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On finite rank deformations of Wigner matrices

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Abstract. We study the distribution of the outliers in the spectrum of finite rank deformations of Wigner random matrices under the assumption that the absolute values of the off-diagonal matrix entries have uniformly bounded fifth moment and the absolute values of the diagonal entries have uniformly bounded third moment. Using our recent results on the fluctuation of resolvent entries (On fluctuations of matrix entries of regular functions of Wigner matrices with non-identically distributed entries, Unpublished manuscript; Fluctuations of matrix entries of regular functions of Wigner matrices, Unpublished manuscript) and ideas from (Fluctuations of the extreme eigenvalues of finite rank deformations of random matrices, Unpublished manuscript), we extend the results by Capitaine, Donati-Martin, and Féral (*Ann. Probab.* **37** (2009) 1–47; *Ann. Inst. Henri Poincaré Probab. Stat.* **48** (2012) 107–133).

Résumé. Nous étudions la distribution des valeurs propres qui sortent de l'amas du spectre de matrices de Wigner déformées par une matrice de rang fini sous l'hypothèse que les valeurs absolues des éléments non diagonaux aient un moment d'ordre cinq uniformément borné et que valeurs absolues des éléments diagonaux aient un moment d'ordre trois uniformément borné. En utilisant des travaux récents (On fluctuations of matrix entries of regular functions of Wigner matrices with non-identically distributed entries, Unpublished manuscript; Fluctuations of matrix entries of regular functions of Wigner matrices, Unpublished manuscript) et des idées de (Fluctuations of the extreme eigenvalues of finite rank deformations of random matrices, Unpublished manuscript), nous étendons les résultats de Capitaine, Donati-Martin et Féral (*Ann. Probab.* **37** (2009) 1–47; *Ann. Inst. Henri Poincaré Probab. Stat.* **48** (2012) 107–133).

1. Introduction and formulation of main results

Let $X_N = \frac{1}{\sqrt{N}} W_N$ be a random real symmetric (Hermitian) Wigner matrix with independent entries up from the diagonal. In the real symmetric case, we assume that the off-diagonal entries

$$(W_N)_{ij}, \quad 1 \le i < j \le N, \tag{1.1}$$

are independent random variables such that

$$\mathbb{E}(W_N)_{ij} = 0, \qquad \mathbb{V}(W_N)_{ij} = \sigma^2, \qquad m_5 := \sup_{i \ne j, N} \mathbb{E}\big|(W_N)_{ij}\big|^5 < \infty, \tag{1.2}$$

where $\mathbb{E}\xi$ denotes the mathematical expectation and $\mathbb{V}\xi$ the variance of a random variable ξ. The diagonal entries

$$(W_N)_{ii}, \quad 1 \le i \le N, \tag{1.3}$$

are independent random variables (that are also independent from the off-diagonal entries), such that

$$\mathbb{E}(W_N)_{ii} = 0, \qquad c_3 := \sup_{i,N} \mathbb{E}|(W_N)_{ii}|^3 < \infty. \tag{1.4}$$

In a similar fashion, in the Hermitian case, we assume that the off-diagonal entries

$$\mathfrak{Re}(W_N)_{ij}, \qquad \mathfrak{Im}(W_N)_{ij}, \quad 1 \le i < j \le N, \tag{1.5}$$

are independent random variables such that

$$\mathbb{E}(W_N)_{ij} = 0, \qquad \mathbb{V}[\mathfrak{Re}(W_N)_{ij}] = \mathbb{V}[\mathfrak{Im}(W_N)_{ij}] = \frac{\sigma^2}{2}, \qquad m_5 := \sup_{i \ne j, N} \mathbb{E}|(W_N)_{ij}|^5 < \infty. \tag{1.6}$$

The diagonal entries

$$(W_N)_{ii}, \quad 1 \le i \le N, \tag{1.7}$$

are independent centered random variables, independent from the off-diagonal entries, with uniformly bounded third moment of the absolute values.

For a real symmetric (Hermitian) matrix M of order N, its empirical distribution of the eigenvalues is defined as $\mu_M = \frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i}$, where $\lambda_1 \ge \cdots \ge \lambda_N$ are the (ordered) eigenvalues of M. Wigner semicircle law (see e.g. [1,2,7]) states that almost surely the empirical distribution μ_{X_N} of a random real symmetric (Hermitian) Wigner matrix X_N converges weakly to the nonrandom limiting distribution μ_{sc}. The limiting distribution μ_{sc} is known as the semicircle distribution. It is absolutely continuous with respect to the Lebesgue measure and has the compact support $[-2\sigma, 2\sigma]$. The density of the Wigner semicircle distribution is given by

$$\frac{\mathrm{d}\mu_{sc}}{\mathrm{d}x}(x) = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - x^2}\,\mathbf{1}_{[-2\sigma, 2\sigma]}(x). \tag{1.8}$$

Wigner semicircle law can be reformulated as follows. For any bounded continuous test function $\varphi : \mathbb{R} \to \mathbb{R}$, the linear statistics

$$\frac{1}{N}\sum_{i=1}^{N}\varphi(\lambda_i) = \frac{1}{N}\mathrm{Tr}(\varphi(X_N)) =: \mathrm{tr}_N(\varphi(X_N))$$

converges to $\int \varphi(x)\,\mathrm{d}\mu_{sc}(\mathrm{d}x)$ almost surely; here and throughout the paper, we use the notation $\mathrm{tr}_N = \frac{1}{N}\mathrm{Tr}$ to denote the normalized trace.

The Stieltjes transform of the semicircle law is

$$g_\sigma(z) := \int \frac{\mathrm{d}\mu_{sc}(x)}{z - x} = \frac{z - \sqrt{z^2 - 4\sigma^2}}{2\sigma^2}, \quad z \in \mathbb{C} \setminus [-2\sigma, 2\sigma]. \tag{1.9}$$

It is the solution to

$$\sigma^2 g_\sigma^2(z) - zg_\sigma(z) + 1 = 0 \tag{1.10}$$

that decays to 0 as $|z| \to \infty$.

In this paper, we study the fluctuations of the outliers in the spectrum of finite-dimensional deformations of Wigner matrices. Starting with the pioneering work by Füredi and Komlós [19], there have been several results on finite

rank perturbations of matrices with i.i.d. entries, in particular [8–13,18,24,27,32]. We also note several papers on the eigenvalues of sample covariance matrices of spiked population models ([3,5,6,26]).

This manuscript can be viewed as a companion paper to our recent works [25] and [28] on the non-Gaussian fluctuation of the matrix entries of regular functions of Wigner matrices. However, no knowledge of the machinery used in [25] and [28] is required, and the paper can be read independently from these papers.

Let us consider a deformed Wigner matrix

$$M_N := \frac{1}{\sqrt{N}} W_N + A_N = X_N + A_N. \tag{1.11}$$

Here W_N is a random real symmetric (Hermitian) Wigner matrix as defined in (1.1)–(1.4) ((1.5)–(1.7)), and A_N is a deterministic Hermitian matrix of fixed finite rank r. We assume that the eigenvalues of A_N and their multiplicities are fixed. Let

$$\theta_1 > \cdots > \theta_{j_0} = 0 > \cdots > \theta_J$$

be the eigenvalues of A_N each with fixed multiplicity k_j. Clearly, the eigenvalue $\theta_{j_0} = 0$ has multiplicity $N - r$ and $\sum_{j \neq j_0} k_j = r$.

The first theorem of this section, Theorem 1.1, concerns the convergence of the extreme eigenvalues of the deformed random matrix. Let us denote $\rho_\theta = \theta + \frac{\sigma^2}{\theta}$. We shall use the shorthand notation ρ_j for ρ_{θ_j}. Theorem 1.1 was originally proved by Capitaine, Donati-Martin and Feral in [11] in the case when the common marginal distribution of the matrix entries is symmetric and satisfies a Poincaré inequality.

Theorem 1.1. *Let* $X_N = \frac{1}{\sqrt{N}} W_N$ *be a random real symmetric* (*Hermitian*) *Wigner matrix satisfying* (1.1)–(1.4) (*respectively* (1.5)–(1.7)). *Let* $J_{\sigma+}$ *be the number of j's such that* $\theta_j > \sigma$ *and* $J_{\sigma-}$ *be the number of j's such that* $\theta_j < -\sigma$.

(a) *For all* $j = 1, \ldots, J_{\sigma+}$ *and* $i = 1, \ldots, k_j$, $\lambda_{k_1 + \cdots + k_{j-1} + i} \to \rho_j$,
(b) $\lambda_{k_1 + \cdots + k_{J_{\sigma+}} + 1} \to 2\sigma$,
(c) $\lambda_{k_1 + \cdots + k_{J - J_{\sigma-}}} \to -2\sigma$,
(d) *for all* $j = J - J_{\sigma-} + 1, \ldots, J$ *and* $i = 1, \ldots, k_j$, $\lambda_{k_1 + \cdots + k_{j-1} + i} \to \rho_j$.

The convergence is in probability.

In other words, the first k_1 largest eigenvalues of M_N converge to ρ_1, the next k_2 largest eigenvalues converge to $\rho_2, \ldots$, the $J_{\sigma+}$th bunch of the largest eigenvalues converge to $\rho_{J_{\sigma+}}$, the next largest eigenvalue converges to 2σ (since it corresponds to a nonnegative eigenvalue of A_N which is not bigger than σ), etc.

Remark 1.1. *If random variables* $(W_N)_{ij}$, $1 \leq i \leq j \leq N$, *satisfy a Poincaré inequality* (1.12) *with constant* $\upsilon_{i,j,N}$ *uniformly bounded from zero,* $\upsilon_{i,j,N} \geq \upsilon > 0$, *the convergence holds with probability one.*

We recall that a probability measure $\mathbb{P}$ on $\mathbb{R}^M$ satisfies a Poincaré inequality with constant $\upsilon > 0$ if, for all continuously differentiable functions $f : \mathbb{R}^M \to \mathbb{C}$,

$$\mathbb{V}_{\mathbb{P}}(f) = \mathbb{E}_{\mathbb{P}}\big(\big|f(x) - \mathbb{E}_{\mathbb{P}}\big(f(x)\big)\big|^2\big) \leq \frac{1}{\upsilon} \mathbb{E}_{\mathbb{P}}\big[\big|\nabla f(x)\big|^2\big]. \tag{1.12}$$

Note that the Poincaré inequality tensorizes and the probability measures satisfying the Poincaré inequality have subexponential tails ([1,20]). In particular, if the marginal distributions of the matrix entries of W_N satisfy the Poincaré inequality with constant $\upsilon > 0$, then the joint distribution of $(W_N)_{ij}$, $1 \leq i \leq j \leq N$, also satisfies the Poincaré inequality with the same constant υ. By a standard scaling argument, we note that if the marginal distributions of the matrix entries of W_N satisfy the Poincaré inequality with $\upsilon > 0$ then the marginal distributions of the matrix entries of $X_N = \frac{1}{\sqrt{N}} W_N$ satisfy the Poincaré inequality with constant $N\upsilon$.

Theorem 1.1 follows from Theorem 1.2 formulated below. Theorem 1.2 is concerned with the distribution of the outliers, i.e. the eigenvalues of M_N corresponding to $\theta_j > \sigma$. Namely, we are interested in the fluctuation of the outliers around ρ_j, $1 \le j \le J_{\sigma^+}$. Let us consider a fixed eigenvalue θ_j of A_N such that $\theta_j > \sigma$. In general, if one does not assume some additional information about the structure of the eigenvectors of A_N corresponding to θ_j, the sequence of random vectors

$$\left(\sqrt{N}(\lambda_{k_1+\cdots+k_{j-1}+i} - \rho_j), i = 1, \ldots, k_j\right) \tag{1.13}$$

does not converge in distribution as $N \to \infty$ (see Theorems 1.3 and 1.5 below). However, one can prove that the sequence (1.13) is bounded in probability (tight). We recall that a sequence $\{\xi_N\}_{N \ge 1}$ of $\mathbb{R}^m$-dimensional random vectors is bounded in probability if for any $\varepsilon > 0$ there exists $L(\varepsilon)$ that does not depend on N such that $\mathbb{P}(|\xi_N| > L(\varepsilon)) < \varepsilon$ for all $N \ge 1$.

Theorem 1.2. *Let $X_N = \frac{1}{\sqrt{N}} W_N$ be a random real symmetric (Hermitian) Wigner matrix defined in (1.1)–(1.4) (respectively (1.5)–(1.7)). Let $1 \le j \le J_{\sigma^+}$, so the eigenvalue θ_j of A_N satisfies $\theta_j > \sigma$. Then the sequence of random vectors*

$$\left(\sqrt{N}(\lambda_{k_1+\cdots+k_{j-1}+i} - \rho_j), i = 1, \ldots, k_j\right) \tag{1.14}$$

is bounded in probability. In addition, if the marginal distributions of the matrix entries of W_N satisfy the Poincaré inequality (1.12) with constant $\upsilon_{i,j,N}$ uniformly bounded from zero, the following holds with probability 1

$$\lambda_{k_1+\cdots+k_{j-1}+i} - \rho_j = O\left(\frac{\log N}{\sqrt{N}}\right), \quad i = 1, \ldots, k_j. \tag{1.15}$$

Remark 1.2. *Theorem 1.2 clearly implies parts (a) and (d) of Theorem 1.1. To see that parts (b) and (c) of Theorem 1.1 also follow, we note that for any fixed positive integer $l \ge 1$ the lth largest eigenvalue of X_N converges in probability to 2σ. This is a simple consequence of the convergence of the largest eigenvalue of X_N to 2σ and the semicircle law. Then the interlacing property and Theorem 1.2 imply the desired result.*

Remark 1.3. *The bound (1.15) means that there exists a sufficiently large deterministic constant $C = C(\sigma, \upsilon, \theta_1, \ldots, \theta_r) > 0$, such that with probability 1*

$$|\lambda_{k_1+\cdots+k_{j-1}+i} - \rho_j| \le \frac{C \log N}{\sqrt{N}}, \quad i = 1, \ldots, k_j,$$

for all but finitely many N.

To study the fluctuations of the outliers in more detail, we consider two special cases following [12].

Case A ("The eigenvectors don't spread out"): The orthonormal eigenvectors of A_N corresponding to θ_j depend on a finite number K_j of canonical basis vectors of $\mathbb{C}^N$ (without loss of generality we can assume those canonical vectors to be $e_1, \ldots, e_{K_j}$), and their coordinates are independent of N.

Case B ("The eigenvectors are delocalized"): The l^∞ norm of every orthonormal eigenvector of A_N corresponding to θ_j goes to zero as $N \to \infty$.

Following [12], we denote by $k_{\sigma^+} = k_1 + \cdots + k_{J_{\sigma^+}}$ the number of positive eigenvalues of A_N bigger than σ (counting with multiplicities) and by $k \ge k_{\sigma^+}$ the minimal number of canonical basis vectors $e_1, \ldots, e_N$ of $\mathbb{C}^N$ required to span all the eigenvectors corresponding to the eigenvalues $\theta_1, \ldots, \theta_{J_{\sigma^+}}$.

We also denote

$$c_{\theta_j} := \frac{\theta_j^2}{\theta_j^2 - \sigma^2}. \tag{1.16}$$

The next theorem is a consequence of Proposition 1.1 below and Theorems 1.1 and 1.5 in [28]. We use a standard notation $\beta = 1$ in the real symmetric case and $\beta = 2$ in the Hermitian case.

