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ON FINITE MINIMAL NON-p-SUPERSOLUBLE GROUPS

BY

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If $\mathfrak{F}$ is a class of groups, then a minimal non-$\mathfrak{F}$-group (a *dual minimal non-$\mathfrak{F}$-group* resp.) is a group which is not in $\mathfrak{F}$ but any of its proper subgroups (factor groups resp.) is in $\mathfrak{F}$. In many problems of classification of groups it is sometimes useful to know structure properties of classes of minimal non-$\mathfrak{F}$-groups and dual minimal non-$\mathfrak{F}$-groups. In fact, the literature on group theory contains many results directed to classify some of the most remarkable among the aforesaid classes. In particular, V. N. Semenchuk in [12] and [13] examined the structure of minimal non-$\mathfrak{F}$-groups for $\mathfrak{F}$ a formation, proving, among other results, that if $\mathfrak{F}$ is a saturated formation, then the structure of finite soluble, minimal non-$\mathfrak{F}$-groups can be determined provided that the structure of finite soluble, minimal non-$\mathfrak{F}$-groups with trivial Frattini subgroup is known.

In this paper we use this result with regard to the formation of p-supersoluble groups (p prime), starting from the classification of finite soluble, minimal non-p-supersoluble groups with trivial Frattini subgroup given by N. P. Kontorovich and V. P. Nagrebetskiĭ ([10]). The second part of this paper deals with non-soluble, minimal non-p-supersoluble finite groups. The problem is reduced to the case of simple groups. We classify the simple, minimal non-p-supersoluble groups, p being the smallest odd prime divisor of the group order, and provide a characterization of minimal simple groups.

All the groups considered are finite.

1. Some preliminary results.

We provide some preliminary results; some of them are implicitly contained in the papers mentioned above, so we omit their proofs.

1.1. *Let G be a minimal non-p-supersoluble group with $p = \min \pi(G)$. Then G is soluble.*

Proof. If $p > 2$, then $|G|$ is odd and so G is soluble ([5]). If $p = 2$, the statement follows from a theorem of Ito ([9]), if we recall that 2-supersolubility is equivalent to 2-nilpotency.

1.2. *Let G be a minimal non-p-supersoluble group. If $O_p(G) \not\leq \Phi(G)$, then G is soluble.*

1.3. *Let G be a minimal non-p-supersoluble group with a normal Sylow subgroup. Then G is soluble and G_p is the only normal Sylow subgroup of G.*

1.4. *Let G be a soluble, minimal non-p-supersoluble group. Then $|\pi(G)| \leq 3$. Moreover, if $|\pi(G)| = 3$ then $G_p \triangleleft G$ and $p = \max \pi(G)$.*

1.5. *Let G be a soluble, minimal non-p-supersoluble group without normal Sylow subgroups. Then $p = \max \pi(G)$.*

The following propositions can be obtained using techniques similar to those used in [1], [4], [13].

1.6. *Let G be a minimal non-p-supersoluble group and $G_p \triangleleft G$. Then:*

(i) *$G_p/\Phi(G_p)$ is minimal normal in $G/\Phi(G_p)$;*
(ii) *if M is a maximal subgroup of G whose index is a power of p, then $M = \Phi(G_p)G_{p'}$;*
(iii) *there exists a supersoluble immersion of $\Phi(G_p)$ in G;*
(iv) *$\Phi(G_p) \leq Z(G_p)$ (and so the class of G_p is ≤ 2);*
(v) *the exponent of G'_p is $\leq p$;*
(vi) *the exponent of G_p is p if $p \neq 2$, and is ≤ 4 if $p = 2$.*

1.7. *Let G be a minimal non-p-supersoluble group and $G_p \triangleleft G$. If K is a p-complement of G, then:*

(i) *$K \cap C_G(G_p/\Phi(G_p)) = K \cap \Phi(G) = \Phi(K) \cap \Phi(G)$;*
(ii) *$K/K \cap \Phi(G)$ is minimal non-abelian or cyclic primary;*
(iii) *$\Phi(G) = \Phi(G_p) \times (\bigtimes_{q \neq p} O_q(G))$;*
(iv) *$\Phi(G_p) \leq Z(G)$.*

1.8. *Let G be a soluble, minimal non-p-supersoluble group without normal Sylow subgroups. With $\pi(G) = \{p, q\}$ $\quad (p > q)$ we have:*

(i) *G has no subgroup of index q;*
(ii) *G has only one subgroup M of index p;*
(iii) *$O_p(G) = M_p$.*

1.9. *Let G be a soluble, minimal non-p-supersoluble group without normal Sylow subgroups. Using the notation of 1.8 with $K = N_G(G_q)$ and $P = \Phi(M_p)(K \cap M_p)$, we have:*

(i) *M_p/P is minimal normal in G/P;*
(ii) *$\Phi(G) = P \times O_q(G)$;*
(iii) *$\Phi(G) \leq K$ (and so $P = K \cap M_p$);*
(iv) *$K/\Phi(G)$ is minimal non-abelian;*
(v) *$P \leq Z(M)$ (and so the class of M_p is ≤ 2);*
(vi) *M'_p has exponent $\leq p$;*

(vii) *if* $K_p = P\langle c\rangle$, *then* $P = \langle c^p\rangle \times Q$ *with* Q *elementary abelian and* $M_p = \Omega(M_p)\langle c^p\rangle$ *where* $\Omega(M_p) = \{x \in M_p \mid x^p = 1\}$.

