ON FINITELY INJECTIVE MODULES

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1. Introduction

A right module A over a ring R is called *finitely injective* if every diagram of right R-modules of the form

$$\begin{array}{ccccc} o & \rightarrow & X & \xrightarrow{f} & Y, \\ & & g\downarrow & & \\ & & A & & \end{array}$$

where X is finitely generated and the row is exact, can be imbedded in a commutative diagram

$$\begin{array}{ccccc} o & \rightarrow & X & \xrightarrow{f} & Y. \\ & & g\downarrow & \swarrow h & \\ & & A & & \end{array}$$

The finitely injective modules turn out to have the nice property of being 'strongly pure' in all the containing modules and, in particular, are absolutely pure in the sense of Maddox [8]. For this reason we call them strongly absolutely pure (for short, SAP) modules. Several other characterizations of the SAP modules are obtained. It is shown that every module admits a 'SAP hull'. The concepts of finitely M-injective and finitely quasi-injective modules are then investigated. A subclass of finitely quasi-injective modules, called the strongly regular modules, is studied in some detail and it is shown that, if R/J is right Noetherian, then the strongly regular right R-modules coincide with the semi-simple R-modules.

2. Preliminaries

All the rings that we consider are associative rings with identity and all the modules (unless otherwise specified) are unitary right modules. M_R denotes the category of right R-modules. The injective hull of a module A is denoted by $\hat{A}$.

A submodule A of a right R-module B is said to be *pure* ([3]) in B if the canonical map $A \otimes_R M \to B \otimes_R M$ (induced by the inclusion $A \subseteq B$) is a monomorphism for each left R-module M. This is equivalent to saying that any finite system of equations $\sum_{j=1}^{n} x_j \gamma_{ij} = a_i$ $(i = 1, \cdots, t)$, where the $\gamma_{ij} \in R$ and the $a_i \in A$, is solvable for $x_1, \cdots, x_n$ in A whenever it is solvable in B.

We call a submodule A of an R-module B *strongly pure* if to each element $a \in A$ (equivalently, any finite set $\{a_1, \cdots, a_n\}$ of elements of A) there exists a morphism $\alpha : B \to A$ such that $\alpha(a) = a$ $(\alpha(a_i) = a_i,\ i = 1, \cdots, n)$. The name strongly pure is appropriate. For any strongly pure submodule is clearly pure; but the converse is not true. The Z-module of all p-adic integers has uncountably many pure submodules, but has no non-zero proper strongly pure submodules. The strongly pure submodules of Z-modules have been studied in [7]. Let $A \subset B \subset C$. If A is strongly pure in B and B is strongly pure in C, then A is strongly pure in C. If A is strongly pure in C, then A is strongly pure in B. The union of an increasing chain of strongly pure submodules is again strongly pure. If B is projective, then $A \subseteq B$ is pure in B if and only if A is strongly pure ([2]).

3. Sap modules

Our first theorem connects finite injectivity with strong purity.

THEOREM 3.1. *A right R-module A is finitely injective if and only if A is strongly pure in every R-module containing A as a submodule.*

PROOF. First note that A is strongly pure in every containing module if and only if A is strongly pure in its injective hull $\hat{A}$.

IF: Suppose we have a row exact diagram

$$\begin{array}{ccccc} o & \to & X & \to & Y \\ & & \alpha\downarrow & & \\ & & A & & \end{array}$$

with X finitely generated by, say, $x_1, \cdots, x_n$. By the injectivity of $\hat{A}$, α extends to a homomorphism α' from Y to $\hat{A}$. If $f : \hat{A} \to A$ is a morphism satisfying $f(\alpha(x_k)) = \alpha(x_k)$, $k = 1, \cdots, n$, then $\beta = f\alpha'$ makes the diagram

$$\begin{array}{ccccc} o & \to & X & \to & Y \\ & & \alpha\downarrow & \swarrow\beta & \\ & & A & & \end{array}$$

commutative. Thus A is finitely injective.