Theorem 1.3. *Let* $X_N = \frac{1}{\sqrt{N}} W_N$ *be a random real symmetric* (*Hermitian*) *Wigner matrix defined in* (1.1)–(1.4) (*respectively* (1.5)–(1.7)) *such that the off-diagonal entries* $(W_N)_{ij}, 1 \le i < j \le N,$ *are i.i.d. real* (*complex*) *random variables with probability distribution* μ *and the diagonal entries* $(W_N)_{ii}, 1 \le i < N,$ *are i.i.d. random variables with probability distribution* μ_1. *In Case* A, *the* k_j-*dimensional vector*

$$\left(c_{\theta_j} \sqrt{N}(\lambda_{k_1+\cdots+k_{j-1+i}} - \rho_j), i = 1, \ldots, k_j\right)$$

converges in distribution to the distribution of the ordered eigenvalues of the $k_j \times k_j$ *random matrix* V_j *defined as*

$$V_j := U_j^*(W_j + H_j)U_j, \tag{1.17}$$

where

(i) W_j *is a Wigner random matrix of size* K_j *with the same marginal distribution of the matrix entries as* W_N,

(ii) H_j *is a real symmetric* (*Hermitian*) *Gaussian matrix of size* K_j, *independent of* W_j, *with centered independent entries* $H_{st}, 1 \le s \le t \le K_j$ ($\mathfrak{Re}\, H_{st}, \mathfrak{Im}\, H_{st}, 1 \le s < t \le K_j, H_{pp}, 1 \le p \le K_j,$ *in the Hermitian case*), *with the variance of the entries given by*

$$\mathbb{E}\left(H_{ss}^2\right) = \frac{\kappa_4(\mu)}{\theta_j^2} + \frac{2}{\beta} \frac{\sigma^4}{\theta_j^2 - \sigma^2}, \quad s = 1, \ldots, K_j, \tag{1.18}$$

$$\mathbb{E}\left(H_{st}^2\right) = \frac{\sigma^4}{\theta_j^2 - \sigma^2}, \quad 1 \le s < t \le K_j, \text{ in the real symmetric case,} \tag{1.19}$$

$$\mathbb{E}\left((\mathfrak{Re}\, H_{st})^2\right) = \mathbb{E}\left((\mathfrak{Im}\, H_{st})^2\right) = \frac{\sigma^4}{2(\theta_j^2 - \sigma^2)}, \quad 1 \le s < t \le K_j, \text{ in the Hermitian case,} \tag{1.20}$$

where

$$\kappa_4(\mu) := \int |x|^4 \, \mathrm{d}\mu(x) - (4 - \beta)\left(\int |x|^2 \, \mathrm{d}\mu(x)\right)^2, \tag{1.21}$$

is the fourth cumulant of μ, *and*

(iii) U_j *is a* $K_j \times k_j$ *such that the* (K_j-*dimensional*) *columns of* U_j *are written from the first* K_j *coordinates of the orthonormal eigenvectors corresponding to* θ_j.

In [12], Theorem 1.3 was proved for symmetric marginal distribution satisfying the Poincaré inequality (1.12) under an additional technical assumption that $k = \mathrm{o}(\sqrt{N})$, where k is defined in the paragraph above (1.16).

Using Theorems 4.1 and 4.2 from [25], one can extend the results of Theorem 1.3 to the case when the entries of W_N are not identically distributed provided the distribution of the entries $(W_N)_{il}, 1 \le i, l \le K_j$, does not depend on N.

Theorem 1.4. *Let* $X_N = \frac{1}{\sqrt{N}} W_N$ *be a random real symmetric* (*Hermitian*) *Wigner matrix defined in* (1.1)–(1.4) (*respectively* (1.5)–(1.7)) *such that the distribution of the entries* $(W_N)_{il}, 1 \le i, l \le K_j$, *does not depend on* N. *Let us assume that the limits*

$$m_4(i) := \lim_{N \to \infty} \frac{1}{N} \sum_{l:l \neq i} \mathbb{E}\left|(W_N)_{il}\right|^4 \tag{1.22}$$

exist for $1 \le i \le K_j$.

Then in Case A, *the results of Theorem* 1.3 *hold with* $\kappa_4(\mu)$ *in* (1.18) *replaced by*

$$\kappa_4(s) := m_4(s) - (4 - \beta)\sigma^2, \quad s = 1, \ldots, K_j. \tag{1.23}$$

The next theorem deals with the Case B.

Theorem 1.5. *Let $X_N = \frac{1}{\sqrt{N}} W_N$ be a random real symmetric (Hermitian) Wigner matrix defined in (1.1)–(1.4) (respectively (1.5)–(1.7)) such that the off-diagonal entries $(W_N)_{ij}, 1 \leq i < j \leq N$, are i.i.d. random variables with probability distribution μ and the diagonal entries $(W_N)_{ii}, 1 \leq i < N$, are i.i.d. random variables with probability distribution μ_1. In Case B, the k_j-dimensional vector*

$$\left(c_{\theta_j} \sqrt{N} (\lambda_{k_1 + \cdots + k_{j-1} + i} - \rho_j), i = 1, \ldots, k_j \right)$$

converges in distribution to the distribution of the (ordered) eigenvalues of a $k_j \times k_j$ GOE (GUE) matrix with the variance of the matrix entries given by $\frac{\theta_j^2 \sigma^2}{\theta_j^2 - \sigma^2}$ provided $k = \mathrm{o}(\sqrt{N})$.

Remark 1.4. *We recall that k has been defined above as the minimal number of canonical basis vectors $e_1, \ldots, e_N$ required to span the eigenvectors corresponding to the eigenvalues $\theta_1, \ldots, \theta_{J_{\sigma}+}$.*

Theorem 1.5 is an immediate extension of the result of Capitaine, Donati-Martin and Féral from [12] to our setting since their arguments apply essentially unchanged as soon as Theorem 1.1 is established.

It should be noted that Benaych-Georges, Guionnet and Maida consider in [9] perturbations of a random Wigner matrix by a finite rank random matrix with eigenvectors that are either independent copies of a random vector v with i.i.d. centered components satisfying the log-Sobolev inequality or are obtained by Gram–Schmidt orthonormalization of such independent copies. The distribution of the outliers is given in Proposition 5.3. of [9]. Let us denote the distribution of the first component of v by ν. If the fourth cumulant $\kappa_4(\nu)$ of ν vanishes, the limiting distribution of the outliers is similar to the result of Theorem 1.5, and given by the distribution of the ordered eigenvalues of a GOE (GUE) matrix. If the fourth cumulant does not vanish, one has to add a diagonal matrix with i.i.d. real Gaussian entries to a GOE (GUE) matrix.

One of the most important results of [9,10] concerns the distribution of the "sticking" eigenvalues (i.e. the eigenvalues that correspond to $|\theta_j| < \sigma$). In Theorem 5.3 of [9], Benaych-Georges, Guionnet and Maida prove that their limiting distribution is given by the Tracy–Widom law.

Let us briefly describe a key ingredient of the proofs of Theorems 1.2–1.4. We use the notation

$$R_N(z) := (z I_N - X_N)^{-1}, \quad z \in \mathbb{C} \setminus [-2\sigma, 2\sigma], \tag{1.24}$$

for the resolvent of X_N. Clearly, $R_N(z)$ is well defined for $z \in \mathbb{C} \setminus \mathbb{R}$. Since the spectral norm of X_N converges to 2σ in probability (see e.g. [4], and Proposition 2.1 in [25]), $R_N(x)$ is well defined for a fixed $x \in \mathbb{R} \setminus [-2\sigma, 2\sigma]$ with probability going to one. Since our results will deal with the limiting distribution of random variables $\sqrt{N}(\langle u, R_N(x) v \rangle - g_\sigma(x) \langle u, v \rangle)$ in the limit $N \to \infty$, this should not lead to ambiguity.

Let us consider a fixed eigenvalue θ_j of A_N such that $\theta_j > \sigma$ and denote by $v^{(1)}, \ldots, v^{(k_j)}$ the orthonormal eigenvectors of A_N that correspond to the eigenvalue θ_j. Denote by $\Xi_N^{(j)}$ the $k_j \times k_j$ matrix with the entries

$$\Xi_{il}^{(j)} := \sqrt{N} \left(\langle v^{(i)}, R_N(\rho_j) v^{(l)} \rangle - g_\sigma(\rho_j) \delta_{il} \right) = \sqrt{N} \left(\langle v^{(i)}, R_N(\rho_j) v^{(l)} \rangle - \frac{1}{\theta_j} \delta_{il} \right), \tag{1.25}$$

where we recall that $\rho_j = \theta_j + \frac{\sigma^2}{\theta_j}$. The following proposition plays an important part in our proofs.

Proposition 1.1. *Let $y_1 \geq \cdots \geq y_{k_j}$ be the ordered eigenvalues of the matrix $\Xi_N^{(j)}$. Then*

$$\sqrt{N} (\lambda_{k_1 + \cdots + k_{j-1} + i} - \rho_j) + \frac{1}{g'_\sigma(\rho_j)} y_i \to 0, \quad i = 1, \ldots, k_j, \tag{1.26}$$

in probability.

Remark 1.5. *A simple computation gives*

$$-\frac{1}{g'_\sigma(\rho_j)} = \theta_j^2 - \sigma^2. \tag{1.27}$$

It should be mentioned that the key part of the proof of Proposition 1.1 is a lemma from [9] which is stated as Lemma 4.2 in Section 4. Proposition 1.1 indicates that the question of the limiting distribution of the outliers of the spectrum of the deformed Wigner matrix M_N can be reduced to the question about the limiting distribution of the entries of (1.25).

Let us denote by $\langle u, v \rangle = \sum_1^N \bar{u}_i v_i$ the standard Euclidean scalar product in $\mathbb{C}^N$. The next theorem deals with the values of the sesquilinear form $\langle u^{(N)}, f(X_N)v^{(N)} \rangle$ where f is a sufficiently nice test function on $\mathbb{R}$ and $u^{(N)}$, $v^{(N)} \in \mathbb{C}^N$ are nonrandom unit vectors in $\mathbb{C}^N$, i.e.

$$\left\| u^{(N)} \right\| = \left\| v^{(N)} \right\| = 1, \quad N \geq 1,$$

where

$$\|u\|^2 = \langle u, u \rangle = \sum_1^N |u_i|^2.$$

Without additional assumptions on $u^{(N)}$ and $v^{(N)}$, the sequence

$$\sqrt{N}\left(\langle u^{(N)}, f(X_N)v^{(N)} \rangle - \mathbb{E}\langle u^{(N)}, f(X_N)v^{(N)} \rangle\right)$$

does not necessarily converge in distribution. However, one can show that it is tight.

We say that a function $f : I \subset \mathbb{R} \to \mathbb{R}$ belongs to $C^n(I)$ if f and its first n derivatives are continuous on I. Define

$$\|f\|_{C^n(I)} := \max\left(\left|\frac{d^l f}{dx^l}(x)\right|, x \in I, 0 \leq l \leq n\right).$$

We use the notation $C_c^n(\mathbb{R})$ for the space of n times continuously differentiable functions on $\mathbb{R}$ with compact support. Define

$$\|f\|_{n,1} := \max\left(\int_{-\infty}^{\infty} \left| d^k f/dx^k(x) \right| dx, 0 \leq k \leq n\right), \tag{1.28}$$

$$\|f\|_{n,1,+} := \max\left(\int_{\mathbb{R}} (|x| + 1)\left|\frac{d^l f}{dx^l}(x)\right| dx, 0 \leq l \leq n\right). \tag{1.29}$$

We recall that a function $f : \mathbb{R} \to \mathbb{R}$ is called Lipschitz continuous on an interval $I \subset \mathbb{R}$ if there exists a constant C such that

$$\left| f(x) - f(y) \right| \leq C|x - y| \quad \text{for all } x, y \in I. \tag{1.30}$$

We define

$$|f|_{\mathcal{L},\mathbb{R}} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|} \tag{1.31}$$

and

$$|f|_{\mathcal{L},\delta} = \sup_{x \neq y, x, y \in [-2\sigma - \delta, 2\sigma + \delta]} \frac{|f(x) - f(y)|}{|x - y|}. \tag{1.32}$$

Theorem 1.6. *Let $X_N = \frac{1}{\sqrt{N}} W_N$ be a random real symmetric* (*Hermitian*) *Wigner matrix defined in* (1.1)–(1.4) (*respectively* (1.5)–(1.7)). *Then the following statements hold*:

(i) *If $f : \mathbb{R} \to \mathbb{R}$ is a $C^5(\mathbb{R})$ function such that $\|f\|_{5,1}$ is finite, and $u^{(N)}, v^{(N)} \in \mathbb{C}^N, N \geq 1$, are two nonrandom sequences of unit vectors (in standard Euclidean norm), then there exists a constant $\mathrm{Const}(\sigma^2, m_5, c_3)$ such that*

$$\mathbb{V}\big(\langle u^{(N)}, f(X_N)v^{(N)}\rangle\big) \leq \mathrm{Const}\,\frac{\|f\|_{5,1}}{N}. \tag{1.33}$$

In particular, the sequence $\sqrt{N}(\langle u^{(N)}, f(X_N)v^{(N)}\rangle - \mathbb{E}\langle u^{(N)}, f(X_N)v^{(N)}\rangle)$ is bounded in probability.

(ii) *If $f \in C_c^8(\mathbb{R})$, with $\mathrm{supp}(f) \subset [-L, +L]$, where L is some positive number then there exists a constant $\mathrm{Const}(L, \sigma^2, m_5, c_3)$ such that*

$$\left| \mathbb{E}\langle u^{(N)}, f(X_N)v^{(N)}\rangle - \langle u^{(N)}, v^{(N)}\rangle \int_{-2\sigma}^{2\sigma} f(x)\,\mathrm{d}\mu_{sc}(\mathrm{d}x) \right|$$

$$\leq \mathrm{Const}\big(L, \sigma^2, m_5, c_3\big)\|f\|_{C^8([-L,+L])}\frac{1}{\sqrt{N}}. \tag{1.34}$$

If, in addition, $f \in C^9(\mathbb{R})$ and $\|f\|_{9,1,+}$ is finite, then

$$\left| \mathbb{E}\langle u^{(N)}, f(X_N)v^{(N)}\rangle - \langle u^{(N)}, v^{(N)}\rangle \int_{-2\sigma}^{2\sigma} f(x)\,\mathrm{d}\mu_{sc}(\mathrm{d}x) \right| \leq \mathrm{Const}\big(\sigma^2, m_5, c_3\big)\|f\|_{9,1,+}\frac{1}{\sqrt{N}}, \tag{1.35}$$

where $\mathrm{Const}(\sigma^2, m_5, c_3)$ depends on σ^2, m_5 and c_3.

(iii) *If the marginal distributions of the entries of W_N satisfy the Poincaré inequality (1.12) with a uniform constant $\upsilon > 0$, and f is a Lipschitz continuous function on $[-2\sigma - \delta, 2\sigma + \delta]$ that satisfies a subexponential growth condition*

$$\big|f(x)\big| \leq a\exp\big(b|x|\big) \quad \text{for all } x \in \mathbb{R} \tag{1.36}$$

for some positive constants a and b, then

$$\mathbb{P}\big(\big|\langle u^{(N)}, f(X_N)v^{(N)}\rangle - \mathbb{E}\langle u^{(N)}, f(X_N)v^{(N)}\rangle\big| \geq t\big)$$

$$\leq 2K\exp\left(-\frac{\sqrt{\upsilon N}t}{2|f|_{\mathcal{L},\delta}}\right) + \big(2K + \mathrm{o}(1)\big)\exp\left(-\frac{\sqrt{\upsilon N}}{2}\delta\right), \tag{1.37}$$

where $|f|_{\mathcal{L},\delta}$ is defined in (1.32),

$$K = -\sum_{i \geq 0} 2^i \log\big(1 - 2^{-1}4^{-i}\big), \tag{1.38}$$

and υ is the constant in the Poincaré inequality (1.12).

(iv) *If the marginal distributions of the entries of W_N satisfy the Poincaré inequality (1.12) with a uniform constant $\upsilon > 0$, and f is a Lipschitz continuous function on $\mathbb{R}$, then*

$$\mathbb{P}\big(\big|\langle u^{(N)}, f(X_N)v^{(N)}\rangle - \mathbb{E}\langle u^{(N)}, f(X_N)v^{(N)}\rangle\big| \geq t\big)$$

$$\leq 2K\exp\left(-\frac{\sqrt{\upsilon N}t}{2|f|_{\mathcal{L},\mathbb{R}}}\right), \tag{1.39}$$

where $|f|_{\mathcal{L},\mathbb{R}}$ is defined in (1.31).

(v) *If the marginal distributions of the entries of W_N satisfy the Poincaré inequality (1.12) with a uniform constant $\upsilon > 0$, $f \in C^8(\mathbb{R})$, and f satisfies the subexponential growth condition (1.36), then*

$$\mathbb{E}\langle u^{(N)}, f(X_N)v^{(N)}\rangle = \langle u^{(N)}, v^{(N)}\rangle \int_{-2\sigma}^{2\sigma} f(x)\,\mathrm{d}\mu_{sc}(\mathrm{d}x) + \mathrm{O}\left(\frac{1}{\sqrt{N}}\right). \tag{1.40}$$

We finish this section by formulating our last theorem, Theorem 1.7, which allows us to extend Theorem 1.3 (see Remark 5.1 in Section 5). Assume that the off-diagonal entries $(W_N)_{ij}$, $1 \le i < j \le N$, are i.i.d. random variables with probability distribution μ and the diagonal entries $(W_N)_{ii}$, $1 \le i < N$, are i.i.d. random variables with probability distribution μ_1.