2. Classification of the soluble, minimal non-p-supersoluble groups with trivial Frattini subgroup ([10])**.** We report this classification, modifying the notation of [10] according to that used in Section 1.

(A) Let p, q, s be primes such that $q \mid p-1$ and $s \neq q$. Let $K = K_q K_s$ be the subgroup of $\mathrm{GL}(s,p)$ defined as follows (equating indexes modulo s):

$$K_s = \langle [\gamma_i \delta_{i,j+i}]_{1\le i,j\le s}\rangle$$

where $\gamma_i = 1$ for $i = 1, \ldots, s-1$ and γ_s is of order $s^k \mid p-1$ $(k \geq 0)$. If $s \mid q-1$, then

$$K_q = \langle [m^{t^{i-1}} \delta_{i,j}]_{1\le i,j\le s}\rangle$$

where m is a primitive qth root of unity, $2 \le t \le q-1$ and $t^s \equiv 1 \pmod q$. If $s \nmid, q-1$, then

$$K_q = \bigtimes_{t=0}^{r-1} \langle [m_{i+t}\delta_{i,j}]_{1\le i,j\le s}\rangle$$

where $r = \exp(q,s)$, $m_{i+t}^q = 1$ $(i = 1, \ldots, s;\ t = 0, \ldots, r-1)$ and $m_{i+r} = m_i^{\beta_1} \ldots m_{i+r-1}^{\beta_r}$ $(i = 1, \ldots, s)$, $x^r - \beta_r x^{r-1} - \ldots - \beta_1$ being the minimal polynomial over $\mathrm{GF}(q)$ of an element of $\mathrm{GF}\,(q^r)^\times$ of order s. The holomorph of an elementary abelian group of order p^s by K will be denoted by $\Gamma(p,q,s)$. If $s = p$, then $\Gamma(p,q,s)$ will sometimes be denoted by $\Gamma(p,q)$.

(B) Let p, q be primes and h an integer such that $q^h \mid p-1$. Let $K = \langle a, b\rangle$ be the q-subgroup of $\mathrm{GL}(q,p)$ defined as follows (equating indexes modulo q):

$$a = [m^{(1+q^{h-1})^{i-1}} \delta_{i,j}]_{1\le i,j\le q}\,, \qquad b = [\delta_{i,j+1}]_{1\le i,j\le q}$$

where m is a primitive qth root of unity. The holomorph of an elementary abelian group of order p^q by K will be denoted by $\Delta(p,q,h)$.

(C) Let p, q be primes such that $q \mid p-1$ and $q \neq 2$. Let K be the subgroup (extraspecial of order q^3) of $\mathrm{GL}(q,p)$ defined as follows (equating indexes modulo q): $K = \langle a, b\rangle$ where

$$a = [ml^{i-1}\delta_{i,j}]_{1\le i,j\le q}\,, \qquad b = [\delta_{i,j+1}]_{1\le i,j\le q}$$

with $m^q = 1$ and l a primitive qth root of unity. The holomorph of an elementary abelian group of order p^q by K will be denoted by $\Delta(p,q)$.

(D) Let p be a prime such that $4 \mid p-1$ and let K be the subgroup

($\simeq Q_8$) of $\mathrm{GL}(2,p)$ defined by

$$K = \left\langle \begin{bmatrix} m & 0 \\ 0 & m^{-1} \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\rangle$$

where m is a primitive 4th root of unity. The holomorph of an elementary abelian group of order p^2 by K will be denoted by $\Theta(p)$.

(E) Let p, q be different primes and m a positive integer. With $n = \exp(p, q^m)$, $\Lambda(p, q, m)$ will denote the holomorph of the additive group of the Galois field $\mathrm{GF}(p^n)$ by the subgroup $\langle \tau \rangle$ of order q^m of the Singer cycle of $\mathrm{GL}(n, p) \simeq \mathrm{Aut}\,\mathrm{GF}(p^n)(+)$; i.e. $x^\tau = \lambda x$ ($x \in \mathrm{GF}(p^n)$) where λ is a primitive q^mth root of unity in $\mathrm{GF}(p^n)$.