ONLY IF: Let A be a submodule of B. Then for any finitely generated submodule C of A, the diagram

$$\begin{array}{ccc} 0 \to C & \xrightarrow{f} & B, \\ g\downarrow & & \\ A & & \end{array}$$

where f and g are the inclusion maps, yields a morphism $h: B \to A$ such that $hf = g$, that is, $h|C = 1_C$. This implies that A is strongly pure in B.

A module which is pure in every containing module is called an absolutely pure module by Maddox [8]. In view of theorem (3.1), the finitely injective modules may be called *strongly absolutely pure* (for short, SAP) modules.

REMARK 3.2. Any injective module is trivially a SAP module but the converse is not true. For example, let $R = \Pi_{p \in P} Z/pZ$, where Z is the ring of integers and P is the set of all positive prime integers and let $T = \Sigma \oplus_{p \in P} Z/pZ$. The R-module T is SAP but is not injective. It is also clear that a SAP module is absolutely pure but not conversely. For instance, the ring R above is von Neumann regular and so every R-module is absolutely pure. But $A = R/T$ is not injective (see [10]) and so it is not SAP as an R-module as the following corollary (3.4i) shows.

PROPOSITION 3.3. *$A \in M_R$ is SAP if and only if A contains an injective hull of each of its finitely generated submodules.*

PROOF. $\Rightarrow$ If S is a finitely generated submodule of A with an injective hull $\hat{S}$, the diagram

$$\begin{array}{ccc} o \to S & \xrightarrow{f} & \hat{S}, \\ g\downarrow & & \\ A & & \end{array}$$

where f, g are inclusion maps, yields a $g': \hat{S} \to A$ such that $g'f = g$. Since g is monic and S is essential in $\hat{S}$, g' is a monomorphism.

$\Leftarrow$ Obvious.

COROLLARY 3.4. (i) *Any finitely generated SAP module is injective.*

(ii) *Any countably generated SAP module is a direct sum of injective modules.*

(iii) *Direct sums and direct products of SAP modules are SAP and a direct summand of a SAP module is again SAP.*

(iv) *A projective module is SAP if and only if it is a direct sum of injective modules.*

(v) *An indecomposable or uniform SAP module is simple and injective.*

PROOF. We prove only (ii) and (iv). Let A be a countably generated SAP module. Then $A = \bigcup_{k=1}^{\infty} A_k$, where $\{A_k\}$ is an increasing sequence of finitely generated submodules of A. Since, for each k, A contains the injective hull $\hat{A}_k$, $A = \bigcup_{k=1}^{\infty} \hat{A}_k$, where we may assume that $\{\hat{A}_k\}$ is again an ascending chain. Let $B_1 = \hat{A}_1$ and $A_{k+1} = B_k \oplus \hat{A}_k$, for all k. Then $A = \sum \oplus_{k=1}^{\infty} B_k$, where each B_k is injective.

Now (iv) follows from (ii), (iii) and the well known ([6]) fact that a projective module is a direct sum of countably generated modules.

The example in Remark (3.2) shows that, in general, a factor module of a SAP module is not SAP. One may ask: when all the factor modules of the SAP modules over a ring R are again SAP? Note that all the factor modules of SAP modules over a ring R are SAP if and only if all the factor modules of each injective R-module are SAP. The following proposition adds some more information.

PROPOSITION 3.5. (i) *If the homomorphic images of SAP right R-modules are SAP, then R is right semi-hereditary.* (ii) *If R is right hereditary, then the homomorphic images of SAP modules are again SAP.*

PROOF. (i) Observe that, by hypothesis, the homomorphic images of injective R-modules are SAP. Hence any diagram

$$\begin{array}{ccccc} 0 \to & I & \xrightarrow{g} & R & \\ & f\downarrow & & & \\ A & \xrightarrow{p} & B & \to 0 & \end{array}$$

where the rows are exact, I is a finitely generated ideal and A injective, yields a morphism $h: R \to B$ satisfying $hg = f$. Since R is projective, there exists $\alpha: R \to A$ satisfying $p\alpha = h$. If $f' = \alpha g$, then $f': I \to A$ satisfies $pf' = f$. Thus I is projective and R is semi-hereditary.