Let us consider $u^{(N)}, v^{(N)} \in \mathbb{C}^N$ that are independent of N for all $N \ge N_0$, in a sense that only a fixed finite number of the coordinates of $u^{(N)}, v^{(N)}$ are nonzero and the coordinates do not change with N for $N \ge N_0$. In this case, we can write $u^{(N)} = u, v^{(N)} = v$, with the understanding that as the dimension N grows, one just adds more zero coordinates to u and v. As an immediate consequence of the results of Theorem 1.1 (real symmetric case) and Theorem 1.5 (Hermitian case) in [28], the random sequence

$$\sqrt{N}\big(\langle u, R_N(z)v\rangle - g_\sigma(z)\langle u, v\rangle\big) \tag{1.41}$$

converges in distribution as $N \to \infty$. Without loss of generality, we will consider the real symmetric case; the Hermitian case is essentially identical. Let m be an arbitrary fixed positive integer. Denote by $R^{(m)}(z)$ the $m \times m$ upper-left corner of the matrix $R_N(z)$. Theorem 1.1 in [28] states that a matrix-valued random field

$$\Upsilon_N(z) = \sqrt{N}\big(R^{(m)}(z) - g_\sigma(z)I_m\big), \quad z \in \mathbb{C} \setminus [-2\sigma, 2\sigma], \tag{1.42}$$

with values in the space of complex symmetric $m \times m$ matrices, converges in finite-dimensional distributions to a random field

$$\Upsilon(z) = g_\sigma^2(z)\big(W^{(m)} + Y(z)\big), \tag{1.43}$$

where $W^{(m)}$ is the $m \times m$ upper-left corner submatrix of a Wigner matrix W_N, $g_\sigma(z)$ is the Stieltjes transform (1.9) of the Wigner semicircle law, and

$$Y(z) = \big(Y_{ij}(z)\big), \quad Y_{ij}(z) = Y_{ji}(z),\ 1 \le i, j \le m,$$

is a Gaussian random field with the covariance matrix given by the formulas (1.18)–(1.23) in the real-symmetric case and (1.50)–(1.55) in the Hermitian case in [28]. It is important to note that $Y_{ij}(z)$, $1 \le i \le j \le m$, are independent random processes for different indices (ij).

Let us extend the definition of $\Upsilon(z)$ to that of an infinite-dimensional matrix $\Upsilon(z)_{pq}$, $1 \le p, q < \infty$, using the formulas (1.18)–(1.23) (respectively (1.50)–(1.55)) from [28]. Thus, the r.h.s. in (1.43) defines now the $m \times m$ upper-left corner of the infinite matrix $\Upsilon(z)$. Then Theorem 1.1 of [28] implies that

$$\sqrt{N}\big(\langle u, R_N(z)v\rangle - g_\sigma(z)\langle u, v\rangle\big) \to \langle u, \Upsilon(z)v\rangle, \tag{1.44}$$

in distribution.

Let $u, v \in l^2(\mathbb{N})$. It follows from the Kolmogorov three-series theorem (see e.g. [16]) that $\langle u, \Upsilon(\theta_j)v\rangle$ is well defined as an infinite sum of centered random variables with summable variances. For our analysis of the outliers in the spectrum of finite-rank deformations of Wigner matrices, it will be useful to have the following result.

Theorem 1.7. *Let $X_N = \frac{1}{\sqrt{N}}W_N$ be a random real symmetric (Hermitian) Wigner matrix defined in (1.1)–(1.4) (respectively (1.5)–(1.7)) such that the off-diagonal entries $(W_N)_{ij}$, $1 \le i < j \le N$, are i.i.d. random variables with probability distribution μ and the diagonal entries $(W_N)_{ii}$, $1 \le i < N$, are i.i.d. random variables with probability distribution μ_1.*

Let l be a fixed positive integer, $u_1, \ldots, u_l$, be a collection of nonrandom vectors in $l^2(\mathbb{N})$, and let $u_p^{(N)}$, $1 \le p \le l$, denote the projection of u_p onto the subspace spanned by the first N standard basis vectors $e_1, \ldots, e_N$. Then the joint distribution of

$$\sqrt{N}\big(\langle u_p^{(N)}, R_N(z)u_q^{(N)}\rangle - g_\sigma(z)\langle u_p^{(N)}, u_q^{(N)}\rangle\big), \quad 1 \le p, q \le l,$$

converges weakly to the joint distribution of $\langle u_p, \Upsilon(z)u_q\rangle$, $1 \le p, q \le l$.

The rest of the paper is organized as follows. Section 2 is devoted to the estimates on the mathematical expectation and the variance of the values of the resolvent sesquilinear form $\langle u^{(N)}, R_N(z)v^{(N)}\rangle$, where $u^{(N)}, v^{(N)}$ are arbitrary nonrandom unit vectors in $\mathbb{C}^N$. Using the estimates obtained in Section 2, we prove Theorem 1.6 in Section 3. Theorem 1.2 is proved in Section 4. Finally, Theorems 1.3, 1.4 and 1.7 are proved in Section 5. In the Appendix, we discuss tools used throughout the paper.

2. Mathematical expectation and variance of resolvent sesquilinear form

This section is devoted to the proof of the main building block Theorem 1.6, namely Proposition 2.1.

Without loss of generality, we can restrict our attention to the real symmetric case. Let $u^{(N)} = (u_1, \ldots, u_N)$, $v^{(N)} = (v_1, \ldots, v_N)$ be nonrandom unit vectors in $\mathbb{C}^N$. When it does not lead to ambiguity, we will omit the superscript in $u^{(N)}$ and $v^{(N)}$. Define

$$\eta_N := \langle u^{(N)}, R_N(z)v^{(N)}\rangle = \sum_{ij} \bar{u}_i R_{ij} v_j. \tag{2.1}$$

When it does not lead to ambiguity we will use the shorthand notation, R_{ij}, for the ijth entry $(R_N(z))_{ij}$, of the resolvent matrix $R_N(z)$.

Proposition 2.1. *Let* $X_N = \frac{1}{\sqrt{N}} W_N$ *be a random real symmetric* (*Hermitian*) *Wigner matrix defined in* (1.1)–(1.4) ((1.5)–(1.7)), $R_N(z) = (zI_N - X_N)^{-1}$, $z \in \mathbb{C} \setminus \mathbb{R}$, *and* $u^{(N)} = (u_1, \ldots, u_N)$, $v^{(N)} = (v_1, \ldots, v_N)$ *be nonrandom unit vectors in* $\mathbb{C}^N$. *Then*

$$\mathbb{E}\eta_N = \mathbb{E}\langle u^{(N)}, R_N(z)v^{(N)}\rangle = g_\sigma(z)\langle u^{(N)}, v^{(N)}\rangle + \mathrm{O}\left(\frac{1}{|\Im z|^7 \sqrt{N}}\right), \tag{2.2}$$

uniformly on bounded subsets of $\mathbb{C} \setminus \mathbb{R}$,

$$\mathbb{E}\langle u^{(N)}, R_N(z)v^{(N)}\rangle = g_\sigma(z)\langle u^{(N)}, v^{(N)}\rangle + \mathrm{O}\left((|z| + M)\frac{P_8(|\Im z|^{-1})}{\sqrt{N}}\right), \tag{2.3}$$

$$\mathbb{V}\eta_N = \mathbb{V}\langle u^{(N)}, R_N(z)v^{(N)}\rangle = \mathrm{O}\left(\frac{P_8(|\Im z|^{-1})}{N}\right), \tag{2.4}$$

uniformly on $\mathbb{C} \setminus \mathbb{R}$, *where* $P_l(x), l \geq 1$, *denotes a polynomial of degree* l *with fixed positive coefficients, and* M *is some constant.*

Remark 2.1. *In the case when* $u^{(N)}$ *and* $v^{(N)}$ *are standard basis vectors,* $u = e_i, v = e_j$, *the mathematical expectation and the variance of* $\langle u^{(N)}, R_N(z)v^{(N)}\rangle$ *have been studied in* [28]. *In particular, it has been shown there in Proposition* 2.1 *and* (3.27) *that*

$$\mathbb{E}R_{ii} = g_\sigma(z) + \mathrm{O}\left(\frac{1}{|\Im z|^6 N}\right), \tag{2.5}$$

uniformly on bounded subsets of $\mathbb{C} \setminus \mathbb{R}$, *and*

$$\mathbb{E}R_{ii} = g_\sigma(z) + \mathrm{O}\left((|z| + M)\frac{P_7(|\Im z|^{-1})}{N}\right), \tag{2.6}$$

$$\mathbb{E}R_{ij} = \mathrm{O}\left(\frac{P_5(|\Im z|^{-1})}{N}\right), \qquad \mathbb{E}R_{ij} = \mathrm{O}\left(\frac{P_9(|\Im z|^{-1})}{N^{3/2}}\right), \quad i \neq j, \tag{2.7}$$

$$\mathbb{V}R_{ij} = \mathrm{O}\left(\frac{P_6(|\Im z|^{-1})}{N}\right), \quad 1 \leq i, j \leq N, \tag{2.8}$$

uniformly on $\mathbb{C} \setminus \mathbb{R}$.

Remark 2.2. *In* [17], *Erdös, Yau and Yin studied generalized Wigner matrices* (*defined at the beginning of Section* 2 *of* [17]), *and obtained the following estimates provided the marginal distributions have subexponential tails*

$$\mathbb{P}\left\{\max_i \left| R_{ii}(z) - g_\sigma(z) \right| \geq \frac{(\log N)^l}{(N|\Im z|)^{1/3}}\right\} \leq C \exp\left[-c(\log N)^{\phi l}\right], \tag{2.9}$$

$$\mathbb{P}\left\{\max_{i \neq j} |R_{ij}(z)| \geq \frac{(\log N)^l}{(N|\Im z|)^{1/2}}\right\} \leq C \exp\left[-c(\log N)^{\phi l}\right], \tag{2.10}$$

where $0 < \phi < 1, C \geq 1, c > 0$ *are some constants,* $4/\phi \leq l \leq C \log N / \log \log N$, $N^{-1}(\log N)^{10l} < \Im z \leq 10, |\Re z| \leq 5\sigma$, *and* N *is sufficiently large.*

Remark 2.3. *It follows from our proofs that the error term on the r.h.s. of* (2.2) *can be replaced by* $\mathrm{O}(\frac{\min(\|u\|_1, \|v\|_1)}{|\Im z|^7 N})$, *where* $\|u\|_1 = \sum_{i=1}^N |u_i|$.

The rest of the section is devoted to the proof of Proposition 2.1.

Proof of Proposition 2.1. Without loss of generality, we can restrict our attention to the real symmetric case. The proof in the Hermitian case is very similar. We start by proving (2.2). Using $(zI_N - X_N)R_N(z) = I_N$, we write

$$z\mathbb{E}\sum_{ij} \bar{u}_i R_{ij} v_j = \mathbb{E}\sum_{ijk} \bar{u}_i (\delta_{ij} + X_{ik}R_{kj})v_j = \langle u, v \rangle + \sum_{ijk} \bar{u}_i v_j \mathbb{E}(X_{ik}R_{kj}). \tag{2.11}$$

Applying the decoupling formula (A.1) and (A.4)–(A.5) to the term $\mathbb{E}(X_{ik}R_{kj})$ in (2.11), we obtain

$$z\mathbb{E}\eta_N = \langle u, v \rangle + \sigma^2 \mathbb{E}(\eta_N \operatorname{tr}_N R) + \frac{\sigma^2}{N}\mathbb{E}\big(\langle u, \big(R_N(z)\big)^2 v \rangle\big) \tag{2.12}$$

$$+ \sum_{i,j} \frac{\mathbb{V}[(W_N)_{ii}] - 2\sigma^2}{N}\bar{u}_i v_j \mathbb{E}(R_{ii}R_{ij}) + r_N, \tag{2.13}$$

where η_N is defined in (2.1), and r_N contains the third and the fourth cumulant terms corresponding to $p = 2$ and $p = 3$ in the decoupling formula (A.1) for $i \neq k$, and the error terms due to the truncation of the decoupling formula (A.1) for $i \neq k$ at $p = 3$ and for $i = k$ at $p = 1$.

It follows from

$$\left|\mathbb{V}\big[(W_N)_{ii}\big] - 2\sigma^2\right| \leq \operatorname{const}(\sigma^2, c_3),$$

that the first term in (2.13) can be written as the mathematical expectation of $\frac{1}{N}\langle a, R_N(z)v \rangle$, where the vector a has coordinates $(\mathbb{V}[(W_N)_{ii}] - 2\sigma^2)\overline{R_{ii}}u_i, 1 \leq i \leq N$. Using (A.7), one obtains by estimating $\|a\|$ from above that

$$\sum_{i,j} \frac{\mathbb{V}[(W_N)_{ii}] - 2\sigma^2}{N}\bar{u}_i v_j \mathbb{E}(R_{ii}R_{ij}) = \mathrm{O}\left(\frac{1}{N|\Im z|^2}\right). \tag{2.14}$$

The third cumulant terms ($p = 2$) give

$$\frac{1}{2!N^{3/2}}\left[4\mathbb{E}\left(\sum_{i,j,k:i \neq k} \kappa_3(i,k)\bar{u}_i R_{ij}R_{ik}R_{kk}v_j\right) + 2\mathbb{E}\left(\sum_{i,j,k:i \neq k} \kappa_3(i,k)\bar{u}_i R_{ii}R_{kk}R_{kj}v_j\right)\right.$$

$$\left. + 2\mathbb{E}\left(\sum_{i,j,k:i \neq k} \kappa_3(i,k)\bar{u}_i R_{ki}R_{ki}R_{kj}v_j\right)\right], \tag{2.15}$$

where by $\kappa_3(i,k)$ we denote the third cumulant of $(W_N)_{ik}$. We note that

$$\left|\kappa_3(i,k)\right| \le \mathrm{Const}(m_5),$$

uniformly in $i \ne k$ and N. To estimate the absolute value of the first term in (2.15), we first sum with respect to j and then use the Cauchy–Schwarz inequality and (A.7) to obtain

$$\left|\mathbb{E}\sum_{i,j,k:i\ne k}\kappa_3(i,k)\bar{u}_i R_{ij}R_{ik}R_{kk}v_j\right| = \left|\mathbb{E}\sum_{i\ne k}\kappa_3(i,k)\bar{u}_i R_{ik}R_{kk}(Rv)_i\right|$$

$$\le \mathrm{Const}(m_5)\mathbb{E}\left(\sqrt{\sum_k |R_{kk}|^2}\sum_{i=1}^N |u_i|\|Re_i\|\big|(Rv)_i\big|\right)$$

$$\le \sqrt{N}\frac{\mathrm{Const}(m_5)}{|\Im z|^3}. \tag{2.16}$$

To estimate the absolute value of the second term in (2.15), we write

$$\left|\mathbb{E}\sum_{i,j,k:i\ne k}\kappa_3(i,k)\bar{u}_i R_{ii}R_{kk}R_{kj}v_j\right| = \left|\mathbb{E}\sum_{i\ne k}\kappa_3(i,k)\bar{u}_i R_{ii}R_{kk}(Rv)_k\right|$$

$$\le \mathrm{Const}(m_5)\mathbb{E}\left(\sum_{i,k}|u_i|\|R\|^2\big|(Rv)_k\big|\right)$$

$$\le \mathrm{Const}(m_5)\sqrt{N}\sum_i |u_i|\|v\|\mathbb{E}\|R_N(z)\|^3$$

$$\le N\frac{\mathrm{Const}(m_5)}{|\Im z|^3}. \tag{2.17}$$

Finally, we bound the last of the third cumulant terms in (2.15) as

$$\left|\mathbb{E}\sum_{i,j,k:i\ne k}\kappa_3(i,k)\bar{u}_i R_{ki}R_{ki}R_{kj}v_j\right| = \left|\mathbb{E}\sum_{i\ne k}\kappa_3(i,k)\bar{u}_i (R_{ki})^2(Rv)_k\right|$$

$$\le \mathrm{Const}(m_5)\mathbb{E}\sum_{ik}|u_i||R_{ki}|^2\|R_N(z)\|\|v\|$$

$$\le \sqrt{N}\frac{\mathrm{Const}(m_5)}{|\Im z|^3}, \tag{2.18}$$

where we again used (A.7) and

$$\sum_k |R_{ki}|^2 = \|R_N(z)e_i\|^2 \le \|R_N(z)\|^2 \le \frac{1}{|\Im z|^2}.$$