2.1. Theorem (Kontorovich–Nagrebetskiĭ [10]). *Let p be a prime. A group G is soluble, minimal non-p-supersoluble with $\Phi(G) = 1$ if and only if G is isomorphic to one of the following groups*:

(A) $\Gamma(p, q, s)$ *with* $s = p$ *or* $s \mid p - 1$;
(B) $\Delta(p, q, h)$; (C) $\Delta(p, q)$; (D) $\Theta(p)$;
(E) $\Lambda(p, q, m)$ *with* $q^{m-1} \mid p - 1$ *and* $q^m \nmid p - 1$.

3. Structure of the soluble, minimal non-p-supersoluble groups

3.1. *Let G be a soluble, minimal non-p-supersoluble group without normal Sylow subgroups. With the notation of* 1.9 *we have* $P = \Phi(M_p)\langle c^p \rangle$.

Proof. Without loss of generality, assume $O_q(G) = 1$ ($q \neq p$). Since $P \leq Z(M)$ (see 1.9) and $G/P \simeq \Gamma(p, q)$ (Theorem 2.1), we can assume, with $|P| = p^n$ ($n \geq 0$), that $G_q = \bigtimes_{t=0}^{r-1} \langle a_t \rangle$ where

$$a_t = \begin{bmatrix} [m_{i+t}\delta_{i,j}]_{1\leq i,j\leq p} & 0 \\ 0 & [\delta_{i,j}]_{1\leq i,j\leq n} \end{bmatrix}$$

and

$$c = \begin{bmatrix} [\delta_{i,j+1}]_{1\leq i,j\leq p} & [\lambda_{i,j}]_{1\leq i\leq p, 1\leq j\leq n} \\ 0 & [\gamma_{i,j}]_{1\leq i,j\leq n} \end{bmatrix}$$

with

$$(3.1)\qquad c^{-1}a_t c = a_{t+1} \quad (t = 0, \ldots, r-2), \qquad c^{-1}a_{r-1}c = a_1^{\beta_1} \ldots a_{r-1}^{\beta_{r-1}}$$

and with the same notation as in (A) of Section 2. From (3.1) it follows that, for each $t = 0, \ldots, r-1$,

$$[\delta_{i,j-1}]_{1\leq i,j\leq p}[(m_{i+t} - 1)\delta_{i,j}]_{1\leq i,j\leq p}[\lambda_{i,j}]_{1\leq i\leq p, 1\leq j\leq n} = 0\,,$$

from which we deduce

$$(m_{i+t+1} - 1)\lambda_{i+1,j} = 0$$

for each: $i = 1, \dots, p$; $j = 1, \dots, n$; $t = 0, \dots, r-1$ (equating indexes modulo p). Since the indexes are arbitrary, as in (A) of Section 2 we conclude that $\lambda_{i,j} = 0$ for each $i = 1, \dots, p$ and $j = 1, \dots, n$. Thus G splits on P and so, as $P = \Phi(G)$, we get $P = 1$.

3.2. *Let G be a minimal non-p-supersoluble group such that $G/\Phi(G)$ is one of the groups* (A), (B), (C) *of Theorem* 2.1. *Then $O_p(G)$ is abelian.*

Proof. Without loss of generality, assume $O_{p'}(G) = 1$. We examine separately the different cases.

Case 1: $G/\Phi(G) \simeq \Gamma(p, q, s)$ and $\exp(q, s) = r > 1$. We can assume (see 3.1 and 1.6, 1.9) $G = O_p(G)G_q\langle c\rangle$ where $O_p(G) = \langle x_1, \dots, x_s, c^{\varepsilon p}\rangle$, with $\varepsilon = 0$ if $s \neq p$, $\varepsilon = 1$ if $s = p$, and $\langle x_1, \dots, x_s\rangle$ of exponent p; $G_q = \bigtimes_{t=0}^{r-1}\langle a_t\rangle$ with

(3.2) $$a_t^{-1} x_i a_t = x^{m_{i+t}} y_{i,t}$$
$$(i = 1, \dots, s;\ t = 0, \dots, r-1;\ y_{i,t} \in \Phi(O_p(G))\langle c^{\varepsilon p}\rangle)$$

and

(3.3)
$$c^{-1} x_i c = x_{i-1} z_i \qquad (i = 1, \dots, s-1;\ z_i \in \Phi(O_p(G))\langle c^{\varepsilon p}\rangle),$$
$$c^{-1} x_s c = x_{s-1}^{(\gamma_s)^\eta} z_s \qquad (\eta = 0 \text{ if } s = p; \eta = 0, 1 \text{ if } s \neq p;\ z_s \in \Phi(O_p(G))\langle c^{\varepsilon p}\rangle)$$

and with the same notation as in (A) of Section 2. As $\Phi(O_p(G))\langle c^{\varepsilon p}\rangle \leq Z(O_p(G)G_q)$ (see 1.9), we have

$$a_t^{-1}[x_i, x_j]a_t = [x_i, x_j] = [x_i, x_j]^{m_{i+t}m_{j+t}}$$

for each $i, j = 1, \dots, s$ and $t = 0, \dots, r-1$. It follows that if $[x_i, x_j] \neq 1$ then