(ii) Let A be a SAP R-module and $f: A \to B$ epic. Let $S \subseteq B$ be finitely generated and let T be a finitely generated submodule of A with $f(T) = S$. Now $A \supseteq \hat{T}$ and so $B \supseteq f(\hat{T}) \supseteq S$. Since $f(\hat{T})$ is injective, B is SAP by Proposition (3.3).

REMARK 3.6. The statement (3.5ii) fails to be true if we replace 'heredity' by 'semi-heredity'. If R is the ring stated in (3.2), R is von Neumann regular and so is semi-hereditary. R/T is not a SAP module; but R is.

Let $\{A_i, \phi_{ij}\}$, $i, j \in I$, be a directed system of injective modules with $\phi_{ij}: A_i \to A_j$, for all $i, j \in I$ with $i \leqq j$, being monic. Then by using Proposition (3.3), one easily shows that $\varinjlim A_i$ is a SAP module. The following proposition shows that every SAP module is such a direct limit.

PROPOSITION 3.7. *An R-module A is SAP* $\Leftrightarrow A = \varinjlim A_i$, *where the* A_i *are injective modules and the morphisms of the directed system* $\{A_i\}$ *are monic.*

PROOF. Let A be a SAP module. Now $A = \lim_{\rightarrow} \{A_i, \phi_{ij}\}$, $i, j \in I$, where $\{A_i\}$ is the family of all the finitely generated submodules of A and the ϕ_{ij} are simply the inclusion maps. Since A is SAP, for each $i \in I$, $A \supseteq \hat{A}_i$, an injective hull of A_i and clearly $A = \cup \hat{A}_i$. The $\phi_{ij}: A_i \rightarrow A_j$ induce monomorphisms $\hat{\phi}_{ij}: \hat{A}_i \rightarrow \hat{A}_j$ so that $\{\hat{A}_i, \hat{\phi}_{ij}\}$, $i, j \in I$ is a directed system and $A = \lim_{\rightarrow} \hat{A}_i$.

REMARK 3.8. (i) Proposition (3.7) remains valid if we replace 'injective' by 'SAP'. On the other hand, it fails to be true if we do not assume that the morphisms of the directed system are monic. Indeed if $A = R/T$ is the module mentioned in (3.2), A is not a SAP module. However, since R is von Neumann regular, $A = \lim_{\rightarrow} A_i$, where the A_i are finitely generated free. Moreover, each A_i is injective, since R is self-injective. Thus a direct limit of SAP modules need not be SAP.

(ii) We wish to mention, without proof, that if R is hereditary, then the SAP R-modules coincide with the direct limits of injective R-modules and are closed with respect to taking direct limits.

(3.9) SAP Hulls. Let A be any R-module. Suppose S is a SAP module and $A \subseteq S \subseteq \hat{A}$, where A is an injective hull of $\hat{A}$. If $\{A_i\}$ is the system of all the finitely generated submodules of A directed by inclusion, then $S \supseteq \bar{A} = \lim_{\rightarrow} \hat{A}_i \supseteq A$. Now $\bar{A}$ is a SAP module containing A and is minimal in the sense that given any diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & \bar{A}, \\ \beta\downarrow & & \\ B & & \end{array}$$

where α is the natural inclusion map, B a SAP R-module and β monic, there is a monomorphism $\gamma: \bar{A} \rightarrow B$ such that $\gamma\alpha = \beta$. We call $\bar{A}$ a *SAP hull* of A. Any two SAP hulls of A are isomorphic. Observe that if A is countably generated, then its SAP hull $\bar{A}$ is a direct sum of injective modules. Hence the SAP hulls of projective modules are always direct sums of injectives.

(3.10) Rings determined by the SAP modules.

(a) Every (finitely generated) R-module is SAP if and only if R is Artinian semi-simple. This follows from (3.4i) and the fact ([10]) that if all the cyclic R-modules are injective, then R is Artinian semi-simple.