Combining the bounds (2.16)–(2.18), we see that the contribution of the third cumulant terms to r_N in (2.12)–(2.13) is bounded from above by $\mathrm{O}(\frac{1}{|\Im z|^3\sqrt{N}})$. The fourth cumulant terms give

$$\frac{1}{3!N^2}\Bigg[18\mathbb{E}\bigg(\sum_{i,j,k:i\ne k}\kappa_4(i,k)\bar{u}_i R_{ii}R_{ik}R_{kk}R_{kj}v_j\bigg) + 6\mathbb{E}\bigg(\sum_{i,j,k:i\ne k}\kappa_4(i,k)\bar{u}_i R_{ii}(R_{kk})^2 R_{ij}v_j\bigg)$$

$$+ 18\mathbb{E}\bigg(\sum_{i,j,k:i\ne k}\kappa_4(i,k)\bar{u}_i (R_{ki})^2 R_{kk}R_{ij}v_j\bigg) + 6\mathbb{E}\bigg(\sum_{i,j,k:i\ne k}\kappa_4(i,k)\bar{u}_i (R_{ki})^3 R_{kj}v_j\bigg)\Bigg]. \tag{2.19}$$

To estimate the absolute value of the first term in (2.19), we note that

$$
\left| \mathbb{E} \sum_{i,j,k:i\neq k} \kappa_4(i,k)\bar{u}_i R_{ii} R_{ik} R_{kk} R_{kj} v_j \right| = \left| \mathbb{E} \sum_{i\neq k} \kappa_4(i,k)\bar{u}_i R_{ii} R_{ik} R_{kk} (Rv)_k \right|
$$

$$
\leq \mathrm{Const}(m_5)\mathbb{E}\left(\sum_{ik} |\bar{u}_i| \|R\|^2 |R_{ik}| |(Rv)_k| \right) \leq \sqrt{N}\frac{\mathrm{Const}(m_5)}{|\mathfrak{Im}\, z|^4}, \tag{2.20}
$$

where we used the bound

$$
\sum_k |R_{ik}| |(Rv)_k| \leq \|R_N(z)e_i\| \|R_N(z)v\| \leq \|R_N(z)\|^2 \|v\|,
$$

(A.7), and the fact that the fourth cumulants of $(W_N)_{ik}$ are uniformly bounded in absolute value by some constant $\mathrm{Const}(m_5)$.

To estimate the second term in (2.19), we write

$$
\left| \mathbb{E} \sum_{i,j,k:i\neq k} \kappa_4(i,k)\bar{u}_i R_{ii}(R_{kk})^2 R_{ij} v_j \right| = \left| \mathbb{E} \sum_{i\neq k} \kappa_4(i,k)\bar{u}_i R_{ii}(R_{kk})^2 (Rv)_i \right|
$$

$$
\leq \mathrm{Const}(m_5)\mathbb{E}\left(\sum_k |R_{kk}|^2 \|R_N(z)\| \sum_i |u_i| |(Rv)_i| \right)
$$

$$
\leq N\frac{\mathrm{Const}(m_5)}{|\mathfrak{Im}\, z|^4}. \tag{2.21}
$$

The other two terms in (2.19) are estimated in a similar fashion. Each of them is $\mathrm{O}(\frac{N\|u\|\|v\|}{|\mathfrak{Im}\, z|^2})$. Therefore, the fourth cumulant terms give the contribution $\mathrm{O}(\frac{1}{N|\mathfrak{Im}\, z|^4})$ to r_N in (2.12)–(2.13).

Finally, we estimate the error terms due to the truncation of the decoupling formula at $p=3$ for $i\neq k$ and at $p=1$ for $i=k$. Here, we treat the error term due to the truncation of the decoupling formula at $p=3$ for $i\neq k$. The second error term can be treated in a similar way. To estimate the error term, we have to consider expressions of the following form

$$
N^{-5/2}\mathbb{E}\left(\sum_{ik} |\kappa_5(i,k)| \sup |u_i| |R_{ab}^{(1)}| |R_{cd}^{(2)}| |R_{ef}^{(3)}| |R_{pq}^{(4)}| |(R^{(5)}v)_s| \right), \tag{2.22}
$$

where $a, b, c, d, e, f, p, q, s \in \{i, k\}$, the supremum in (2.22) is considered over the resolvents $R^{(l)} = (z - X_N^{(l)})^{-1}, l = 1, \ldots, 5$, of rank two perturbations $X_N^{(l)} = X_N + x E_{ik}$ of X_N with $(E_{ik})_{jh} = \delta_{ij}\delta_{kh} + \delta_{ih}\delta_{kj}$. Estimating each entry of $R^{(l)}$ by $\frac{1}{|\mathfrak{Im}\, z|}$, taking into account that

$$
\sum_{i=1}^N |u_i| \leq \sqrt{N}\|u\| = \sqrt{N},
$$

and using the fact that the fifth cumulants of the off-diagonal entries of W_N are uniformly bounded, we bound (2.22) from above by $\mathrm{O}(\frac{1}{N|\mathfrak{Im}\, z|^5})$.

Combining the estimates of the third and the fourth cumulant terms and the truncation error term, we can rewrite the Master equation (2.12) as

$$
z\mathbb{E}\eta_N = \langle u, v \rangle + \sigma^2 \mathbb{E}(\eta_N \mathrm{tr}_N R) + \frac{\sigma^2}{N}\mathbb{E}\big(\langle u, (R_N(z))^2 v \rangle\big) + \mathrm{O}\left(\frac{P_5(|\mathfrak{Im}\, z|^{-1})}{\sqrt{N}} \right), \tag{2.23}
$$

where we recall that by P_l we denote a polynomial of degree l with positive coefficients that do not depend on N.

Since

$$\big|\langle u, \big(R_N(z)\big)^2 v\rangle\big| \le \|u\|\|v\|\frac{1}{|\Im z|^2},$$

we obtain

$$z\mathbb{E}\eta_N = \langle u, v\rangle + \sigma^2\mathbb{E}(\eta_N \operatorname{tr}_N R) + \mathrm{O}\bigg(\frac{P_5(|\Im z|^{-1})}{\sqrt{N}}\bigg). \tag{2.24}$$

Finally, we have to estimate the term $\mathbb{E}(\eta_N \operatorname{tr}_N R)$ in the Master equation. We write

$$\big|\mathbb{E}\big(\operatorname{tr}_N R\langle u, R_N(z)v\rangle\big) - g_\sigma(z)\mathbb{E}\langle u, R_N(z)v\rangle\big| \le \big(\mathbb{V}(\langle u, R_N(z)v\rangle)\big)^{1/2}\big(\mathbb{V}(\operatorname{tr}_N R)\big)^{1/2} \tag{2.25}$$

$$+ \big|g_N(z) - g_\sigma(z)\big|\|u\|\|v\|\frac{1}{|\Im z|}, \tag{2.26}$$

where we use the notation

$$g_N(z) := \mathbb{E}\operatorname{tr}_N R_N(z). \tag{2.27}$$

The variance $\mathbb{V}(\operatorname{tr}_N R_N(z))$ has been estimated in Proposition 2 of [30] as

$$\mathbb{V}\big(\operatorname{tr}_N R_N(z)\big) = \mathrm{O}\bigg(\frac{1}{|\Im z|^4 N^2}\bigg), \tag{2.28}$$

uniformly on $\mathbb{C}\setminus\mathbb{R}$. It follows from the proof of (2.28) that the bound is valid provided the fourth moments of the off-diagonal entries are uniformly bounded and the second moments of the diagonal entries are uniformly bounded ([31]). Applying the bound $|\langle u, R_N(z)v\rangle| \le \frac{\|u\|\|v\|}{|\Im z|} = \frac{1}{|\Im z|}$ and (2.5), we obtain

$$\mathbb{E}\big(\operatorname{tr}_N R\langle u, R_N(z)v\rangle\big) = g_\sigma(z)\mathbb{E}\langle u, R_N(z)v\rangle + \mathrm{O}\bigg(\frac{P_7(|\Im z|^{-1})}{N}\bigg),$$

uniformly on bounded subsets of $\mathbb{C}\setminus\mathbb{R}$. This allows us to write the Master equation for $\eta_N = \langle u^{(N)}, R_N(z)v^{(N)}\rangle$ as

$$z\mathbb{E}\eta_N = \langle u, v\rangle + \sigma^2 g_\sigma(z)\mathbb{E}\eta_N + \mathrm{O}\bigg(\frac{P_7(|\Im z|^{-1})}{\sqrt{N}}\bigg), \tag{2.29}$$

uniformly on bounded subsets of $\mathbb{C}\setminus\mathbb{R}$. Since $z - \sigma^2 g_\sigma(z) = 1/g_\sigma(z)$ and $g_\sigma(z)$ is bounded, we arrive at

$$\mathbb{E}\langle u, R_N(z)v\rangle = g_\sigma(z)\langle u, v\rangle + \mathrm{O}\bigg(\frac{P_7(|\Im z|^{-1})}{\sqrt{N}}\bigg), \tag{2.30}$$

which is exactly the estimate (2.2) of Proposition 2.1.

To prove (2.3), we note that (2.25)–(2.26), (2.28) and (2.6) imply

$$\mathbb{E}\big(\operatorname{tr}_N R\langle u, R_N(z)v\rangle\big) = g_\sigma(z)\mathbb{E}\langle u, R_N(z)v\rangle + \mathrm{O}\bigg((|z| + M)\frac{P_8(|\Im z|^{-1})}{N}\bigg),$$

uniformly on $\mathbb{C}\setminus\mathbb{R}$. Therefore, one can rewrite (2.24) as

$$z\mathbb{E}\eta_N = \langle u, v\rangle + \sigma^2 g_\sigma(z)\mathbb{E}\eta_N + \mathrm{O}\bigg((|z| + M)\frac{P_8(|\Im z|^{-1})}{\sqrt{N}}\bigg), \tag{2.31}$$

uniformly on $\mathbb{C}\setminus\mathbb{R}$, which implies (2.3).

Now, we turn our attention to the proof of (2.4). The key part of the proof is the following lemma.

Lemma 2.1. *Let $X_N = \frac{1}{\sqrt{N}} W_N$ be a random real symmetric (Hermitian) Wigner matrix defined in* (1.1)–(1.4) *(*(1.5)–(1.7)*),* $R_N(z) = (zI_N - X_N)^{-1}$, $z \in \mathbb{C} \setminus \mathbb{R}$, *and* $u^{(N)} = (u_1, \ldots, u_N)$, $v^{(N)} = (v_1, \ldots, v_N)$ *be nonrandom unit vectors in* $\mathbb{C}^N$. *Then*

$$\left(z - \sigma^2 g_N(z)\right) \mathbb{V}\left(\langle u^{(N)}, R_N(z) v^{(N)} \rangle\right) = \sqrt{\mathbb{V}\left(\langle u^{(N)}, R_N(z) v^{(N)} \rangle\right)} \mathrm{O}\left(\frac{P_3(|\mathfrak{Im}\, z|^{-1})}{\sqrt{N}}\right)$$

$$+ \mathrm{O}\left(\frac{P_6(|\mathfrak{Im}\, z|^{-1})}{N}\right), \tag{2.32}$$

uniformly in $z \in \mathbb{C} \setminus \mathbb{R}$, *where* $g_N(z)$ *is defined in* (2.27).

Proof. As always, we will suppress the dependence on N in $u = u^{(N)}$ and $v = v^{(N)}$, and use the notation $\eta_N = \langle u, R_N(z) v \rangle$. Clearly, $\mathbb{V}(\eta_N) = \mathbb{E}|\eta_N|^2 - |\mathbb{E}\eta_N|^2$, and $\overline{\langle u, R_N(z) v \rangle} = \langle v, R_N(\bar{z}) u \rangle$. We start with the following form of the Master equation for η_N,

$$z \mathbb{E}\eta_N = \langle u, v \rangle + \sigma^2 g_N(z) \mathbb{E}\eta_N \tag{2.33}$$

$$+ \frac{1}{N^{3/2}} \mathbb{E}\left(\sum_{i,j,k: i \neq k} \kappa_3(i, k) \bar{u}_i R_{ii}(z) R_{kk}(z) R_{kj}(z) v_j\right) + \mathrm{O}\left(\frac{P_5(|\mathfrak{Im}\, z|^{-1})}{N}\right), \tag{2.34}$$

uniformly on $\mathbb{C} \setminus \mathbb{R}$. We singled out in (2.34) the only term in r_N which is $\mathrm{O}(N^{-1/2})$, namely (2.17). As we have shown above, all other terms in r_N can be estimated as $\mathrm{O}(\frac{P_5(|\mathfrak{Im}\, z|^{-1})}{N})$. Multiplying both sides of the equation by $\overline{\mathbb{E}\eta_N}$, we obtain

$$z |\mathbb{E}\eta_N|^2 = \langle u, v \rangle \overline{\mathbb{E}\eta_N} + \sigma^2 |\mathbb{E}\eta_N|^2 g_N(z)$$

$$+ \frac{1}{N^{3/2}} \mathbb{E}\left(\sum_{i,j,k: i \neq k} \kappa_3(i, k) \bar{u}_i R_{ii}(z) R_{kk}(z) R_{kj}(z) v_j\right) \overline{\mathbb{E}\eta_N} + \mathrm{O}\left(\frac{P_6(|\mathfrak{Im}\, z|^{-1})}{N}\right). \tag{2.35}$$

Our next goal is to obtain the Master equation for $z\mathbb{E}(|\eta_N|^2)$. As before, we use the resolvent identity (A.3) to write

$$z \mathbb{E}\left(|\eta_N|^2\right) = z \mathbb{E}\left[\sum_{ij} \bar{u}_i R_{ij}(z) v_j \overline{\eta_N}\right] = \mathbb{E}\left[\sum_{ijk} \bar{u}_i \left(\delta_{ij} + X_{ik} R_{kj}(z)\right) v_j \overline{\eta_N}\right] \tag{2.36}$$

$$= \langle u, v \rangle \mathbb{E}\overline{\eta_N} + \sum_{ijk} \bar{u}_i v_j \mathbb{E}\left(X_{ik} R_{kj}(z) \overline{\eta_N}\right). \tag{2.37}$$

Applying the decoupling formula (A.1) and (A.4)–(A.5) to the term $\mathbb{E}(X_{ik} R_{kj}(z) \overline{\eta_N})$ in (2.36)–(2.37), we obtain

$$z \mathbb{E}\left(|\eta_N|^2\right) = \langle u, v \rangle \overline{\mathbb{E}\eta_N} + \sigma^2 \mathbb{E}\left(|\eta_N|^2 \mathrm{tr}_N R_N(z)\right) + \frac{\sigma^2}{N} \mathbb{E}\left(\langle u, \left(R_N(z)\right)^2 v \rangle \overline{\eta_N}\right) \tag{2.38}$$

$$+ \sum_{i,j} \frac{\mathbb{V}[(W_N)_{ii}] - 2\sigma^2}{N} \bar{u}_i v_j \mathbb{E}[R_{ii} R_{ij} \overline{\eta_N}] + \frac{\sigma^2}{N} \mathbb{E}\left(\sum_{i,j,k: i \neq k} \bar{u}_i R_{kj}(z) v_j \frac{\partial \langle v, R_N(\bar{z}) u \rangle}{\partial X_{ik}}\right) \tag{2.39}$$

$$+ \sum_{i,j} \frac{\mathbb{V}[(W_N)_{ii}]}{N} \mathbb{E}\left(\bar{u}_i R_{ij}(z) v_j \frac{\partial \langle v, R_N(\bar{z}) u \rangle}{\partial X_{ii}}\right) + r_N, \tag{2.40}$$

where r_N contains the third and the fourth cumulant terms corresponding to $p = 2$ and $p = 3$ in (A.1) for $k = i$, and the error due to the truncation of the decoupling formula (A.1) at $p = 3$ for $k \neq i$ and at $p = 1$ for $k = i$. Clearly,

$$\frac{\sigma^2}{N} \mathbb{E}\left(\langle u, \left(R_N(z)\right)^2 v \rangle \overline{\eta_N}\right) = \mathrm{O}\left(\frac{1}{|\mathfrak{Im}\, z|^3 N}\right). \tag{2.41}$$