(3.4) $$m_{i+t}m_{j+t} \equiv 1 \pmod p$$

for each $t = 0, \dots, r-1$. On the other hand, from (3.3) it follows that, for every integer k, we have (equating indexes modulo s)

$$c^k[x_i, x_j]c^{-k} = [x_{i+k}, x_{j+k}]^\beta \qquad (0 \leq \beta \leq p-1)$$

for each $i, j = 1, \dots, s$; we deduce that if $[x_i, x_j] \neq 1$ then also $[x_{i+k}, x_{j+k}] \neq 1$, and so, by (3.4), we obtain

(3.5) $$m_{i+k}m_{j+k} \equiv 1 \pmod p$$

for each integer k (equating indexes modulo s).

Now, suppose $O_p(G)$ is non-abelian. As $c^{\varepsilon p} \in Z(G)$ and $G/\Phi(G) \simeq \Gamma(p, q, s)$, for any $i = 1, \dots, s$ there exists $j = 1, \dots, s$ such that (3.5) holds. As k is arbitrary, it follows that $m_i^2 \equiv 1 \pmod p$ for each $i = 1, \dots, s$ and so, as $s \neq 2$, we have $q = 2$. We can then assume

$$\begin{matrix} m_1 \equiv \ldots \equiv m_h \equiv 1 \\ m_{h+1} \equiv \ldots \equiv m_s \equiv -1 \end{matrix} \quad (\operatorname{mod} p) \quad (1 \leq h \leq s-1).$$

As $m_i m_j \equiv -1 \ (\operatorname{mod} p)$ ($i = 1, \ldots, h$; $j = h+1, \ldots, s$), it follows that $[x_i, x_j] = 1$ for each $i = 1, \ldots, h$ and $j = h+1, \ldots, s$, from which we get

$$1 = c^k [x_i, x_j] c^{-k} = [x_{i+k}, x_{j+k}]$$

for every integer k and for each $i = 1, \ldots, h$ and $j = h+1, \ldots, s$. It follows, obviously, that $[x_i, x_j] = 1$ for each $i, j = 1, \ldots, s$ and so $O_p(G)$ is abelian, which contradicts the hypothesis.

Case 2: $G/\Phi(G) \simeq \Gamma(p, q, s)$ and $s \mid p-1$. In this case $O_p(G) = G_p = \langle x_1, \ldots, x_s \rangle$ is of exponent p, $G_q = \langle a \rangle$ is of order q and we can assume

$$a^{-1} x_i a = x_i^{m^{t^{i-1}}} y_i \quad (i = 1, \ldots, s;\ y_i \in \Phi(G_p))$$

with the same notation as in (A) of Section 2. As $\Phi(G_p) \leq Z(G)$, we have

$$a^{-1} [x_i, x_j] a = [x_i, x_j] = [x_i, x_j]^{m^{t^{i+j-2}}} \quad (i, j = 1, \ldots, s).$$

It follows that if $[x_i, x_j] \neq 1$ then

$$m^{t^{i+j-2}} \equiv 1 \ (\operatorname{mod} p),$$

from which, as $m \not\equiv 1 \ (\operatorname{mod} p)$ and so $\exp(m, p) = q$, we obtain $t^{i+j-2} \equiv 0 \ (\operatorname{mod} q)$, which is false, since $2 \leq t \leq q-1$. Thus $[x_i, x_j] = 1$ for each $i, j = 1, \ldots, s$, that is, G_p is abelian.

Case 3: $G/\Phi(G) \simeq \Delta(p, q, h)$. As in the previous case, $O_p(G) = G_p = \langle x_1, \ldots, x_q \rangle$ is of exponent p. Moreover, $G_q = \langle a, b \mid a^{q^h} = b^q = 1, b^{-1} a b = a^{1+q^{h-1}} \rangle$ and we can assume

$$\begin{aligned} a^{-1} x_i a &= x_i^{m(1+q^{h-1})^{i-1}} y_i && (i = 1, \ldots, q;\ y_i \in \Phi(G_p)), \\ b^{-1} x_i b &= x_{i-1} z_i && (i = 1, \ldots, q;\ z_i \in \Phi(G_p)), \end{aligned} \tag{3.6}$$

with the same notation as in (B) of Section 2. As $\Phi(G_p) \leq Z(G)$, we have

$$a^{-1} [x_i, x_j] a = [x_i, x_j] = [x_i, x_j]^{m^{(1+q^{h-1})^{i-1} + (1+q^{h-1})^{j-1}}},$$

from which, if $[x_i, x_j] \neq 1$ ($i < j$), we obtain

$$m^{(1+q^{h-1})^{i-1}((1+q^{h-1})^{j-i}+1)} \equiv 1 \ (\operatorname{mod} p)$$

and so, as $\exp(m, p) = q^h$, we get $(1+q^{h-1})^{j-i} + 1 \equiv 0 \ (\operatorname{mod} q^h)$, therefore, obviously, $q = 2$. Thus $G_p = \langle x_1, x_2 \rangle$, and from (3.6) we get

$$[x_1, x_2] = b^{-1} [x_1, x_2] b = [x_2, x_1] = [x_1, x_2]^{-1},$$

hence $[x_1, x_2] = 1$, that is, G_p is abelian.