(b) Every SAP module over R is injective if and only if R is right Noetherian. If R is Noetherian, then trivially every SAP is injective. If we assume the converse, then any direct sum of injective R-modules (being SAP) is injective so that R is Noetherian by a well-known theorem of Bass (see [2]).

(c) Every projective R-module is SAP if and only if R is self-injective.

REMARK 3.11. (i) A SAP module (even though finitely injective) need not be quasi-injective and a quasi-injective module need not be SAP. Let R and T be

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as mentioned in (3.2) and $M = R \oplus T$. Since R is self-injective, M is SAP as an R-module. But M is not quasi-injective. On the other hand, any simple Z-module is quasi-injective but is evidently not SAP.

(ii) An extension of a SAP module by another SAP module need not be SAP. For example, let S be the regular subring of the ring R of (3.2) containing $T = \sum \oplus_{p \in P} Z/pZ$ and the identity of R such that $S/T \cong Q$, the field of rational numbers. Now T and S/T are SAP as S-modules, but S is not a SAP S-module, since S is not injective.

4. A generalisation

Let M be a fixed R-module. We call an R-module A *finitely M-injective* if every homomorphism from each finitely generated submodule N of M to A extends to a homomorphism from M to A. Following closely the ideas of G. Azumaya [1], we give below a characterization of finitely M-injective modules.

THEOREM 4.1. *Let M be a fixed R-module. Then the following are equivalent for any R-module A.*

(i) *A is finitely M-injective*

(ii) *Any exact sequence $0 \to A \xrightarrow{\alpha} B$ splits, whenever there exists a $\beta: M \to B$ with $\alpha(A) + \beta(M) = B$ and $\beta^{-1}(\alpha(A))$ is finitely generated.*

PROOF. (i) $\Rightarrow$ (ii). Let $N = \beta^{-1}(\alpha(A))$. By hypothesis, the homomorphism $\alpha^{-1}\beta: N \to A$ extends to a $\gamma: M \to A$. Now, for any $b \in B$, if $b = \alpha(a) + \beta(m) = \alpha(a') + \beta(m')$, with $a,\ a' \in A$ and $m, m' \in M$, then $\alpha(a - a') = \beta(m' - m) \in \alpha(A) \cap \beta(M)$ so that $a - a' \in \alpha^{-1}\beta(m' - m) = \gamma(m' - m)$ since $m' - m \in N$. Thus $a + \gamma\ m = a' + \gamma(m')$. Hence the correspondence δ: $B \to A$ defined by $\delta(b) = a + \gamma(m)$ whenever $b = \alpha(a) + \beta(m)$ is well defined and is indeed a homomorphism satisfying $\delta\alpha = 1_A$. Hence the sequence $0 \to A \xrightarrow{\alpha} B$ splits.

(ii) $\Rightarrow$ (i). Let N be a finitely generated submodule of M and $g: N \to A$ be a given morphism. Consider the direct sum $M \xrightarrow{i} M \oplus A \xleftarrow{j} A$, with i and j being natural injections. Define $h: N \to M \oplus A$ by $h(n) = n - g(n)$, for each $n \in N$. If $p: M \oplus A \to (M \oplus A)/(h(N)) = B$ is the canonical quotient map, then pi extends pjg and pj is a monomorphism. Further $pi(M) + pj(A) = B$ and $(pi)^{-1}(pj(A)) = N$ is finitely generated. Hence, by hypothesis, there exists $\phi: B \to A$ such that $\phi pj = 1_A$. Let $f = \phi pi: M \to A$. Then for all $n \in N$, $f(n) = \phi pi(n) = \phi pjg(n) = g(n)$. Hence A is finitely M-injective.

Note that an R-module A is SAP if and only if A is finitely M-injective for every R-module M. Hence Theorem (4.1) will yield another characterization of SAP modules. Megibben [9] has shown that A is absolutely pure exactly when A is finitely P-injective for every projective module P. Thus the absolute purity of A is equivalent to the condition (ii) of (4.1) if B is allowed to vary over projective modules

(4.2). Let M be a fixed right R-module. We state some simple properties of finitely M-injective modules:

(i) A direct summand of a finitely M-injective module is again finitely M-injective

(ii) Direct sums of finitely M-injective modules are again finitely M-injective.

