not extendible to G. It is easy to check that $(\lambda\delta)_{G''} = (\lambda\delta)^g_{G''} = \cdots = (\lambda\delta)^{g^{p-1}}_{G''} = \varphi$ for each $g \in G \setminus G'$, where $\lambda\delta \in L$. So $(\lambda\delta)^G$ vanishes at the elements where φ vanishes. Hence, G has exactly p irreducible characters lying over δ and k irreducible characters lying over other characters in L, and the $p+k$ irreducible characters vanish on the elements where φ vanishes, contrary to $k \geq 2$. Thus $\delta_{G''}$ is reducible. If $\delta \neq \gamma \in \mathrm{Irr}(G')$ is extendible to G, by the same arguments as above, one has that γ is reducible, which implies that both δ and γ vanish at $G' \setminus G''$. Moreover, G has $2p$ non-linear irreducible characters lying over δ and γ, and the $2p$ characters vanish on the elements of $G' \setminus G''$, which leads to a contradiction since G is a $V(p + 1)$ group. Hence, δ is the unique character of G' which is extendible to G.

Let $\eta \in \mathrm{Irr}_1(G')$ such that η is reducible. If $\eta = \theta^{G'}$ for some $\theta \in \mathrm{Irr}(G'')$, then $t_1 = |G : I_G(\theta)| \mid |G : G''| = p \cdot kp + 1$. Since $t = |G' : I_{G'}(\theta)| = kp + 1$, we have that $t_1 = |G : G''| = p(kp + 1)$, which implies that η^g is reducible for any $g \in G$. On the other hand, if $\eta_{G''}$ is irreducible, by the same reason as above, one has that η^g is irreducible for any $g \in G$. It is easy to check that if two of the characters $\eta, \eta^g, \ldots, \eta^{g^{p-1}}$ are extensions of μ for some $g \in G - G'$ and $\mu \in \mathrm{Irr}(G'')$, then all the characters $\eta, \eta^g, \ldots, \eta^{g^{p-1}}$ are extensions of μ. Hence, we can get that $\mathrm{Irr}_1(G') = \{\delta\}$ or $\mathrm{Irr}_1(G') = \{\delta, \eta, \eta^g, \ldots, \eta^{g^{p-1}}\}$. If $\mathrm{Irr}_1(G') = \{\delta\}$, then G'' is abelian and $|G''| = kp + 2$ which is an even number, a contradiction. So $\mathrm{Irr}_1(G') = \{\delta, \eta, \eta^g, \ldots, \eta^{g^{p-1}}\}$. Let $\delta = \lambda^{G'}$ and $\eta = \theta^{G'}$ where $\lambda, \theta \in \mathrm{Irr}(G'')$. If $\lambda(1) = 1$ and $\theta(1) > 1$, then $|G''/G'''| = \delta(1) + 1$, which is an even number, a contradiction. Also, if $\lambda(1) > 1$ and $\theta(1) = 1$, then $|G''/G'''| = p(kp + 1) + 1$, a contradiction. Hence, $\lambda(1) = \theta(1) = 1$, so G'' is abelian and $|G''| = (kp + 1)(p + 1) + 1$ (co-prime to $kp + 1$), $|G/G''| = (kp + 1)p$, $\eta(1) = \eta^g(1) = \cdots = \eta^{g^{p-1}}(1) = kp + 1$, $\delta(1) = kp + 1$, and so $\mathrm{cd}_1(G) = \{p(kp + 1), kp + 1\}$. Therefore, G has a normal $(kp + 1)$-complement H. Moreover, we have that $G'' < H$ and $|H/G''| = p$, and so G/G'' is abelian, a contradiction.

Subsubcase 2.2.2. Suppose all the non-linear irreducible characters of G' are not extendible to G.

By Lemma 2.3, G is a *Frobenius* group with a kernel G' and a cyclic complement L of order p. Furthermore, we have that G has at most $p+1$ non-linear irreducible characters. It follows by Lemma 2.6 that G' is an elementary q-group and $|G'| \leq mp+1$, where $m \leq p+1$. But G is a non-solvable group, an obvious contradiction.

Conversely, it is easy to check that all these groups listed in Theorem B are obvious $V(p+1)$-groups. This completes the proof of Theorem B. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
YY carried out the study of the zeros in the character table of group of odd order. HX and GC carried out the study of the zeros in the character table of group of even order. All authors read and approved the final manuscript.

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