For $k \neq i$, we have

$$\frac{\partial \langle v, R_N(\bar{z})u \rangle}{\partial X_{ik}} = \left(R_N(\bar{z})\bar{v}\right)_i \left(R_N(\bar{z})u\right)_k + \left(R_N(\bar{z})\bar{v}\right)_k \left(R_N(\bar{z})u\right)_i, \tag{2.42}$$

$$\frac{\partial^2 \langle v, R_N(\bar{z})u \rangle}{\partial X_{ik}^2} = 2R_{ii}(\bar{z})\left(R_N(\bar{z})\bar{v}\right)_k \left(R_N(\bar{z})u\right)_k \tag{2.43}$$

$$+ 2R_{ik}(\bar{z})\left(R_N(\bar{z})\bar{v}\right)_i \left(R_N(\bar{z})u\right)_k + 2R_{ik}(\bar{z})\left(R_N(\bar{z})\bar{v}\right)_k \left(R_N(\bar{z})u\right)_i \tag{2.44}$$

$$+ 2R_{kk}(\bar{z})\left(R_N(\bar{z})\bar{v}\right)_i \left(R_N(\bar{z})u\right)_i, \tag{2.45}$$

$$\frac{\partial (R_N(\bar{z})w)_i}{\partial X_{ik}} = R_{ii}(\bar{z})\left(R_N(\bar{z})w\right)_k + R_{ik}(\bar{z})\left(R_N(\bar{z})w\right)_i. \tag{2.46}$$

For $k = i$, we have

$$\frac{\partial \langle v, R_N(\bar{z})u \rangle}{\partial X_{ii}} = \left(R_N(\bar{z})\bar{v}\right)_i \left(R_N(\bar{z})u\right)_i, \tag{2.47}$$

$$\frac{\partial^2 \langle v, R_N(\bar{z})u \rangle}{\partial X_{ii}^2} = 2R_{ii}(\bar{z})\left(R_N(\bar{z})\bar{v}\right)_i \left(R_N(\bar{z})u\right)_i, \tag{2.48}$$

$$\frac{\partial (R_N(\bar{z})w)_i}{\partial X_{ii}} = R_{ii}(\bar{z})\left(R_N(\bar{z})w\right)_i. \tag{2.49}$$

Using (2.42) and (2.47), one can write the last term in (2.39) as

$$\frac{\sigma^2}{N}\mathbb{E}\left(\sum_{i,j,k:i\neq k} \bar{u}_i R_{kj}(z) v_j \frac{\partial \langle v, R_N(\bar{z})u \rangle}{\partial X_{ik}}\right) \tag{2.50}$$

$$= \frac{\sigma^2}{N}\mathbb{E}\left(\sum_{i,j,k:i\neq k} \bar{u}_i R_{kj}(z) v_j \left[\left(R_N(\bar{z})\bar{v}\right)_i \left(R_N(\bar{z})u\right)_k + \left(R_N(\bar{z})\bar{v}\right)_k \left(R_N(\bar{z})u\right)_i\right]\right) = \mathrm{O}\left(\frac{1}{|\Im m\, z|^3 N}\right). \tag{2.51}$$

The third cumulant terms in r_N in (2.40) can be written as

$$\frac{1}{2N^{3/2}}\sum_{i,j,k:i\neq k} \kappa_3(i,k)\bar{u}_i v_j \mathbb{E}\left(\frac{\partial^2 (R_{kj}(z)\langle v, R_N(\bar{z})u \rangle)}{\partial X_{ik}^2}\right) \tag{2.52}$$

$$= \frac{1}{2N^{3/2}}\sum_{i,j,k:i\neq k} \kappa_3(i,k)\bar{u}_i v_j \mathbb{E}\left(\frac{\partial^2 R_{kj}(z)}{\partial X_{ik}^2}\langle v, R_N(\bar{z})u \rangle\right) \tag{2.53}$$

$$+ \frac{1}{N^{3/2}}\sum_{i,j,k:i\neq k} \kappa_3(i,k)\bar{u}_i v_j \mathbb{E}\left(\frac{\partial R_{kj}(z)}{\partial X_{ik}}\frac{\partial \langle v, R_N(\bar{z})u \rangle}{\partial X_{ik}}\right) \tag{2.54}$$

$$+ \frac{1}{2N^{3/2}}\sum_{i,j,k:i\neq k} \kappa_3(i,k)\bar{u}_i v_j \mathbb{E}\left(R_{kj}(z)\frac{\partial^2 \langle v, R_N(\bar{z})u \rangle}{\partial X_{ik}^2}\right). \tag{2.55}$$

We are going to estimate the terms (2.53)–(2.55) separately. We start with the last two. We claim that both (2.54) and (2.55) are $\mathrm{O}(\frac{1}{|\Im m\, z|^4 N})$. Indeed, consider first (2.54). It follows from (A.4)–(A.5), (2.42) and (2.47), that it is equal to

$$\frac{1}{2N^{3/2}}\sum_{i,j,k:i\neq k} \kappa_3(i,k)\bar{u}_i v_j$$

$$\times \mathbb{E}\left(\left[R_{kk}(z)R_{ij}(z) + R_{ik}(z)R_{kj}(z)\right]\left[\left(R_N(\bar{z})\bar{v}\right)_i \left(R_N(\bar{z})u\right)_k + \left(R_N(\bar{z})\bar{v}\right)_k \left(R_N(\bar{z})u\right)_i\right]\right). \tag{2.56}$$

Let us estimate the term

$$\frac{1}{2N^{3/2}} \sum_{i,j,k:i\neq k} \kappa_3(i,k)\bar{u}_i v_j \mathbb{E}\big(R_{kk}(z)R_{ij}(z)\big(R_N(\bar{z})\bar{v}\big)_i\big(R_N(\bar{z})u\big)_k\big) \tag{2.57}$$

in (2.56).

We note that the Euclidean norm of the vector in $\mathbb{C}^N$ with the coordinates $\kappa_3(i,k)u_i(R_N(\bar{z})\bar{v})_i$, $1 \leq i \leq N, i \neq k$, and 0 for $i = k$ is bounded from above by $\frac{\text{Const}(m_5)}{|\Im m\, z|}$. Thus, it follows from (A.7) and $\|v\| = 1$ that

$$\left| \sum_{i,j:i\neq k} \kappa_3(i,k)\bar{u}_i\big(R_N(\bar{z})\bar{v}\big)_i v_j R_{ij}(z) \right| \leq \frac{\text{Const}(m_5)}{|\Im m\, z|^2}. \tag{2.58}$$

In addition,

$$\sum_{k=1}^{N} \big|R_{kk}(z)\big|\big|\big(R_N(\bar{z})u\big)_k\big| \leq \frac{1}{|\Im m\, z|} \sum_{k=1}^{N} \big|\big(R_N(\bar{z})u\big)_k\big| = \text{O}\bigg(\frac{\sqrt{N}}{|\Im m\, z|^2}\bigg). \tag{2.59}$$

Combining (2.59) and (2.58), we estimate (2.57) as $\text{O}(\frac{1}{|\Im m\, z|^4 N})$. The other terms in (2.56) can be estimated in a similar way, which implies that (2.54) is $\text{O}(\frac{1}{|\Im m\, z|^4 N})$.

Now, we turn our attention to (2.55). Using (2.43)–(2.45) and (2.48), one can rewrite (2.55) as

$$\frac{1}{N^{3/2}} \sum_{i,j,k:i\neq k} \kappa_3(i,k)\bar{u}_i v_j \mathbb{E}\big[R_{kj}(z)R_{ii}(\bar{z})\big(R_N(\bar{z})\bar{v}\big)_k\big(R_N(\bar{z})u\big)_k\big] \tag{2.60}$$

$$+ \frac{1}{N^{3/2}} \sum_{i,j,k:i\neq k} \kappa_3(i,k)\bar{u}_i v_j \mathbb{E}\big[R_{kj}(z)R_{ik}(\bar{z})\big(R_N(\bar{z})\bar{v}\big)_i\big(R_N(\bar{z})u\big)_k\big] \tag{2.61}$$

$$+ \frac{1}{N^{3/2}} \sum_{i,j,k:i\neq k} \kappa_3(i,k)\bar{u}_i v_j \mathbb{E}\big[R_{kj}(z)R_{ik}(\bar{z})\big(R_N(\bar{z})\bar{v}\big)_k\big(R_N(\bar{z})u\big)_i\big] \tag{2.62}$$

$$+ \frac{1}{N^{3/2}} \sum_{i,j,k:i\neq k} \kappa_3(i,k)\bar{u}_i v_j \mathbb{E}\big[R_{kj}(z)R_{kk}(\bar{z})\big(R_N(\bar{z})\bar{v}\big)_i\big(R_N(\bar{z})u\big)_i\big]. \tag{2.63}$$

We estimate (2.60). The subsums (2.61)–(2.63) can be estimated in a similar way. The summation with respect to j in (2.60) gives

$$\frac{1}{N^{3/2}} \sum_{i\neq k} \mathbb{E}\big[\kappa_3(i,k)\bar{u}_i\big(R_N(z)v\big)_k R_{ii}(\bar{z})\big(R_N(\bar{z})\bar{v}\big)_k\big(R_N(\bar{z})u\big)_k\big].$$

Now, we estimate

$$\sum_k \big|\kappa_3(i,k)\big(R_N(z)v\big)_k\big|\big|\big(R_N(\bar{z})\bar{v}\big)_k\big|\big|\big(R_N(\bar{z})u\big)_k\big| \leq \frac{\text{Const}(m_5)}{|\Im m\, z|^3}$$

and

$$\sum_i |u_i|\big|R_{ii}(\bar{z})\big| \leq \frac{\sqrt{N}}{|\Im m\, z|}.$$

Combining the last two bounds, we obtain that (2.60) is $\text{O}(\frac{1}{|\Im m\, z|^4 N})$.

Finally, let us estimate (2.53). It can be written as

$$\frac{1}{2!N^{3/2}}4\mathbb{E}\left(\sum_{i,j,k:i\neq k}\kappa_3(i,k)\bar{u}_i R_{ij}(z)R_{ik}(z)R_{kk}(z)v_j\langle v, R_N(\bar{z})u\rangle\right) \tag{2.64}$$

$$+\frac{1}{2!N^{3/2}}2\mathbb{E}\left(\sum_{i,j,k:i\neq k}\kappa_3(i,k)\bar{u}_i R_{ii}(z)R_{kk}(z)R_{kj}(z)v_j\langle v, R_N(\bar{z})u\rangle\right) \tag{2.65}$$

$$+\frac{1}{2!N^{3/2}}2\mathbb{E}\left(\sum_{i,j,k:i\neq k}\kappa_3(i,k)\bar{u}_i R_{ki}(z)R_{ki}(z)R_{kj}(z)v_j\langle v, R_N(\bar{z})u\rangle\right). \tag{2.66}$$

The subsums (2.64) and (2.66) are bounded from above by $O(\frac{1}{|\Im z|^4 N})$. The calculations are very similar to the ones used above and are left to the reader. The subsum (2.65) can be written as

$$\mathbb{E}\left(\frac{1}{N^{3/2}}\sum_{i,j,k:i\neq k}\kappa_3(i,k)\bar{u}_i R_{ii}(z)R_{kk}(z)R_{kj}(z)v_j\overline{\eta_N}\right).$$

To estimate it, we write

$$\left|\mathbb{E}\left(\frac{1}{N^{3/2}}\left(\sum_{i,j,k:i\neq k}\kappa_3(i,k)\bar{u}_i R_{ii}(z)R_{kk}(z)R_{kj}(z)v_j\right)\overline{\eta_N}\right)\right. \tag{2.67}$$

$$\left.-\frac{1}{N^{3/2}}\mathbb{E}\left(\sum_{i,j,k:i\neq k}\kappa_3(i,k)\bar{u}_i R_{ii}(z)R_{kk}(z)R_{kj}(z)v_j\right)\mathbb{E}\overline{\eta_N}\right| \tag{2.68}$$

$$\leq \frac{1}{N^{3/2}}\left(\mathbb{V}\left(\sum_{i,j,k:i\neq k}\kappa_3(i,k)\bar{u}_i R_{ii}(z)R_{kk}(z)R_{kj}(z)v_j\right)\right)^{1/2}\left(\mathbb{V}(\eta_N)\right)^{1/2}. \tag{2.69}$$

It follows from the estimates in (2.17) that one has a deterministic upper bound

$$\left|\frac{1}{N^{3/2}}\sum_{i,j,k:i\neq k}\kappa_3(i,k)\bar{u}_i R_{ii}(z)R_{kk}(z)R_{kj}(z)v_j\right| \leq \text{const}\,\frac{1}{|\Im z|^3\sqrt{N}}.$$

Thus,

$$\mathbb{E}\left(\frac{1}{N^{3/2}}\sum_{i,j,k:i\neq k}\kappa_3(i,k)\bar{u}_i R_{ii}(z)R_{kk}(z)R_{kj}(z)v_j\overline{\eta_N}\right) \tag{2.70}$$

$$=\frac{1}{N^{3/2}}\mathbb{E}\left(\sum_{i,j,k:i\neq k}\kappa_3(i,k)\bar{u}_i R_{ii}(z)R_{kk}(z)R_{kj}(z)v_j\right)\mathbb{E}\overline{\eta_N}+O\left(\frac{1}{|\Im z|^3\sqrt{N}}\right)(\mathbb{V}\eta_N)^{1/2}. \tag{2.71}$$

Combining the estimates (2.53)–(2.71), we obtain that the third cumulant term (2.52) contributing to r_N in (2.38) can be written as

$$\frac{1}{N^{3/2}}\mathbb{E}\left(\sum_{i,j,k:i\neq k}\kappa_3(i,k)\bar{u}_i R_{ii}(z)R_{kk}(z)R_{kj}(z)v_j\right)\mathbb{E}\overline{\eta_N} \tag{2.72}$$

$$+O\left(\frac{1}{|\Im z|^3\sqrt{N}}\right)(\mathbb{V}\eta_N)^{1/2}+O\left(\frac{1}{|\Im z|^4 N}\right). \tag{2.73}$$

Somewhat long but straightforward calculations using (A.4)–(A.5) and (2.42)–(2.51) show that the fourth cumulant term in r_N in (2.38) can be estimated from above by $O(\frac{1}{|\Im z|^5 N})$. Since the calculations are very similar to those

in (2.19)–(2.21), we leave the details to the reader. In a similar fashion, the error terms in r_N, due to the truncation of the decoupling formula at $p = 3$ for $i \neq k$ and at $p = 1$ for $i = k$ are bounded from above by $\mathrm{O}(\frac{1}{|\Im z|^6 N})$. The considerations are similar to those given in the analysis of (2.22).