Recall that, in the definition of finitely M-injective modules, if we drop the condition that N is finitely generated, we get the definition of M-injective modules. In contrast to (ii), direct sums of M-injective modules may not be M-injective. In fact, bodily lifting the proof of a result of H. Bass (see Proposition (4.1) in [2]) one could show that if M is finitely generated, then direct sums of M-injective modules are again M-injective exactly when M is Noetherian, that is, M satisfies the ascending chain condition for submodules. This remark, together with (ii), gives at once the following:

(iii) Let M be finitely generated. Then every finitely M-injective module is M-injective if and only if M is Noetherian.

Imitating the proof of (3.5ii) we get a condition for a projective module to be 'semi-hereditary':

(iv) Let M be a projective R-module. Then every finitely generated submodule of M is projective if and only if homomorphic images of finitely M-injective modules are again finitely M-injective.

It is not clear if A is finitely $(M_1 \oplus M_2)$-injective whenever A is finitely injective with respect to both M_1 and M_2.

5. Finitely quasi-injective modules and strongly regular modules

Modules M which are finitely M-injective may be called — in line with the prevailing terminology — *finitely quasi-injective* modules. Clearly any quasi-injective module is finitely quasi-injective, but not conversely. Indeed the module M in (3.11i) justifies this statement. However, the following proposition shows that the distinction between quasi-injectives and finitely quasi-injectives vanishes for modules over Noetherian rings.

PROPOSITION 5.1. *Let R be right Noetherian. Then an R-module A is finitely quasi-injective if and only if A is quasi-injective.*

PROOF. We prove just the 'only if' part. Fuchs [5] has shown that A is quasi-injective if given any cyclic submodule S of A and a submodule $T \subseteq S$, every homomorphism $\alpha: T \to A$ extends to a $\beta: S \to A$. If R is Noetherian, a submodule of a cyclic module is finitely generated and so a finitely quasi-injective R-module becomes quasi-injective.

If A is a module in which every finitely generated submodule is a direct summand, then clearly A is finitely quasi-injective. We call such modules *strongly regular*. This is justified by the fact that *A is strongly regular if and only if every submodule of A is strongly pure*. As a first step towards the study of the finitely quasi-injective modules, we investigate the strongly regular modules. In general, a finitely quasi-injective module may not be strongly regular, as is clear by considering the Z-module Q of rational numbers. It is easy to see that every finitely quasi-injective module over a ring R is strongly regular if and only if R is Artinian semi-simple.

The following proposition gives a (rather unsatisfactory) condition for a finitely quasi-injective module to be strongly regular.

PROPOSITION 5.2. *An R-module A is strongly regular if and only if A is finitely quasi-injective and every finitely generated submodule is isomorphic to a summand of A.*

The proof follows immediately from the following lemma.

LEMMA 5.3. *If A is a finitely quasi-injective module, then a finitely generated submodule S is a summand of A if and only if S is isomorphic to a summand of A.*

PROOF. Let $A = B \oplus C$ with $\eta: A \to B$ the canonical projection and $\sigma: S \to B$ an isomorphism. If σ extends to an endomorphism ϕ of S, then $\sigma^{-1}\eta\phi: A \to S$ is an idempotent endomorphism of A, whence S is a direct summand.

COROLLARY 5.4. a) *Let R be right semi-hereditary (or more generally, let every principal right ideal of M be projective). Then*

(i) *A free R-module is strongly regular if and only if it is finitely quasi-injective.*

(ii) *R is von Neumann regular if and only if R is finitely quasi-injective as an R-module.*

b) *If the ring R is finitely quasi-injective as an R-module, then any element of R which is not a right zero divisor has a right inverse.*

(5.5) Observe that if A is strongly regular, then every submodule of A is again strongly regular. It is immediate from the definition that a finitely generated strongly regular module is a direct sum of strongly regular cyclic modules. Hence, proceeding as in the proof of (3.4 ii), we also get that a *countably generated strongly regular module is a direct sum of strongly regular cyclic modules*. If A is any strongly regular module, then every countably generated submodule is a direct sum of strongly regular cyclic modules.