Case 4: $G/\Phi(G) \simeq \Delta(p,q)$. As in the previous cases, $O_p(G) = G_p = \langle x_1, \dots, x_q \rangle$ is of exponent p. G_q is extraspecial of order q^3 and exponent q, and we can assume, if $G_q = \langle a, b \rangle$,

$$a^{-1}x_i a = x_i^{ml^{i-1}} y_i \qquad (i = 1, \dots, q;\ y_i \in \Phi(G_p)),$$
$$b^{-1}x_i b = x_{i-1} z_i \qquad (i = 1, \dots, q;\ z_i \in \Phi(G_p)),$$

with the same notation as in (C) of Section 2. As $\Phi(G_p) \le Z(G)$, we have

$$[a,b]^{-1}[x_i, x_j][a,b] = [x_i, x_j] = [x_i, x_j]^{l^2}.$$

It follows that if $[x_i, x_j] \ne 1$ then $l^2 \equiv 1 \pmod p$, which is false, since $l \ne 1$ and $q \ne 2$. Thus $[x_i, x_j] = 1$, so G_p is abelian.

3.3. Remark. Proposition 3.2 assures that if G is soluble, minimal non-p-supersoluble without normal Sylow subgroups then $O_p(G) = M_p$ is abelian (the notation is that of 1.9). As $M_p = \Omega(M_p)\langle c^p \rangle$ we then have $M_p = N \times \langle c^p \rangle$ where N is elementary abelian of order p^p. Let now p and q be primes such that $q \mid p-1$, let $K = K_q K_p$ ($K_q \lhd K$) be a minimal non-abelian group and let $\psi = \pi\sigma$ be the homomorphism $K \to \mathrm{GL}(p,p)$, where π and σ are respectively the canonical homomorphism $K \to K/\Phi(K_p)$ and the immersion of $K/\Phi(K_p)$ in $\mathrm{GL}(p,p)$ considered in (A) of Section 2. If N is an elementary abelian group of order p^p, let G be the semidirect product $K \ltimes_\psi N$. Then G is soluble, minimal non-p-supersoluble and without normal Sylow subgroups. Such a semidirect product will be denoted by $\Gamma^*(p,q,n)$, where $p^n = |K_p|$ (if $n = 1$, then $\Gamma^*(p,q,n) = \Gamma(p,q)$).

The following proposition provides the structure of the soluble, minimal non-p-supersoluble groups without normal Sylow subgroups in terms of $\Gamma^*(p,q,n)$.

3.4. *Let G be a group without normal Sylow subgroups. Then G is soluble and minimal non-p-supersoluble if and only if $G/O_q(G) \simeq \Gamma^*(p,q,n)$ and $O_q(G) = \Phi(G)_q$ ($\pi(G) = \{p,q\}$, $p > q$).*

Proof. The condition is obviously sufficient. Let now G be soluble, minimal non-p-supersoluble without normal Sylow subgroups. We have (see 1.9 and Theorem 2.1) $G/\Phi(G) \simeq \Gamma(p,q)$, $\Phi(G) = \Phi(M_p)\langle c^p \rangle \times O_q(G)$ (see 3.1) and so, by 3.3, $\Phi(G) = \langle c^p \rangle \times O_q(G)$. Again by Remark 3.3, we get $\Omega(M_p) = N \times \langle c^p \rangle$ (N elementary abelian of order p^p, $o(c) = p^n$). Arguing as in the proof of 3.1, with $O_q(G) = 1$ and supposing $n > 1$, we can assume, as $c^p \in Z(G)$, that

$$G_q\langle c \rangle / \langle c^p \rangle = \langle d \rangle \Big(\bigtimes_{t=0}^{r-1} \langle a_t \rangle \Big) \underset{\simeq}{\le} \mathrm{Aut}\, \Omega(M_p) = \mathrm{GL}(p+1, p),$$

where

$$a_t = \begin{bmatrix} [m_{i+t}\delta_{i,j}]_{1\le i,j\le p} & 0 \\ 0 & 1 \end{bmatrix}, \qquad d = \begin{bmatrix} [\delta_{i,j+1}]_{1\le i,j\le p} & [\lambda_i]_{1\le i\le p} \\ 0 & 1 \end{bmatrix}$$

and

$$d^{-1}a_t d = a_{t+1} \quad (t = 0,\dots,r-2)\,, \qquad d^{-1}a_{r-1}d = a_0^{\beta_1}\dots a_{r-1}^{\beta_r}\,,$$

with the same notation as in (A) of Section 2. Arguing exactly as in the proof of 3.1 we obtain $\lambda_i = 0$ for each $i = 1,\dots,p$, and so, obviously, $G \simeq \Gamma^*(p,q,n)$.