Combining (2.41), (2.50)–(2.51), (2.72)–(2.73), and the bounds on the fourth cumulant term and the error terms discussed in the above paragraph, one rewrites the Master equation (2.38)–(2.39) as

$$z\mathbb{E}\big(|\eta_N|^2\big) = \langle u, v\rangle \overline{\mathbb{E}\eta_N} + \sigma^2 \mathbb{E}\big(|\eta_N|^2 \operatorname{tr}_N R_N(z)\big) \tag{2.74}$$

$$+ \frac{\kappa_3}{N^{3/2}} \mathbb{E}\bigg(\sum_{ijk} \bar{u}_i R_{ii}(z) R_{kk}(z) R_{kj}(z) v_j\bigg) \mathbb{E}\overline{\eta_N} + \mathrm{O}\bigg(\frac{1}{|\Im z|^3 \sqrt{N}}\bigg)(\mathbb{V}\eta_N)^{1/2} \tag{2.75}$$

$$+ \mathrm{O}\bigg(\frac{P_6(|\Im z|^{-1})}{N}\bigg). \tag{2.76}$$

Using (2.28), we estimate

$$\big|\mathbb{E}\big(|\eta_N|^2 \operatorname{tr}_N R_N(z)\big) - g_N(z)\mathbb{E}|\eta_N|^2\big| \leq \big(\mathbb{V}\big(|\eta_N|^2\big)\big)^{1/2}\big(\mathbb{V}\operatorname{tr}_N R_N(z)\big)^{1/2} = \mathrm{O}\bigg(\frac{1}{|\Im z|^4 N}\bigg).$$

This allows us to write

$$z\mathbb{E}\big(|\eta_N|^2\big) = \langle u, v\rangle \overline{\mathbb{E}\eta_N} + \sigma^2 g_N(z)\mathbb{E}|\eta_N|^2 + \frac{1}{N^{3/2}} \mathbb{E}\bigg(\sum_{i,j,k:i\neq k} \kappa_3(i,k)\bar{u}_i R_{ii}(z) R_{kk}(z) R_{kj}(z) v_j\bigg) \mathbb{E}\overline{\eta_N}$$

$$+ \mathrm{O}\bigg(\frac{1}{|\Im z|^3 \sqrt{N}}\bigg)(\mathbb{V}\eta_N)^{1/2} + \mathrm{O}\bigg(\frac{P_6(|\Im z|^{-1})}{N}\bigg). \tag{2.77}$$

Subtracting the r.h.s. in (2.35) from the r.h.s. in (2.77), we obtain (2.32). Lemma 2.1 is proven. □

Now, we are ready to finish the proof of Proposition 2.1. To obtain the estimate (2.4) from (2.32), we use the same arguments as in Section 3 of [25] and Section 2 of [28]. We note (see e.g. (3.9) in [25]) that

$$g_N(z)\big(z - \sigma^2 g_N(z)\big) = 1 + \mathrm{O}\bigg(\frac{P_4(|\Im z|^{-1})}{N}\bigg). \tag{2.78}$$

We define

$$\mathcal{O}_N := \big\{z\colon |\Im z| > LN^{-1/4}\big\},$$

where the constant L is chosen sufficiently large so that the $\mathrm{O}(\frac{P_4(|\Im z|^{-1})}{N})$ term on the r.h.s. of (2.78) is at most $1/2$ in absolute value. Multiplying both sides of (2.32) by $g_N(z)$, and using (A.8), we obtain that

$$\mathbb{V}\big(\langle u^{(N)}, R_N(z)v^{(N)}\rangle\big) = \sqrt{\mathbb{V}\big(\langle u^{(N)}, R_N(z)v^{(N)}\rangle\big)}\mathrm{O}\bigg(\frac{P_4(|\Im z|^{-1})}{\sqrt{N}}\bigg) + \mathrm{O}\bigg(\frac{P_7(|\Im z|^{-1})}{N}\bigg) \tag{2.79}$$

for $z \in \mathcal{O}_N$. It follows from (2.79) that

$$\mathbb{V}\big(\langle u^{(N)}, R_N(z)v^{(N)}\rangle\big) = \mathrm{O}\bigg(\frac{P_8(|\Im z|^{-1})}{N}\bigg) \quad \text{for } z \in \mathcal{O}_N. \tag{2.80}$$

On the other hand, if $|\Im z| \leq LN^{-1/4}$, then $\frac{L^4}{N|\Im z|^4} \geq 1$. Since $|\langle u^{(N)}, R_N(z)v^{(N)}\rangle| \leq \frac{1}{|\Im z|}$, we have

$$\mathbb{V}\big(\langle u^{(N)}, R_N(z)v^{(N)}\rangle\big) \leq \frac{1}{|\Im z|^2} \leq \frac{L^4}{N|\Im z|^6} \tag{2.81}$$

for z such that $|\Im m \, z| \leq LN^{-1/4}$. Combining (2.80) and (2.81), we obtain (2.4). This finishes the proof of Proposition 2.1. □

3. Proof of Theorem 1.6

Our exposition follows closely the ones in Section 3 of [28] and Section 4 of [25]. In order to extend the estimates of Proposition 2.1 to a more general class of test functions, we use the Helffer Sjöstrand functional calculus (see [15,21]).

Let l be some nonnegative integer, and $f \in C^{l+1}(\mathbb{R})$ decay at infinity sufficiently fast. For any self-adjoint operator X we can write

$$f(X) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}} \frac{1}{z - X} \, dx \, dy, \quad \frac{\partial \tilde{f}}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial \tilde{f}}{\partial x} + i\frac{\partial \tilde{f}}{\partial y}\right), \tag{3.1}$$

where:

 (i) $z = x + iy$ with $x, y \in \mathbb{R}$;
(ii) $\tilde{f}(z)$ is the extension of the function f defined as follows

$$\tilde{f}(z) := \left(\sum_{n=0}^{l} \frac{f^{(n)}(x)(iy)^n}{n!}\right)\sigma(y); \tag{3.2}$$

here $\sigma \in C^{\infty}(\mathbb{R})$ is a nonnegative function equal to 1 for $|y| \leq 1/2$ and equal to zero for $|y| \geq 1$.

The integral in (3.1) does not depend on the choice of l and the cut-off function $\sigma(y)$ (see e.g. [15]). Using the definition of $\tilde{f}$ in (3.2) one can easily calculate

$$\frac{\partial \tilde{f}}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial \tilde{f}}{\partial x} + i\frac{\partial \tilde{f}}{\partial y}\right) \tag{3.3}$$

$$= \frac{1}{2}\left(\sum_{n=0}^{l} \frac{f^{(n)}(x)(iy)^n}{n!}\right)i\frac{d\sigma}{dy} + \frac{1}{2}f^{(l+1)}(x)(iy)^l\frac{\sigma(y)}{l!} \tag{3.4}$$

and derive the crucial bound

$$\left|\frac{\partial \tilde{f}}{\partial \bar{z}}(x + iy)\right| \leq C_1 \max\left(\left|\frac{d^j f}{dx^j}(x)\right|, 1 \leq j \leq l+1\right)|y|^l. \tag{3.5}$$

For $X = X_N$, (3.1) implies

$$\langle u, f(X_N)v\rangle = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}}\langle u, R_N(z)v\rangle \, dx \, dy. \tag{3.6}$$

To prove (1.34), we let $l = 7$ in (3.2) and assume that f has compact support. It follows from (2.2) that

$$\mathbb{E}\langle u, f(X_N)v\rangle = -\mathbb{E}\frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}}\langle u, R_N(z)v\rangle \, dx \, dy \tag{3.7}$$

$$= -\frac{1}{\pi}\langle u, v\rangle \int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}} g_\sigma(z) \, dx \, dy - \frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}}\epsilon_{u,v}(z) \, dx \, dy \tag{3.8}$$

$$= \langle u, v\rangle \int f(x) \, d\mu_{sc}(x) - \frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}}\epsilon_{u,v}(z) \, dx \, dy, \tag{3.9}$$

where

$$|\epsilon_{u,v}(z)| \leq C_2 \frac{1}{\sqrt{N}} \frac{1}{|\Im m \, z|^7}, \tag{3.10}$$

uniformly on $\{z\colon \mathfrak{Re}\, z \in \operatorname{supp}(f), |\mathfrak{Im}\, z| \le 1\}$, and C_2 is a constant depending on $\operatorname{supp}(f)$. We conclude that the second term on the r.h.s. of (3.8) can be estimated as follows

$$\left| \frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}} \epsilon_{u,v}(z) \, dx \, dy \right| \le \frac{1}{\pi} \int_{\mathbb{C}} \left| \frac{\partial \tilde{f}}{\partial \bar{z}} \epsilon_{u,v}(z) \right| dx \, dy \tag{3.11}$$

$$\le C_1 C_2 \|f\|_{C^8([-L,L])} \frac{1}{\sqrt{N}} \int dx\, \chi_f(x) \int dy\, \chi_\sigma(y), \tag{3.12}$$

where χ_f and χ_σ are the characteristic functions of the support of f and of σ respectively, and L is such that $\operatorname{supp}(f) \subset [-L, L]$. This proves (1.34).

To prove (1.35), one considers $f \in C^9(\mathbb{R})$ (so $l = 8$) such that $\|f\|_{9,1,+}$ is finite. Using (2.3), one replaces the estimate (3.10) with

$$\left| \epsilon_{u,v}(z) \right| \le C_3 \frac{|z| + M}{\sqrt{N}} P_8\big(|\mathfrak{Im}\, z|^{-1}\big), \tag{3.13}$$

valid on $\mathbb{C} \setminus \mathbb{R}$, which leads to

$$\left| \frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}} \epsilon_{u,v}(z) \, dx \, dy \right| \le \frac{1}{\pi} \int_{\mathbb{C}} \left| \frac{\partial \tilde{f}}{\partial \bar{z}} \epsilon_{u,v}(z) \right| dx \, dy \le C_1 C_3 \|f\|_{9,1,+} \frac{1}{\sqrt{N}}. \tag{3.14}$$

To prove (1.33), we consider $f \in C^5(\mathbb{R})$ such that $\|f\|_{5,1} < \infty$, and let $l = 4$ in (3.2). Then

$$\mathbb{V}\big(\langle u, f(X_N)v \rangle\big) = \mathbb{V}\left(-\frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}} \langle u, R_N(z)v \rangle \, dx \, dy \right)$$

$$= \frac{1}{\pi^2} \int_{\mathbb{C}} \int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}} \frac{\partial \tilde{f}}{\partial \bar{w}} \, \mathbb{Cov}\big(\langle u, R_N(z)v \rangle, \langle u, R_N(w)v \rangle\big) \, dx \, dy \, ds \, dt,$$

where $z = x + \mathrm{i}y,\, w = s + \mathrm{i}t$. Taking into account (2.4), we get

$$\mathbb{V}\big(\langle u, f(X_N)v \rangle\big) \le \frac{1}{\pi^2} \int_{\mathbb{C}} \int_{\mathbb{C}} \left| \frac{\partial \tilde{f}}{\partial \bar{z}} \right| \left| \frac{\partial \tilde{f}}{\partial \bar{w}} \right| \sqrt{\mathbb{V}\big(\langle u, R_N(z)v \rangle\big)} \sqrt{\mathbb{V}\big(\langle u, R_N(w)v \rangle\big)} \, dx \, dy \, ds \, dt$$

$$\le \frac{\operatorname{Const}}{N} \left(\int_{\mathbb{C}} \left| \frac{\partial \tilde{f}}{\partial \bar{z}} \right| P_4\big(|\mathfrak{Im}\, z|^{-1}\big) \, dx \, dy \right)^2. \tag{3.15}$$

Plugging (3.5) with $l = 4$ in (3.15), we prove (1.33). Thus, we have proved the parts (i) and (ii) of Theorem 1.6.

Now, let us assume that the marginal distributions of the entries of W_N satisfy the Poincaré inequality (1.12) with a uniform constant υ and prove the parts (iii)–(v), i.e. the estimates (1.37), (1.39) and (1.40). Since the proof of (1.37)–(1.40) is very similar to the proof of Proposition 3.3 in [28], we discuss here only the main ingredients.

The first important observation is that, if $f(x)$ is a Lipschitz continuous function on $\mathbb{R}$ with the Lipschitz constant $|f|_{\mathcal{L},\mathbb{R}}$ then on the space of the $N \times N$ real symmetric (Hermitian) matrices, the matrix-valued function $f(X)$ is also Lipschitz continuous with respect to the Hilbert–Schmidt norm ([14], Proposition 4.6(c)). Namely,

$$\big\| f(X) - f(Y) \big\|_{\mathrm{HS}} \le |f|_{\mathcal{L},\mathbb{R}} \|X - Y\|_{\mathrm{HS}}, \tag{3.16}$$

where the Hilbert–Schmidt norm is defined as

$$\|X\|_{\mathrm{HS}} = \big(\mathrm{Tr}\big(|X|^2\big)\big)^{1/2}. \tag{3.17}$$

In particular, if u and v are unit vectors, then

$$G(X_N) = \langle u, f(X_N)v \rangle \tag{3.18}$$

is a complex-valued Lipschitz continuous function on the space of $N \times N$ real symmetric (Hermitian) matrices with the Lipschitz constant

$$|G|_{\mathcal{L}} := \sup_{X \neq Y} \frac{|G(X) - G(Y)|}{\|X - Y\|_{\mathrm{HS}}} = |f|_{\mathcal{L}, \mathbb{R}}.$$

The second observation is that the joint distribution of the matrix entries

$$\left\{ X_{ii}, 1 \leq i \leq N, \sqrt{2} X_{jk}, 1 \leq j < k \leq N \right\}$$

of X_N satisfies the Poincaré inequality with the constant $\frac{1}{2} N \upsilon$ since the Poincaré inequality tensorizes ([1,20]). Therefore, for any complex-valued Lipschitz continuous function of the matrix entries with the Lipschitz constant $|G|_{\mathcal{L}}$, the distribution of $G(X_N)$ has exponential tails (see e.g. Lemma 4.4.3 and Exercise 4.4.5 in [1]), i.e.

$$\mathbb{P}\big(\big|G(X_N) - \mathbb{E}G(X_N)\big| \geq t\big) \leq 2K \exp\left(-\frac{\sqrt{\upsilon N}}{2|G|_{\mathcal{L}}} t\right), \tag{3.19}$$

where K is a universal constant,

$$K = -\sum_{i \geq 0} 2^i \log\big(1 - 2^{-1} 4^{-i}\big).$$

This proves (1.39).

Applying (3.19) to the spectral norm $\|X\|$ of the matrix X_N and using the universality results for the largest eigenvalues (see [22] and references therein), we obtain

$$\mathbb{P}\big(\big| \|X_N\| - 2\sigma \big| \geq t\big) \leq \big(2K + \mathrm{o}(1)\big) \exp\left(-\frac{\sqrt{\upsilon N}}{2} t\right) \tag{3.20}$$

and, in particular,

$$\mathbb{P}\big(\|X_N\| > 2\sigma + \delta\big) \leq \big(2K + o(1)\big) \exp\left(-\frac{\sqrt{\upsilon N}}{2} \delta\right). \tag{3.21}$$

Let $f(x)$ be a real-valued Lipschitz continuous function on $[-2\sigma - \delta, 2\sigma + \delta]$. Then, we can find a function $f_1(x)$ that is Lipschitz continuous on $\mathbb{R}$, coincides with f on $[-2\sigma - \delta, 2\sigma + \delta]$, and satisfies $|f_1|_{\mathcal{L}, \mathbb{R}} = |f|_{\mathcal{L}, \delta}$. It follows from (3.21) that $\langle u, f(X_N) v \rangle$ does not coincide with $\langle u, f_1(X_N) v \rangle$ on a set of probability at most $(2K + \mathrm{o}(1)) \exp(-\frac{\sqrt{\upsilon N}}{2} t)$, which implies (1.37). The details are left to the reader.

4. Outliers in the spectrum of finite rank perturbations of Wigner matrices

This section is devoted to the proof of Theorem 1.2

Proof of Theorem 1.2. For $x \in (2\sigma, +\infty)$,

$$g_\sigma(x) = \frac{x}{2\sigma^2} \left(1 - \sqrt{1 - \frac{4\sigma^2}{x^2}}\right) \tag{4.1}$$

is decreasing and $g_\sigma(2\sigma + 0) = 1/\sigma$. Let us choose $\delta > 0$ in such a way that

$$\theta_j > \frac{1}{g_\sigma(2\sigma + 2\delta)} \quad \text{for all } 1 \leq j \leq J_{\sigma+}, \tag{4.2}$$

i.e. for all θ_j that correspond to the outliers (so $\theta_j > \sigma$). Let

$$L := \max(\theta_j, 1 \leq j \leq J_{\sigma+}) + 2\sigma + 2\delta.$$

It follows from (1.1)–(1.4) (see e.g. [2,4], and the proof of Proposition 2.1 in [25]) that there exists a random real symmetric Wigner matrix $\tilde{W}_N$ that satisfies (1.1)–(1.4),

$$\mathbb{P}(W_N = \tilde{W}_N) \to 1 \quad \text{as } N \to \infty,$$

and

$$\|\tilde{W}_N/\sqrt{N}\| \to 2\sigma \quad \text{a.s.} \tag{4.3}$$

Without loss of generality, we can assume that $\tilde{W}_N = W_N$, so

$$\|W_N/\sqrt{N}\| \to 2\sigma \quad \text{a.s.} \tag{4.4}$$

It follows from the definition of M_N and (4.4) that, with probability 1, the deformed random matrix $M_N = \frac{1}{N}W_N + A_N$ has no eigenvalues bigger than L for all but finitely many N. Let $u = u^{(N)}$, and $v = v^{(N)}$ be nonrandom unit vectors in $\mathbb{C}^N$. Define

$$\xi_N(x) := \sqrt{N}\big(\langle u, R_N(x)v \rangle - g_\sigma(x)\langle u, v \rangle\big), \quad x \in [2\sigma + 2\delta, \infty), \tag{4.5}$$

$$\zeta_N(x) := \frac{\mathrm{d}\xi_N(x)}{\mathrm{d}x} = -\sqrt{N}\big(\langle u, R_N^2(x)v \rangle + g_\sigma'(x)\langle u, v \rangle\big), \quad x \in [2\sigma + 2\delta, \infty), \tag{4.6}$$

$$\tilde{\xi}_N(x) := \sqrt{N}\big(\langle u, \big(h(X_N)R_N(x)\big)v \rangle - g_\sigma(x)\langle u, v \rangle\big), \quad x \in [2\sigma + 2\delta, \infty), \tag{4.7}$$

$$\tilde{\zeta}_N(x) := \frac{\mathrm{d}\tilde{\xi}_N(x)}{\mathrm{d}x} = -\sqrt{N}\big(\langle u, \big(h(X_N)R_N^2(x)\big)v \rangle + g_\sigma'(x)\langle u, v \rangle\big), \quad x \in [2\sigma + 2\delta, \infty), \tag{4.8}$$

where $h \in C_c^\infty(\mathbb{R})$ such that

$$h(x) \equiv 1 \quad \text{for } x \in [-2\sigma - \delta/2, 2\sigma + \delta/2], \tag{4.9}$$

$$h(x) \equiv 0 \quad \text{for } x \notin [-2\sigma - \delta, 2\sigma + \delta]. \tag{4.10}$$

We claim the following lemma.