PROPOSITION 5.6. *Let R/J be right Noetherian, where J is the Jacobson radical of R. Then a right R-module A is strongly regular if and only if A is semi-simple.*

PROOF. Note that over any ring R, $AJ=0$ for a strongly regular R-module A. This is because for any $a \in A$, aR is strongly regular and so $\mathrm{Rad}(aR) = 0$. This means that $J \subseteq (o: a)$. Thus $J \subseteq (o: A)$ and A becomes a (strongly regular) R/J-module. On the other hand over a Noetherian ring S any finitely generated strongly regular module is semi-simple and so every strongly regular S-module is semi-simple. This proves the proposition.

Strong regularity appears to have some connection with von Neumann regular rings. It is known ([6]) that over a von Neumann regular ring, every projective module is strongly regular. Let R be a commutative ring. Then note that a cyclic R-module $A \cong R/I$ is strongly regular if and only if the ring R/I is von Neumann regular. Let $S = \Pi S_r$, where S_r varies over all the homomorphic images of R which are von Neumann regular. Then S is a von Neumann regular ring and any strongly regular R-module can be made a (strongly regular) S-module. Moreover, by (5.5), any countably generated (hence any projective) strongly regular R-module becomes a projective S-module.

In [4] and [11], the strongly regular projective modules have been studied under the name regular modules. The following lemma enables us to slightly sharpen up the structure theorem of strongly regular projective modules.

LEMMA 5.7. *A cyclic projective module P over a ring R is strongly regular if and only if P is isomorphic to a principal right ideal $\subset M(R)$, the maximal regular ideal of R.*

The proof follows on noting that if a right ideal T is strongly regular as an R-module, then T becomes regular in the sense of von Neumann.

Since any projective module is a direct sum of countably generated modules [6)], by appealing to (5.5) and (5.7) and noting that a right ideal of $M(R)$ is a right ideal of R, we obtain the structure theorem: *A strongly regular projective module over any ring R is a direct sum of principal right ideals of the regular radical $M(R)$.*

Finally, note that a special class of strongly regular modules are those modules A all of whose submodules are SAP. These are just the strongly regular SAP modules, or equivalently, the modules whose finitely generated submodules are injective. Projective modules over a self-injective regular ring are modules of the above type. We do not know if the strongly regular SAP modules must be direct sums of injective modules.

References

[1] G. Azumaya, '*M*-projective and *M*-injective modules', (unpublished).

[2] S. U. Chase, 'Direct products of modules', *Trans. Amer. Math. Soc.* 97 (1960), 457–473.

[3] P. M. Cohn, 'On the free product of associative rings', *Math. Z.* 71 (1959), 380–398.

[4] D. J. Fieldhouse, *Regular modules and rings*, (Queen's Univ., Preprint 1967).

[5] L. Fuchs, 'On quasi-injective modules', *Ann. Scuola. Norm. Sup. Pisa* (3) 23 (1969), 541–546.
[6] I. Kaplansky, 'Projective Modules', *Ann. of Math.* 68 (1958), 372–377.
[7] S. Janakiraman, Ph. D. thesis, Madurai Univ. (1970).
[8] B.H. Maddox, 'Absolutely pure modules', *Proc. Amer. Math. Soc.* 18 (1967), 155–158.
[9] C. Megibben, 'Absolutely pure modules', *Proc. Amer. Math. Soc.* 26 (1970), 561–566.
[10] B.L. Osofsky, 'Noninjective cyclic modules', *Proc. Amer. Math. Soc.* 19 (1968), 1383–1384.
[11] R. Ware, 'Endomorphism rings of projective modules', *Trans. Amer. Math. Soc.* 155 (1971), 233-256.

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