3.5. Remark. The statement of 3.2 is not true if $G/\Phi(G) \simeq \Theta(p)$ or $\Lambda(p,q,m)$, as the following examples show.

3.5.1. EXAMPLE. Let P be an extraspecial group of order p^3 and exponent p, with $4\,|\,p-1$. With $P = \langle x_1, x_2\rangle$, let $\langle\sigma,\tau\rangle \simeq Q_8$ be the subgroup of $\operatorname{Aut} P$ defined as follows:

$$x_1^\sigma = x_1^m\,, \quad x_2^\sigma = x_2^{m^{-1}}\,, \quad x_1^\tau = x_2\,, \quad x_2^\tau = x_1^{-1}\,,$$

where m is a primitive 4th root of unity. The holomorph of P by $\langle\sigma,\tau\rangle$ is minimal non-p-supersoluble and its Sylow p-subgroup is not abelian. Such a holomorph will be denoted by $\Theta^*(p)$.

3.5.2. EXAMPLE. Let P be as in the previous example and $4 \nmid p-1$. Let σ be the automorphism of P defined as follows:

$$x_1^\sigma = x_2[x_1,x_2]^{n_1}\,, \qquad x_2^\sigma = x_1^{-1}[x_1,x_2]^{n_2}\,,$$

with n_1 and n_2 integers (between 0 and $p-1$). The holomorph of P by $\langle\sigma\rangle$ is minimal non-p-supersoluble and its Sylow p-subgroup is not abelian. Such a holomorph will be denoted by $\Lambda^*(p,n_1,n_2)$.

3.5.3. EXAMPLE. Further examples of soluble, minimal non-p-supersoluble groups whose Sylow p-subgroups are not abelian are all minimal non-p-nilpotent groups with G_p non-abelian. As a minimal non-p-supersoluble group is minimal non-p-nilpotent if and only if $\mathrm{G}/\Phi(G) \simeq \Lambda(p,q,1)$, a minimal non-$p$-nilpotent group with $O_q(G) = 1$ $(q \neq p)$ will be denoted by $\Lambda^*(p,q)$. The structure of minimal non-p-nilpotent groups is well known ([11]).

3.6. *Let G be a minimal non-p-supersoluble group such that $G/\Phi(G) \simeq \Theta(p)$. Then $G/O_2(G)$ is isomorphic either to $\Theta(p)$ or to $\Theta^*(p)$.*

Proof. Let $O_2(G) = O_{p'}(G) = 1$. We have $G_p = \langle x_1, x_2\rangle$ of exponent p, $G_2 = \langle a, b\rangle \simeq Q_8$, and we can assume

$$\begin{matrix} a^{-1}x_1a = x_1^m y^{n_1}\,, & b^{-1}x_1b = x_2y^{n_3} \\ a^{-1}x_2a = x_2^{m^{-1}}y^{n_2}\,, & b^{-1}x_2b = x_1^{-1}y^{n_4} \end{matrix} \qquad (y = [x_1,x_2])\,,$$

where m is a primitive 4th root of unity and n_i $(i = 1, \dots, 4)$ are integers (between 0 and $p-1$). Since $y \in Z(G)$, we get

$$a^{-2}x_1a^2 = x_1^{-1}y^{(m+1)n_1} = b^{-2}x_1b^2 = x_1^{-1}y^{n_3+n_4} = (ab)^{-2}x_1(ab)^2$$
$$= x_1^{-1}y^{n_4+mn_3+n_1+mn_2}$$

and

$$a^{-2}x_2a^2 = x_2^{-1}y^{(m^{-1}+1)n_2} = b^{-2}x_2b^2 = x_2^{-1}y^{n_4-n_3} = (ab)^{-2}x_2(ab)^2$$
$$= x_2^{-1}y^{-n_3+m^{-1}n_4+n_2-m^{-1}n_1}\,.$$

It follows that if $y \neq 1$ then $n_1, \dots, n_4$ is a solution of the linear system

$$(m+1)\xi_1 - \xi_3 - \xi_4 = 0\,,$$
$$(m^{-1}+1)\xi_2 + \xi_3 - \xi_4 = 0\,,$$
$$\xi_1 + m\xi_2 + (m-1)\xi_3 = 0\,,$$
$$m^{-1}\xi_1 + m^{-1}\xi_2 + \xi_3 - m^{-1}\xi_4 = 0\,,$$

which, as its matrix is non-singular, has only the trivial solution; hence, obviously, $G \simeq \Theta(p)$ or $\Theta^*(p)$.