Lemma 4.1.

$$\mathbb{P}\big(\max\big(|\zeta_N(x)|, x \in [2\sigma + 2\delta, L]\big) \leq \log(N)N^{1/6}\big) \to 1, \tag{4.11}$$

where $\zeta_N(x)$ is defined in (4.6).

Proof. It follows from (4.4) that almost surely $\xi_N(x) = \tilde{\xi}_N(x)$ and $\zeta_N(x) = \tilde{\zeta}_N(x)$ for all $x \in [2\sigma + 2\delta, L]$ and all but finitely many N almost surely. Thus, it is enough to prove the result of the lemma for $\tilde{\zeta}_N(x)$. Consider an equidistributed finite sequence

$$x_0 = 2\sigma + 2\delta < x_1 < x_2 < \cdots < x_{l(N)},$$

where $x_{i+1} - x_i = N^{-1/3}, 0 \leq i \leq l(N) - 1$, and $x_{l(N)-1} \leq L < x_{l(N)}$. Clearly, the number of elements in the sequence is $O(N^{1/3})$. We have

$$\mathbb{P}\bigg\{\max\big(|\tilde{\zeta}_N(x_i)|, 0 \leq i \leq l(N) - 1\big) > \frac{1}{2}\log(N)N^{1/6}\bigg\} \tag{4.12}$$

$$\leq \sum_{i=0}^{l(N)}\mathbb{P}\bigg\{|\tilde{\zeta}_N(x_i)| > \frac{1}{2}\log(N)N^{1/6}\bigg\} \tag{4.13}$$

$$\leq \frac{1}{(\log N)^2 N^{1/3}}\sum_{i=0}^{l(N)}\big(\mathbb{V}\big(\tilde{\zeta}_N(x_i)\big) + \big(\mathbb{E}\big(\tilde{\zeta}_N(x_i)\big)\big)^2\big). \tag{4.14}$$

It follows from Theorem 1.6 that

$$\mathbb{V}\big(\tilde{\zeta}_N(x_i)\big) = \mathrm{O}(1), \tag{4.15}$$

$$\mathbb{E}\big(\tilde{\zeta}_N(x_i)\big) = \mathrm{O}(1), \tag{4.16}$$

uniformly in $0 \le i \le l(N)$ and $N \ge 1$. Indeed,

$$\tilde{\zeta}_N(x) = \sqrt{N}\bigg(\langle u, f^{(x)}(X_N)v\rangle - \langle u, v\rangle \int_{-2\sigma}^{2\sigma} f^{(x)}(t)\,\mathrm{d}\mu_{sc}(t)\bigg), \tag{4.17}$$

where

$$f^{(x)}(t) = -h(t)\frac{1}{(x-t)^2} \tag{4.18}$$

is a $C_c^\infty(\mathbb{R})$ function such that $\|f^{(x)}\|_{5,1}$ and $\|f^{(x)}\|_{C_c^8(\mathbb{R})}$ are uniformly bounded in $x \in [2\sigma + 2\delta, \infty)$. Thus, (4.15)–(4.16) follow from (1.33)–(1.34). The bounds (4.12)–(4.14) and (4.15)–(4.16) then imply

$$\mathbb{P}\bigg\{\max\big(|\tilde{\zeta}_N(x_i)|, 0 \le i \le l(N) - 1\big) > \frac{1}{2}\log(N)N^{1/6}\bigg\} \le \frac{\mathrm{const}}{(\log N)^2}. \tag{4.19}$$

Taking into account that $|\frac{\mathrm{d}\tilde{\zeta}_N(x_i)}{\mathrm{d}x}| \le \mathrm{const}\sqrt{N}\|u\|\|v\|$, we arrive at (4.11). Lemma 4.1 is proven. $\qquad\square$

Now, we are ready to start the proof of Theorem 1.2. Let us denote by $u^{(1)}, \ldots, u^{(r)}$, the orthonormal eigenvectors of A_N corresponding to the nonzero eigenvalues. We recall that we used the notation $\theta_1 > \cdots > \theta_{j_0} = 0 > \cdots > \theta_J$ for the (fixed) eigenvalues of A_N, and denoted the (fixed) multiplicity of θ_j by k_j. The zero eigenvalue $\theta_{j_0} = 0$ has multiplicity $N - r$. Clearly, $\sum_{j \ne j_0} k_j = r$. Let us denote by Θ the $r \times r$ diagonal matrix built from the nonzero eigenvalues of A_N,

$$\Theta := \mathrm{diag}(\theta_1, \ldots, \theta_1, \ldots, \theta_{j_0-1}, \ldots, \theta_{j_0-1}, \theta_{j_0+1}, \ldots, \theta_{j_0+1}, \ldots, \theta_J, \ldots, \theta_J). \tag{4.20}$$

Let us also denote by U_N the $N \times r$ matrix whose columns are given by the orthonormal eigenvectors $u^{(1)}, \ldots, u^{(r)}$ of A_N. Clearly,

$$A_N = U_N \Theta U_N^*. \tag{4.21}$$

For any $x \in [2\sigma + 2\delta, L]$, we define the $r \times r$ matrix $\Xi_N(x)$ as follows. Let

$$\big(\Xi_N(x)\big)_{ij} = \xi_N^{ij}(x) := \sqrt{N}\big(\langle u^{(i)}, R_N(x)u^{(j)}\rangle - g_\sigma(x)\delta_{ij}\big), \quad 1 \le i, j \le r. \tag{4.22}$$

The first step in the proof of Theorem 1.2 is the following lemma from [9].

Lemma 4.2. *Suppose that x is not an eigenvalue of X_N. Then x is an eigenvalue of $X_N + A_N$ with multiplicity $n \ge 1$ if and only if $g_\sigma(x)$ is an eigenvalue of the $r \times r$ matrix*

$$Z_N(x) := \Theta^{-1} - \frac{1}{\sqrt{N}}\Xi_N(x), \tag{4.23}$$

with the same multiplicity.

For the convenience of the reader, we sketch the proof of Lemma 4.2 below.

Proof of Lemma 4.2. Let $x \notin \mathrm{Sp}(X_N)$. Therefore $R_N(x) = (xI_N - X_N)^{-1}$ is well defined, and

$$\det(X_N + A_N - xI_N) = \det\big((X_N - xI_N)(I_N - R_N(x)A_N)\big). \tag{4.24}$$

We obtain that for $x \notin \mathrm{Sp}(X_N)$ that $x \in \mathrm{Sp}(X_N + A_N)$ if and only if

$$\det\big(I_N - R_N(x)A_N\big) = \det\big(I_N - R_N(x)U_N \Theta U_N^*\big) = \det\big(I_r - \Theta U_N^* R_N(x)U_N\big) = 0, \tag{4.25}$$

where one uses the identity $\det(I - BC) = \det(I - CB)$. Rewriting

$$I_r - \Theta U_N^* R_N(x)U_N = -\Theta\big(U_N^* R_N(x)U_N - \Theta^{-1}\big),$$

one finishes the proof of Lemma 4.2. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proposition 1.1 plays an important role in the proof of Theorem 1.2. Before we prove Proposition 1.1, we need to introduce some notations and prove Lemma 4.3.

Consider a family of $r \times r$ matrices $Z_N(x)$ defined in (4.23) for $x \in [2\sigma + 2\delta, L]$. Fix an eigenvalue θ_j of A_N such that $\theta_j > \sigma$ and use the notation $v^{(1)}, \dots, v^{(k_j)}$ for the eigenvectors of A_N that correspond to the eigenvalue θ_j. Without loss of generality we can assume that $j = 1$. We do it just to simplify notations. The case $1 < j \le J_{\sigma+}$ is identical. We recall that $\Xi_N^{(j)}$ is defined in (1.25) as the $k_j \times k_j$ submatrix of $\Xi_N(\rho_j)$ restricted to the rows and columns corresponding to $v^{(i)}, 1 \le i \le k_j$. The central role in the proof of Proposition 1.1 is played by the following lemma.

Lemma 4.3. *Let $Z_N(x)$, $x \in [2\sigma + 2\delta, L]$, be as in (4.23), with $\Xi_N(x)$ defined in (4.22), and Θ defined in (4.20). Let*

$$z_1(x) \le z_2(x) \le \cdots \le z_r(x) \tag{4.26}$$

be the ordered eigenvalues of $Z_N(x)$. Then, for a sufficiently large constant $C > 0$,

$$\mathbb{P}\bigg(\big|z_i(x) - z_i(y)\big| \le C \frac{\log N}{N^{1/3}} |x - y|, \forall x, y \in [2\sigma + 2\delta, L], i = 1, \dots, r\bigg) \to 1, \tag{4.27}$$

as $N \to \infty$, and

$$z_i(\rho_1) = \frac{1}{\theta_1} + \mathrm{O}\bigg(\frac{1}{\sqrt{N}}\bigg), \quad 1 \le i \le k_1, \tag{4.28}$$

in probability, i.e. $\sqrt{N}(z_i(\rho_1) - \frac{1}{\theta_1})$ is bounded in probability, $1 \le i \le k_1$.

Below, we prove Lemma 4.3.

Proof of Lemma 4.3. We claim that (4.27) follows from Lemma 4.1. Indeed, (4.6) and (4.11) imply that

$$\mathbb{P}\big(\|\Xi_N(x) - \Xi_N(y)\| \le \mathrm{Const}\log(N)N^{1/6}|x - y|, \forall x, y \in [2\sigma + 2\delta, L]\big) \to 1, \tag{4.29}$$

as $N \to \infty$. Since $|z_i(x) - z_i(y)| \le \|Z_N(x) - Z_N(y)\| = \frac{1}{\sqrt{N}}\|\Xi_N(y) - \Xi_N(y)\|, 1 \le i \le r$, we conclude that (4.29) implies (4.27).

To prove (4.28), we use the fact that

$$\big\|Z_N(x) - \Theta^{-1}\big\| = \bigg\|\frac{1}{\sqrt{N}}\Xi_N(x)\bigg\| = \mathrm{O}\bigg(\frac{1}{\sqrt{N}}\bigg), \tag{4.30}$$

in probability. Indeed, the entries of the $r \times r$ matrix $\Xi_N(x)$ are bounded in probability since the expectation and variance of

$$\tilde{\xi}_N^{ij}(x) := \sqrt{N}\big(\langle u^{(i)}, h(X_N)R_N(x)u^{(j)}\rangle - g_\sigma(x)\delta_{ij}\big), \quad 1 \le i, j \le r,$$

is uniformly bounded by Theorem 1.6, and

$$\tilde{\xi}_N^{ij}(x) = \xi_N^{ij}(x), \quad 1 \le i, j \le r,$$

almost surely. Thus, $\|\Xi_N(x)\|$ is also bounded in probability. Since the first k_1 eigenvalues of Θ^{-1} are equal to $\frac{1}{\theta_1}$, we obtain (4.28). Lemma 4.3 is proven. $\qquad\square$

Now, we are ready to prove Proposition 1.1.

Proof of Proposition 1.1. By Lemma 4.2, the outliers of $X_N + A_N$ are given by those values of $x \in [2\sigma + \delta, M]$ such that

$$g_\sigma(x) = z_i(x) \quad \text{for some } 1 \le i \le r. \tag{4.31}$$

We recall that $g_\sigma(x)$ is a monotonically decreasing function on $[2\sigma + \delta, M]$ and

$$g'_\sigma(x) \le \text{const}(\sigma, \delta, M) < 0, \quad x \in [2\sigma + \delta, M]. \tag{4.32}$$

We note that

$$g_\sigma(\rho_1) = \frac{1}{\theta_1}. \tag{4.33}$$

Since for $1 \le i \le k_1$, (4.28) gives us that $z_i(\rho_1) - g_\sigma(\rho_1) = \text{O}(\frac{1}{\sqrt{N}})$ in probability, it follows from (4.27) and (4.31) that with probability going to 1, there exist $M > x_1 \ge x_2 \ge \cdots \ge x_{k_1} > 2\sigma + \delta$ such that $g_\sigma(x_i) = z_i(x_i), 1 \le i \le k_1$, and

$$\sqrt{N}|x_i - \rho_1| = \text{O}(1), \quad 1 \le i \le k_1, \tag{4.34}$$

in probability. Applying (4.27) one more time, we get that

$$g_\sigma(x_i) = z_i(\rho_1) + \text{O}\left(\frac{\log(N)N^{1/6}}{N}\right), \quad 1 \le i \le k_1, \tag{4.35}$$

in probability. By a standard perturbation theory argument (see e.g. Section XII.1 in [29]), one proves that the first k_1 smallest eigenvalues of the matrix $Z_N(\rho_1)$ differ from the (increasingly ordered) eigenvalues of the $k_1 \times k_1$ matrix $\frac{1}{\theta_1}\text{Id} - \frac{1}{\sqrt{N}}\Xi_N^{(m)}$ by at most $\text{O}(\frac{1}{N})$, in probability, where the matrix $\Xi_N^{(m)}$ has been defined in (1.25). To see this, we use the following standard lemma from the perturbation theory

Lemma 4.4. *Let B be an $n \times n$ real symmetric (Hermitian) matrix that can be written in the block form as $B = (B_{ij})_{i,j=1,2}$, where B_{ij} is an $n_i \times n_j$ matrix. Suppose that all eigenvalues of B_{11} are smaller than all eigenvalues of B_{22} and the gap between the spectra of B_{11} and B_{22} is at least $\text{Const} > 0$. In addition, suppose that the operator norm of the offdiagonal block B_{12} is bounded from above by ϵ, so that $\|B_{12}\| = \|B_{21}\| \le \epsilon$.*
Then there exists $\text{const}(\text{Const}, n)$ such that the first n_1 smallest eigenvalues of B differ from the (increasingly ordered) eigenvalues of B_{11} by at most $\text{const}\,\epsilon^2$.

Proof. We sketch the main idea of the proof for the convenience of the reader. First of all, one can assume that the eigenvalues of B_{11} are non-degenerate. In addition, one can assume that the blocks B_{11} and B_{22} are diagonal matrices. If not, one can simultaneously diagonalize them without changing the bound on the operator norms of the off-diagonal blocks. Thus, $B_{11} = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{n_1})$, and $B_{22} = \text{diag}(\lambda_{n_1+1}, \ldots, \lambda_n)$. Then the eigenvectors of B_{11} are given by $e_1, \ldots, e_{n_1}$, and the eigenvectors of B_{22} are given by $e_{n_1+1}, \ldots, e_n$, where $e_i, 1 \le i \le n$, are the standard basis vectors in $\mathbb{C}^n$. We recall that

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_{n_1} < \lambda_{n_1+1} \le \cdots \le \lambda_n,$$

and $\lambda_{n_1+1} - \lambda_{n_1} > \text{Const}$. Then it is easy to see that

$$\tilde{e}_1 = e_1 + \sum_{j > n_1} \frac{\langle(B - \lambda_1)e_1, e_j\rangle}{\lambda_1 - \lambda_j} e_j$$

is an approximate eigenvector of B with the approximate eigenvalue λ_1 such that

$$(B - \lambda_1)\tilde{e}_1 = \sum_{j > n_1} \frac{\langle (B - \lambda_1)e_1, e_j \rangle}{\lambda_1 - \lambda_j}(B - \lambda_j)e_j.$$

Since $\|(B - \lambda_j)e_j\| \le \epsilon, 1 \le j \le n$, and $\lambda_j - \lambda_1 \ge \text{Const}, n_1 < j \le n$, we obtain that

$$\|(B - \lambda_1)\tilde{e}_1\| \le \text{const}\,\epsilon^2,$$

where const depends just on Const and n. The last inequality also holds (albeit with a different value of const) if one replaces $\tilde{e}_1$ by the normalized vector $\frac{\tilde{e}_1}{\|\tilde{e}_1\|}$. Thus, B has an eigenvalue in the const$\,\epsilon^2$-neighborhood of λ_1. □

The result of the lemma can be immediately extended by induction to the case of $m \times m$ block matrices $B = (B_{ij})_{1 \le i, j \le m}$. To apply it in our setting, we note that the $k_1 \times k_1$ matrix $\frac{1}{\theta_1}\text{Id} - \frac{1}{\sqrt{N}}\Xi_N^{(m)}$ is the upper-left block of $Z_N(\rho_1)$. The other diagonal blocks of $Z_N(\rho_1)$ are given by $\Xi_N^{(i)}, 1 \le i \le m - 1$, defined in (1.25). Since the operator norms of the off-diagonal blocks of $Z_N(\rho_1)$ are $\text{O}(N^{-1/2})$ (see (4.30)), the desired statement follows.