3.7. *Let G be a minimal non-p-supersoluble group such that $G/\Phi(G) \simeq \Lambda(p,q,m)$ with $m > 1$. Then either $G/O_q(G) \simeq \Lambda(p,q,m)$ or $G/O_2(G) \simeq \Lambda^*(p,n_1,n_2)$.*

Proof. Let $O_q(G) = 1$. As $m > 1$, and so $\exp(p, q^m) = q$, we have $G_p = \langle x_1, \dots, x_q\rangle$ and, if G_p is abelian, then $G \simeq \Lambda(p,q,m)$. Let now G_p be non-abelian and so (see 1.6) special of exponent p. Then (see, for instance [6], Th. 6.5) q^m divides $p^r + 1$ for some integer $r \le q/2$. As $m > 1$, we get $q = 2$ and so G_p is extraspecial of order p^3 (and exponent p). We can then assume $G_p = \langle x_1, x_2\rangle$, $G_2 = \langle b\rangle$ with

$$b^{-1}x_1b = x_2y^{n_1}\,, \qquad b^{-1}x_2b = x_1^{\beta_1}x_2^{\beta_2}y^{n_2} \qquad (y = [x_1, x_2])\,,$$

where n_1 and n_2 are integers (between 0 and $p-1$) and $x^2 - \beta_2x - \beta_1 \in \mathrm{GF}(p)[x]$ is the minimal polynomial of an element λ of $\mathrm{GF}(p^2)$ of order 2^m. We have obviously $\lambda^2 \in \mathrm{GF}(p)$. On the other hand, as $y \in Z(G)$, we get

$$[x_1, x_2] = b^{-1}[x_1, x_2]b = [x_2, x_1^{\beta_1}] = [x_1, x_2]^{-\beta_1}\,,$$

from which we deduce $\beta_1 = -1$. It follows that $m = 2$ and so $G \simeq \Lambda^*(p, n_1, n_2)$.

From the results of this section a theorem follows that provides the structure of soluble, minimal non-p-supersoluble groups.

3.8. Theorem. *Let p be a prime. A group G is soluble and minimal non-p-supersoluble if and only if $O_{p'}(G) = \Phi(G)_{p'}$ and $G/O_{p'}(G)$ is isomor-*

phic to one of the following groups:

(a) $\Gamma(p,q,s)\,,\ s\,|\,p-1\,;$ (b) $\Gamma^*(p,q,n)\,;$
(c) $\Delta(p,q,h)\,;$ (d) $\Delta(p,q)\,;$
(e) $\Theta(p)\,;$ (f) $\Theta^*(p)\,;$
(g) $\Lambda(p,q,m)\,,\ q^{m-1}\,|\,p-1\,,\ q^m\nmid p-1\,,\ m>1\,;$
(h) $\Lambda^*(p,n_1,n_2)\,;$ (i) $\Lambda^*(p,q)\,.$

4. Non-soluble, minimal non-p-supersoluble groups. Minimal non-p-supersoluble groups are not necessarily soluble. For instance, $\mathrm{PSL}(2,p)$ (p prime >3) is minimal non-p-supersoluble. Now we show how the study of non-soluble, minimal non-p-supersoluble groups can be reduced to that of simple groups.

From 1.2 we deduce immediately the following proposition.

4.1. *Let G be a non-soluble, minimal non-p-supersoluble group. Then $F(G)=\Phi(G)$.*

4.2. *Let G be a non-soluble, minimal non-p-supersoluble group. Then $G/\Phi(G)$ is simple.*

Proof. Let G be a counterexample of least order and so $\Phi(G)=1$. We have, by 4.1, $O_p(G)=1$. If N is a minimal normal subgroup of G, from this it follows that N is a p'-group. If G$=MN$ (M maximal in G) we find that G is p-supersoluble: a contradiction.

4.3. *Let G be a non-soluble, minimal non-p-supersoluble group. Then $O_p(G)\leq Z(G)$.*

Proof. If $C=C_G(O_p(G))<G$, we have, by 4.2, $C\leq\Phi(G)$. Let M be a maximal subgroup of G. Since there exists a supersoluble immersion of $O_p(G)$ in M, we conclude that M/C is supersoluble; hence G/C is soluble, which contradicts the hypothesis.

From 4.2 and 4.3 we immediately deduce the following theorem.

4.4. Theorem. *Let G be a non-soluble group and let p be an odd prime. Then G is minimal non-p-supersoluble if and only if $G/\Phi(G)$ is simple, minimal non-p-supersoluble and $O_p(G)\leq Z(G)$.*

The next results provide a classification of simple, minimal non-p-supersoluble groups if p is the smallest odd prime that divides the order of the group. In the proof of one of the propositions we use the classification of the finite simple groups.