Therefore, we have

$$g_\sigma(\rho_1) + \big(g_\sigma'(\rho_1) + \text{o}(1)\big)(x_i - \rho_1) = \frac{1}{\theta_1} - \frac{1}{\sqrt{N}}y_i + \text{O}\left(\frac{\log(N)N^{1/6}}{N}\right), \quad 1 \le i \le k_1, \tag{4.36}$$

where $y_1 \ge \cdots \ge y_{k_1}$ are the eigenvalues of the matrix $\Xi_N^{(m)}$. The result of Proposition 1.1 now follows from (4.33) and (4.36). □

Since the eigenvalues of the matrix $\Xi_N^{(m)}(\rho_1)$ are bounded in probability, the first part of Theorem 1.2, i.e. (1.14), follows from (1.26) in Proposition 1.1.

Now assume that the marginal distributions of the matrix entries of W_N satisfy the Poincaré inequality (1.12) with a uniform constant υ. Our goal is the almost sure bound (1.15) on the rate of convergence of the outliers. We note that one can improve (4.27) and (4.28) in Lemma 4.3 as follows. Applying (1.39)–(1.40) to (4.17) and taking into account that $\xi_N(x) = \tilde{\xi}_N(x)$ and $\zeta_N(x) = \tilde{\zeta}_N(x)$ for all $x \in [2\sigma + 2\delta, L]$ on a set of probability $1 - \text{O}(\exp(-\frac{\sqrt{\upsilon N}\delta}{2}))$, one proves

$$\max\big(|\zeta_N(x)|, x \in [2\sigma + 2\delta, L]\big) \le \text{Const}_1 \log(N), \tag{4.37}$$

$$\max\left(\left|z_i(x) - \frac{1}{\theta_1}\right|, x \in [2\sigma + 2\delta, L]\right) \le \text{Const}_2 \frac{\log(N)}{\sqrt{N}}, \quad 1 \le i \le k_1, \tag{4.38}$$

almost surely, where $\text{Const}_1 > 0, \text{Const}_2 > 0$ are sufficiently large, improving (4.11). Reasoning as before, (4.37) implies that

$$|z_i(x) - z_i(y)| \le \text{Const}_3 \frac{\log(N)}{\sqrt{N}}|x - y|, \quad \forall x, y \in [2\sigma + 2\delta, L], i = 1, \ldots, r, \tag{4.39}$$

almost surely for sufficiently large constant $\text{Const}_3 > 0$. Thus, we have

$$g_\sigma(\rho_1) + \big(g_\sigma'(\rho_1) + \text{o}(1)\big)(x_i - \rho_1) = \frac{1}{\theta_1} - \frac{1}{\sqrt{N}}y_i + \text{O}\left(\frac{\log(N)}{N}\right), \quad 1 \le i \le k_1, \tag{4.40}$$

almost surely, which implies (1.15) since $g_\sigma(\rho_1) = \frac{1}{\theta_1}$. Theorem 1.2 is proven. □

5. Proof of Theorems 1.3, 1.4 and 1.7

In this section, we prove Theorems 1.3, 1.4 and 1.7. We start with Theorem 1.3.

Proof of Theorem 1.3. Let $\theta_j > \sigma$ be an eigenvalue of A_N with the multiplicity k_j. Let us assume that Case A takes place. Thus, without loss of generality, we can assume that the eigenvectors of A_N corresponding to the eigenvalue θ_j belong to $\mathrm{Span}(e_1, \ldots, e_{K_j})$, where K_j is a fixed positive integer. As always, we consider the real symmetric case. The treatment of the Hermitian case is very similar. Consider a $K_j \times k_j$ matrix U_j such that the (K_j-dimensional) columns of U_j are filled by the first K_j coordinates of the k_j orthonormal vectors of A_N corresponding to the eigenvalue θ_j. We recall that the remaining $N - K_j$ coordinates of these orthonormal vectors are zero. Let us denote by $R_N^{(K_j)}(z)$ the upper-left $K_j \times K_j$ submatrix of the resolvent matrix $R_N(z) = (zI_N - X_N)^{-1}$. Finally, we define the random matrix-valued field

$$\Upsilon_N(z) := \sqrt{N}\big(R_N^{(K_j)}(z) - g_\sigma(z)I_{K_j}\big), \quad z \in \mathbb{C} \setminus [-2\sigma, 2\sigma]. \tag{5.1}$$

It follows from the definition that $\Upsilon_N(z)$ is a random function on $\mathbb{C} \setminus [-2\sigma, 2\sigma]$ with values in the space of complex symmetric $K_j \times K_j$ matrices. In particular, $\Upsilon_N(x)$ is real symmetric for real $x \in \mathbb{R} \setminus [-2\sigma, 2\sigma]$. It follows from (1.25), (5.1) and the definition of U_j in the paragraph above (5.1) that

$$\Xi_N^{(j)} = U_j^* \Upsilon_N(\rho_j) U_j. \tag{5.2}$$

We recall that Theorem 1.1 in [28] states that $\Upsilon_N(z)$ converges weakly in finite-dimensional distributions to a random field

$$\Upsilon(z) := g_\sigma^2(z)\big(W^{(K_j)} + Y(z)\big), \tag{5.3}$$

where $W^{(K_j)}$ is the $K_j \times K_j$ upper-left corner submatrix of the Wigner matrix W_N (1.1)–(1.4), and $Y(z) = (Y_{il}(z))$, $Y_{il}(z) = Y_{li}(z)$, $1 \le i, l \le K_j$, is a centered Gaussian random field with the covariance matrix given in (1.18)–(1.23) in [28]. In particular, for real $x \in \mathbb{R} \setminus [-2\sigma, 2\sigma]$, one has that the entries Y_{il}, $1 \le i \le l \le K_j$, are independent centered Gaussian real random variables such that

$$\mathbb{V}\big(Y_{ii}(x)\big) = \kappa_4(\mu)g_\sigma^2(x) - 2\sigma^4 g_\sigma'(x), \quad 1 \le i \le K_j, \tag{5.4}$$

$$\mathbb{V}\big(Y_{il}(x)\big) = -\sigma^4 g_\sigma'(x), \quad 1 \le i < l \le K_j. \tag{5.5}$$

Now, Theorem 1.3 follows from Proposition 1.1 in this paper, and Theorem 1.1 in [28], since

$$g_\sigma(\rho_j) = \frac{1}{\theta_j} \quad \text{and} \quad g_\sigma'(\rho_j) = -\frac{1}{\theta_j^2 - \sigma^2}. \tag{5.6}$$

Theorem 1.3 is proven. □

The proof of Theorem 1.4 is very similar to the given proof of Theorem 1.3. One has to use Theorems 4.1 and 4.2 and Remark 4.1 in [25] that generalize Theorems 1.1 and 1.5 in [28] to the non-i.i.d. case, and replace $\kappa_4(\mu)$ in (5.4) with $\kappa_4(i)$, $1 \le i \le K_j$.

Now, we turn to the proof of Theorem 1.7.

Proof of Theorem 1.7. Recall that we extended the definition of $\Upsilon(z)$ to that of an infinite-dimensional matrix $\Upsilon(z)_{pq}$, $1 \le p, q < \infty$, in the paragraphs above the formulation of Theorem 1.7. We employ a standard approximation argument. For simplicity, we assume that $\Im m\, z \ne 0$. If z is real, one has to replace $R_N(z)$ by $h(X_N)R_N(z)$, where $h \in C_c^\infty(\mathbb{R})$ is defined in (4.9)–(4.10). Let n be a sufficiently large fixed positive integer and consider $N \ge n$. It follows from Theorem 1.1 in [28] (see the proof of Theorem 1.3 above) that the joint distribution of

$$\big\langle u_p^{(n)}, \Upsilon_N(z)u_q^{(n)}\big\rangle = \sqrt{N}\big(\big\langle u_p^{(n)}, R_N(z)u_q^{(n)}\big\rangle - g_\sigma(z)\big\langle u_p^{(n)}, u_q^{(n)}\big\rangle\big), \quad 1 \le p, q \le l,$$

converges weakly to the joint distribution of $\langle u_p^{(n)}, \Upsilon(z) u_q^{(n)} \rangle$, $1 \le p, q \le l$. Choosing n sufficiently large, we can make

$$\mathbb{V}\big(\langle u_p^{(n)}, \Upsilon(z) u_q^{(n)} \rangle - \langle u_p, \Upsilon(z) u_q \rangle\big), \quad 1 \le p, q \le l, \tag{5.7}$$

and

$$\mathbb{V}\big(\langle u_p^{(n)}, \Upsilon_N(z) u_q^{(n)} \rangle - \langle u_p^{(N)}, \Upsilon(z)_N u_q^{(N)} \rangle\big), \quad 1 \le p, q \le l, \tag{5.8}$$

arbitrary small uniformly in $N \ge n$. Indeed, the variance in (5.7) is bounded by $\mathrm{O}(\|u_p - u_p^{(n)}\|^2 + \|u_q - u_q^{(n)}\|^2)$ since the entries of $\Upsilon(z)$ are i.i.d. random variables with bounded variance on the diagonal and i.i.d. random variables with bounded variance off the diagonal. In addition,

$$\langle u_p^{(n)}, \Upsilon_N(z) u_q^{(n)} \rangle - \langle u_p^{(N)}, \Upsilon_N(z) u_q^{(N)} \rangle = \langle u_p^{(n)} - u_p^{(N)}, \Upsilon_N(z) u_q^{(n)} \rangle + \langle u_p^{(N)}, \Upsilon_N(z) \big(u_q^{(n)} - u_q^{(N)} \big) \rangle$$

and we can use the bounds (1.33) and (1.34) in Theorem 1.6 rewritten as

$$\mathbb{V}\big(\langle u, f(X_N) v \rangle\big) \le \mathrm{const}_1 \frac{\|f\|_{C_c^5(\mathbb{R})} \|u\|^2 \|v\|^2}{N}, \tag{5.9}$$

$$\left| \mathbb{E}\langle u, f(X_N) v \rangle - \langle u, v \rangle \int_{-2\sigma}^{2\sigma} f(x)\, \mathrm{d}\mu_{sc}(\mathrm{d}x) \right| \le \mathrm{const}_2 \|f\|_{C_c^8(\mathbb{R})} \frac{\|u\| \|v\|}{\sqrt{N}}, \tag{5.10}$$

to show that

$$\mathbb{V}\big(\langle u_p^{(n)} - u_p^{(N)}, \Upsilon_N(z) u_q^{(n)} \rangle\big), \qquad \mathbb{V}\big(\langle u_p^{(N)}, \Upsilon_N(z) \big(u_q^{(n)} - u_q^{(N)} \big) \rangle\big)$$

are arbitrary small (uniformly in N) provided one chooses n sufficiently large. This finishes the proof. $\qquad\square$

Theorem 1.7 allows the following extension of Theorem 1.3:

Remark 5.1. *Let $u^{(1)}, \ldots, u^{(r)}$ be a set of orthonormal vectors in $l^2(\mathbb{N})$ such that*

$$\|u^{(p)} - u_N^{(p)}\| = \mathrm{o}\big(N^{-1/2}\big), \quad 1 \le p \le r, \tag{5.11}$$

where $u_N^{(p)}$ denotes the projection of $u^{(p)}$ onto the subspace spanned by the first N standard basis vectors $e_1, \ldots, e_N$. Let U_N be the $N \times r$ matrix whose columns are given by the vectors $u_N^{(1)}, \ldots, u_N^{(r)}$. Also denote by Θ the $r \times r$ diagonal matrix

$$\Theta = \mathrm{diag}(\theta_1, \ldots, \theta_1, \ldots, \theta_{j_0-1}, \ldots, \theta_{j_0-1}, \theta_{j_0+1}, \ldots, \theta_{j_0+1}, \ldots, \theta_J, \ldots, \theta_J).$$

Finally, define

$$A_N = U_N \Theta U_N^*.$$

The result of Theorem 1.3 can be extended for such A_N, with the matrix V_j given by

$$\theta_j^2 \langle u^{(p)}, \Upsilon(\theta_j) u^{(q)} \rangle, \quad 1 \le p, q \le k_j. \tag{5.12}$$

Appendix

The appendix contains several basic formulas used throughout the paper.

First, we recall the decoupling formula from [23]. Let ξ be a real-valued random variable with $p+2$ finite moments, and ϕ be a function from $\mathbb{C} \to \mathbb{R}$ with $p+1$ continuous and bounded derivatives. Then

$$\mathbb{E}\big(\xi\phi(\xi)\big) = \sum_{a=0}^{p} \frac{\kappa_{a+1}}{a!} \mathbb{E}\big(\phi^{(a)}(\xi)\big) + \epsilon, \tag{A.1}$$

where κ_a are the cumulants of ξ, $|\epsilon| \leq C \sup_t |\phi^{(p+1)}(t)| \mathbb{E}(|\xi|^{p+2})$, and C depends only on p. If ξ is a centered Gaussian random variable, the decoupling formula (A.1) becomes

$$\mathbb{E}\big(\xi\phi(\xi)\big) = \mathbb{V}(\xi)\mathbb{E}\big(\phi'(\xi)\big), \tag{A.2}$$

and can be immediately verified by integration by parts.

Next, we write a basic resolvent identity. For any two Hermitian matrices X_1 and X_2 and nonreal z we have:

$$(zI - X_2)^{-1} = (zI - X_1)^{-1} - (zI - X_1)^{-1}(X_1 - X_2)(zI - X_2)^{-1}. \tag{A.3}$$

As a corollary of (A.3), one has the following formulas. If X is a real symmetric matrix with resolvent R then

$$\frac{\partial R_{kl}}{\partial X_{pq}} = R_{kp}R_{ql} + R_{kq}R_{pl} \quad \text{for } p \neq q, \tag{A.4}$$

$$\frac{\partial R_{kl}}{\partial X_{pp}} = R_{kp}R_{pl}. \tag{A.5}$$

In a similar way, if X is a Hermitian matrix then

$$\frac{\partial R_{kl}}{\partial \mathfrak{Re}\, X_{pq}} = R_{kp}R_{ql} + R_{kq}R_{pl}, \quad p \neq q,$$

$$\frac{\partial R_{kl}}{\partial \mathfrak{Im}\, X_{pq}} = \mathrm{i}(R_{kp}R_{ql} - R_{kq}R_{pl}), \quad p \neq q,$$

$$\frac{\partial R_{kl}}{\partial X_{pp}} = R_{kp}R_{pl}.$$

Finally, we will use the following properties of the resolvent:

$$\|R_N(z)\| = \frac{1}{\mathrm{dist}(z, \mathrm{Sp}(X))}, \tag{A.6}$$

where by $\mathrm{Sp}(X)$ we denote the spectrum of a real symmetric (Hermitian) matrix X. The bound (A.6) implies

$$\|R_N(z)\| \leq |\mathfrak{Im}(z)|^{-1}. \tag{A.7}$$

Therefore, all entries of the resolvent matrix are bounded by $|\mathfrak{Im}(z)|^{-1}$. In a similar fashion, we have the following bound for the Stieltjes transform, $g(z)$, of any probability measure:

$$|g(z)| \leq |\mathfrak{Im}(z)|^{-1}. \tag{A.8}$$

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