4.5. *Let G be a minimal non-3-supersoluble group. Then all proper subgroups of G are soluble.*

P r o o f. Let G be a counterexample of least order. Since $G/\Phi(G)$ is, as G, minimal non-3-supersoluble, we have obviously $\Phi(G) = 1$ and so $O_{3'}(G) = 1$, and, by 4.1, $O_3(G) = 1$. If N is a minimal normal subgroup of G and $N \neq G$, we have obviously $3 \nmid |N|$, which is false, as $O_{3'}(G) = 1$. Thus G is simple. Since G is not a minimal simple group, let H be a proper simple non-abelian subgroup of G. As H is 3-supersoluble, we have $3 \nmid |H|$, and therefore H is isomorphic to a Suzuki group $S_z(2^{2n+1})$. Thus the proper simple non-abelian subgroups of G are isomorphic to Suzuki groups; from this, using the classification of the finite simple groups (see for instance [2]), it follows that G itself is a Suzuki group and so G is 3-supersoluble, since $3 \nmid |G|$: a contradiction.

4.6. *The Suzuki group* $S_z(2^{2n+1})$ *is minimal non-*5*-supersoluble if and only if* $2n+1$ *is prime.*

P r o o f. If $2n+1$ is not prime, denote by $2m+1$ a proper divisor ($\neq 1$) of $2n+1$. Then $S_z(2^{2n+1})$ has a subgroup isomorphic to $S_z(2^{2m+1})$ (see for instance [8]), which is not 5-supersoluble. Conversely, let $2n+1 = q$ be prime. Then (see for instance [8]) the only non-supersoluble subgroups of $S_z(2^q)$ are Frobenius groups whose kernel are 2-groups (of order 2^{2q}) and whose complements are cyclic (of order $2^q - 1$). Such groups are obviously 5-supersoluble, and therefore $S_z(2^q)$ is minimal non-5-supersoluble.

4.7. *Let* G *be a minimal non-*p*-supersoluble group, where* p *is the smallest odd prime divisor of* $|G|$*. Then all proper subgroups of* G *are soluble. In particular, if* G *is simple, then* G *is a minimal simple group.*

P r o o f. If $p = 3$, the statement follows from 4.5. Let now $p \geq 5$ and let G be a counterexample of least order. By similar arguments to the proof of 4.5 we show that G is simple, and therefore, as $3 \nmid |G|$, G is a Suzuki group $S_z(2^{2n+1})$. We then have $p = 5$. Since G is not a minimal simple group, $2n+1$ is not prime, which contradicts 4.6.

4.8. THEOREM. *Let* G *be a simple non-abelian group. Then* G *is minimal non-*p*-supersoluble with* p *the smallest odd prime divisor of* $|G|$ *if and only if* G *is isomorphic to one of the following groups*:

(i) $\mathrm{PSL}(2, 2^q)$, *q prime*;
(ii) $\mathrm{PSL}(2, q)$, *q prime* > 3 *and* $q^2 + 1 \equiv 0 \pmod 5$;
(iii) $S_z(2^q)$, *q prime* ($\neq 2$).

Moreover, the groups (i)–(iii) *are, up to isomorphism, the only simple, minimal non-*s*-supersoluble groups for every odd prime* s *that divides their order.*

P r o o f. A direct analysis (see for instance [7] and [8]) proves that the groups (i)–(iii) are minimal non-s-supersoluble for every odd prime s that divides their order. Let now G be simple and minimal non-p-supersoluble, with $p = \min \pi(G) \setminus \{2\}$. By 4.7, G is a minimal simple group. The classification of the minimal simple groups due to J. G. Thompson ([15]) provides, besides the groups (i)–(iii), the groups $\mathrm{PSL}(2, 3^q)$, q odd prime, and $\mathrm{PSL}(3, 3)$.

We can exclude $\mathrm{PSL}(2, 3^q)$, because a Sylow 3-subgroup of $\mathrm{PSL}(2, 3^q)$ is minimal normal in its normalizer (see for instance [7]) and therefore the latter is not 3-supersoluble. As far as $\mathrm{PSL}(3, 3)$ is concerned, if we regard it as an automorphisms group of the projective plane Π over GF(3), the stabilizer G_α (G_r) of a point (of a line) of Π is isomorphic to the complete holomorph of an elementary abelian group P of order 3^2 ($\mathrm{Aut}\, P \simeq \mathrm{GL}(2, 3)$ is the stabilizer in G_α (in G_r) of a line (a point) not containing α (not belonging to r)). Such subgroups are obviously non-3-supersoluble and so $\mathrm{PSL}(3, 3)$ is not minimal non-3-supersoluble.

The following theorem provides a characterization of minimal simple groups.

4.9. *Let G be a simple non-abelian group. Then G is minimal non-p-supersoluble for every prime $p \geq 5$ that divides its order if and only if G is a minimal simple group.*

Proof. A direct analysis proves that the minimal simple groups are minimal non-p-supersoluble for every prime $p \geq 5$ that divides their order. Vice versa, let G be simple and minimal non-p-supersoluble for every prime $p \geq 5$ that divides its order. Let $\omega = \{2, 3\}$. Then $H/O_\omega(H)$ is supersoluble for every proper subgroup H of G. It follows that H is soluble and therefore G is a minimal simple group.

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